F A S C I C U L I   M A T H E M A T I C I

Nr 35 2005

Devendra Kumar

APPROXIMATION OF GROWTH NUMBERS OF GENERALIZED BI-AXIALLY SYMMETRIC POTENTIALS

Abstract: The paper deals with growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called Generalized Bi-Axially Symmetric Potentials (GBASP′s). The GBASP′s are taken to be regular in a finite hyperball and influence of the growth of their maximum moduli on the rate of decay of their approximation errors in sup norm is studied. The author has been obtained the characterizations of the q-growth number and lower q-growth number of a GBASP $H \in H_R$, $0 < R < \infty$ in terms of rate of decay of approximation error $E_n(H, R_0)$, $0 < R_0 < \infty$. Finally we have obtained a necessary condition for a GBASP $H \in H_R$ to be of perfectly regular growth.

Key words: generalized bi-axially symmetric potentials, index q, GBASP polynomial approximates in sup norm.

1. Introduction

Generalized bi-axially symmetric potentials (GBASP′s) are the solutions of the elliptic partial differential equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2\alpha+1}{y}\frac{\partial H}{\partial y} + \frac{2\beta+1}{x}\frac{\partial H}{\partial x} = 0, \quad \alpha, \beta > -\frac{1}{2}, \tag{1.1}$$

which are even in x and$y$ Gilbert [2]. A polynomial of degree n which is even in x and y is said to be a GBASP polynomial of degree n if it satisfies (1.1). A GBASP H, regular about origin, can be expanded as

$$H = H(r, \theta) = \sum_{n=0}^{\infty} a_n\, r^{2n} P_n^{(\alpha,\beta)}(\cos 2\,\theta) \tag{1.2}$$

where $x = r\cos\theta$, $y = r sin\theta$ and $P_n^{(\alpha,\beta)}(t)$ are Jacobi polynomials.

Let $D_R = \{(x,y) : x^2 + y^2 < R^2\}$, $0 \le R \le \infty$ and $\overline{D}_R$ be the closure of D_R. A GBASP H is said to be regular in D_R if series (1.2) converges uniformly on every compact subset of D_R. Let H_R be the class of all GBASP$'s$ H regular in $D_{R'}$, for every $R' \le R$ but for no $R' > R$. The functions in the class H_∞ are called entire GBASP$'s$.

McCoy [7] considered the approximation of pseudonaltyic functions on the disc. Pseudonanlytic functions are constructed as complex combinations of real-valued analytic solutions to be Stokes-Beltrami system. These solutions include the GBASP$'s$. McCoy obtained some coefficient and Bernstein type growth theorems on the disc. It is significant to mension that hear the results and methods are different from those of McCoy.

A GBASP H is said to be regular on $\overline{D}_{R_0}$, $0 < R_0 < \infty$, the closure of D_{R_0}, if it is regular in D_R for some $R < R_0$. Let $\overline{H}_{R_0}$ be the class of all GBASP$'s$ a regular on D_{R_0}. For $\overline{H} \in \overline{H}_{R_0}$, set

$$\|H\|_{R_0} = \max_{(x,y)\in\overline{D}_{R_0}} |H(x,y)|\,, \tag{1.3}$$

where $\|\cdot\|$ is the uniform norm on $\overline{H}_{R_0}$.

For $H \in \overline{H}_{R_0}$, the approximation error $E_n(H, R_0)$ is defined as

$$E_n(H,R_0) = \inf_{g\in\pi_n} \|H - g\|_{R_0} \tag{1.4}$$

where π_n consists of all GBASP polynomials of degree at most $2n$.

The concept of index q, the q-order $\rho(q)$ and lower q-order $\lambda(q)$ are introduced by Bajpai et al. [1] in order to obtain a measure the growth of the maximum modulus, when it is rapidly increasing. Thus, let $M(r,H) \to \infty$, $r \to R$ and $q = 2, 3, \ldots$ set

$$\rho_q(H,R) = \limsup_{r\to R} \frac{\log^{[q]} M(r,H)}{\log(R/(R-r))}$$

where $M(r,H) = \max_\theta |H(r,\theta)|$ is the maximum modulus function and $\log^{[0]} M(r,H) = M(r,H)$, $\log^{[q-1]} M(r,H) = \log\log^{[q-2]} M(r,H)$. The GBASP $H \in H_R$ is said to have the index q if $\rho_q(H,R) < \infty$ and $\rho_{q-1}(H,R) = \infty$. If q is the index of H then $\rho_q(H,R)$ is called the q-order of H.

Let $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$, $(a_k \ne 0)$ and $v(r) = \max\{k : \mu(r) = |a_k| r^k\}$ $(0 < r < R)$. The q-growth number γ and lower q-growth number δ of $f(z)$ are defined analogues to Kapoor [4] as

$$\gamma = \limsup_{r\to R} \frac{\log^{[q-2]} v(r)}{(Rr/(R-r))^{\rho(q)+A(q)}}\,,$$

$$\delta = \liminf_{r\to R} \frac{\log^{[q-2]} v(r)}{(R\,r/(R-r))^{\rho(q)+A(q)}} ,$$

where $A(q) = 1$ if $q = 2$, $A(q) = 0$ if $q = 3, 4, ...$ and $\mu(r)$ is the maximum term of $f(z)$ on $|z| = r$.

In the same manner the q-growth number $\gamma_q(H)$ and lower q-growth number $\delta_q(H)$ of the GBASP $H \in H_R$ are defined as

$$\gamma_q(H) = \limsup_{r\to R} \frac{\log^{[q-2]} v(r,H)}{(R\,r/(R-r))^{\rho_q(H,R)+A(q,H)}} ,$$

$$\delta_q(H) = \liminf_{r\to R} \frac{\log^{[q-2]} v(r,H)}{(R\,r/(R-r))^{\rho_q(H,R)+A(q,H)}} ,$$

where $A(q,H) = 1$, if $q = 2$, $A(q,H) = 0$, otherwise, and $v(r,H) = \max\{k : \mu(r,H)\} = \max_{\theta} \left\{a_k\, r^{2k} P_k^{(\alpha,\beta)}(\cos 2\theta)\right\}$ is the rank of maximum term of $H \in H_R$.

We have the following definitions:

Definition 1. *A GBASP $H \in H_R$, $0 < R < \infty$, having q-order $\rho_q(H,R)(\rho_q(H,R) > 0$, $q = 2,3...$, is said to have q-type $T_q(H,R)$ and lower q-type $t_q(H,R)$ if*

$$T_q(H,R) = \limsup_{r\to R} \frac{\log^{[q-1]} M(r,H)}{(R\,r/(R-r))^{\rho_q(H,R)}} ,$$

$$t_q(H,R) = \liminf_{r\to R} \frac{\log^{[q-1]} M(r,H)}{(R\,r/(R-r))^{\rho_q(H,R)}} ,$$

$(0 \le t_q(H,R) < T_q(H,R) < \infty)$

Definition 2. *A GBASP $H \in H_R$ having regular growth is said to be of perfectly regular growth if $T_q(H,R) = t_q(H,R) < \infty$.*

Generally, the growth of entire functions is measured in terms of its order and type. It has been observed ([1], [3]) that the growth numbers play an important role to studying the growth of entire/analytic functions. Juneja [3] characterized some theorems involving the ratio of the consecutive coefficients occurring in the Taylor series expansion of analytic functions and growth numbers for $q = 2$ in unit disc. The GBASP are natural extensions of analytic/harmonic functions. Hence we anticipate properties similar to those of the harmonic functions found from associated analytic functions by taking its real part [6]. In this article we have obtained the characterizations of the q-growth number and lower q-growth number of a GBASP H in terms of the ratio of approximation errors defined by (1.4) in sup norm in finite disc and a necessary condition for GBASP to be of perfectly regular growth.

2. Auxiliary results

Now we give some auxiliary results which have been used in proving the main theorems.

Lemma 2.1. *Let* $f(z) = \sum_{k=0}^{\infty} a_{n_k} z^{n_k}$ *have q-growth number* γ *and lower q-growth number* δ. *Let* $u(n_k) = \frac{1}{(n_{k+1}-n_k)} \log^+ \left|a_{n_k}/a_{n_{k+1}}\right|, u(n_k) > u(n_{k-1})$ *for* $k > k_0$ *and* $\{n_k\}$ *denotes the sequence of principal indices such that* $n_k \to \infty$ *as* $k \to \infty$. *Then*

$$\gamma = \limsup_{k\to\infty} \log^{[q-2]} n_k \left(\frac{1}{(n_k - n_{k-1})} \log^+ \left| \frac{a_k}{a_{n_{k-1}}} \right| \right)^{\rho(q)+A(q)} \tag{2.11}$$

and

$$\delta \geq \limsup_{k\to\infty} \log^{[q-2]} n_{k-1} \left(\frac{1}{(n_k - n_{k-1})} \log^+ \left| \frac{a_k}{a_{n_{k-1}}} \right| \right)^{\rho(q)+A(q)} \tag{2.12}$$

where $A(q) = 1$ *if* $q = 2$ *and* $A(q) = 0$ *if* $q = 3, 4, \ldots$. *The equality holds in (2.12) provided* $\log^{[q-2]} n_{k+1} \sim \log^{[q-2]} n_k$ *as* $k \to \infty$, $\log^+ t = \max(0, \log t)$ *for* $t > 0$.

Proof. First let $\gamma < \infty$. By definition of γ for any $\varepsilon > 0$ and for all r such that $R > r > r_0 = r_0(\varepsilon)$, we have

$$\log^{[q-2]} v(r) < (\gamma + \varepsilon) \left(\frac{R\,r}{R - r} \right)^{\rho(q)+A(q)}.$$

Since $u(n_k)$ forms a strictly increasing function of k for $k > k_0$,

$$v(r) = n_k \quad \text{for} \quad u(n_{k-1}) \leq r < u(n_k).$$

Hence, for r in the above range

$$\log^{[q-2]} n_k < (\gamma + \varepsilon) \left(\frac{R\,r}{R - r} \right)^{\rho(q)+A(q)}.$$

Since the inequality holds for all $r \geq u(n_{k-1})$, we have

$$\begin{aligned} (\gamma + \varepsilon) \quad &> \quad \log^{[q-2]} n_k \left[\frac{R - u(n_{k-1})}{R\,u(n_{k-1})} \right]^{\rho(q)+A(q)} \\ &\approx \quad \log^{[q-2]} n_k \left[\frac{1}{u(n_{k-1})} \right]^{\rho(q)+A(q)} \qquad \text{as} \quad k \to \infty. \end{aligned}$$

Now $u(n_k) \to R$ as $k \to \infty$, therefore on proceeding to limits we get

$$\gamma \geq \limsup_{k\to\infty} \log^{[q-2]} n_k \left(\frac{1}{n_k - n_{k-1}} \log^+ \left| \frac{a_{n_k}}{a_{n_{k-1}}} \right| \right)^{\rho(q)+A(q)} \tag{2.13}$$

This inequality in obviously true for $\gamma = \infty$.

Next assume $y > 0$. Then for every ε such the $\gamma > \varepsilon > 0$, there exists a sequence $\{r_{p_k}\}$ tending to R as $k \to \infty$ such that

$$\log^{[q-2]} v\,(r_{p_k}) > (\gamma - \varepsilon) \left[\frac{R\, r_{p_k}}{R - r_{p_k}} \right]^{\rho(q)+A(q)} \quad \text{for} \quad k = 1, 2, 3, \dots. \tag{2.14}$$

Since $u(n_k)$ is strictly increasing function of k for $k > k_0$ and tends to R as $k \to \infty$, for every r_{p_k} we can find an integer n_{k_p} such that $u(n_{k_{p-1}}) \leq r_{p_k} < u(n_{p_k})$, and so we have $v(r_{p_k}) = n_{k_p}$. Thus (2.14) gives

$$\begin{aligned} (\gamma - \varepsilon) &< \log^{[q-2]} n_{k_p} \left[\frac{R\, u(n_{p_{k-1}})}{R - u(n_{k_{p+1}})} \right]^{\rho(q)+A(q)} \\ &\approx \log^{[q-2]} n_{k_p} \left[\frac{1}{u(n_{k_{p+1}})} \right]^{\rho(q)+A(q)} \quad \text{as} \quad p \to \infty. \end{aligned}$$

Proceeding to limits we get

$$\gamma \leq \limsup_{k\to\infty} \log^{[q-2]} n_k \left(\frac{1}{n_k - n_{k-1}} \log^+ \left| \frac{a_{n_k}}{a_{n_{k-1}}} \right| \right)^{\rho(q)+A(q)} \tag{2.15}$$

This inequality is obviously true if $\gamma = 0$ (2.13) and (2.15) give (2.11). Now for proving (2.12), we assume that

$$\liminf_{k\to\infty} \log^{[q-2]} n_{k-1} \left(\frac{1}{n_k - n_{k-1}} \log^+ \left| \frac{a_{n_k}}{a_{n_{k-1}}} \right| \right)^{\rho(q)+A(q)} = C$$

the inequality in (2.12) obviously holds if $C = 0$. Hence we assume that $0 < C < \infty$. Then for $\varepsilon > 0$ and taking k sufficiently large we have

$$\left(\frac{1}{n_k - n_{k-1}} \log^+ \left| \frac{a_{n_k}}{a_{n_{k-1}}} \right| \right)^{\rho(q)+A(q)} \log^{[q-2]} n_{k-1} > C - \varepsilon$$

$$\frac{1}{n_k - n_{k-1}} \log^+ \left| \frac{a_{n_k}}{a_{n_{k-1}}} \right| > \left[\frac{C - \varepsilon}{\log^{[q-2]} n_{k-1}} \right]^{1/(\rho(q)+A(q))}$$

or

$$\log^+ \left| \frac{a_{n_k}}{a_{n_{k-1}}} \right| > (n_k - n_{k-1}) \left[\frac{C - \varepsilon}{\log^{[q-2]} n_{k-1}} \right]^{1/(\rho(q)+A(q))} \tag{2.16}$$

Writing the above inequality for $k = N+1, N+2, ..., m$ and adding, we get

$$\text{(2.17)} \qquad \log^{+}\left|\frac{a_{n_k}}{a_{n_{k-1}}}\right| > \sum_{k=N+1}^{m} (n_k - n_{k-1}) \left[\frac{C-\varepsilon}{\log^{[q-2]} n_{k-1}}\right]^{1/(\rho(q)+A(q))}$$

to estimate the right hand side, we put $v(t) = n_k$ for $n_{k-1} < t \le n_k$ and $G(t) = \left[(C-\varepsilon)/\log^{|q-2|} t\right]^{1/(\rho(q)+A(q))}$. Hence right hand side of (2.17), can be written as

$$\begin{aligned}
&\sum_{k=N+1}^{m} G(n_{k-1})(n_k - n_{k-1}) = (n_m - n_{m-1}) G(n_{m-1}) \\
&\quad + (n_{m-1} - n_{m-2}) G(n_{m-2}) + K + (n_{N+2} - n_{N+1}) G(n_{N+1}) \\
&\quad + (n_{N+1} - n_N) G(n_N) = n_m G(n_{m-1}) - n_{m-1}\{G(n_{m-1}) G(n_{m-2})\} \\
&\quad + K - n_{N+1}\{G(n_{N+1}) - G(n_N)\} - n_N G(n_N) \\
&= n_m G(n_{m-1}) - \sum_{k=N+1}^{m} n_{k-1}\{G(n_{k-1})) - G(n_{k-2})\} - n_N G(n_N) \\
&= n_m G(n_{m-1}) - \int_{t_N}^{t_{m-1}} v(t)\, dG(t) - n_N G(n_N) \\
&= n_m G(n_{m-1}) + \frac{1}{\rho(q) + A(q)} \int_{t_N}^{t_{m-1}} \frac{v(t) dG(t)}{t \prod_{i=1}^{q-2} \log^{[i]} t} + O(1).
\end{aligned}$$

Since $\frac{v(t)}{t} \ge 1$ therefore in view of above expression with (2.17) we have

$$\begin{aligned}
&\log^{+}|a_{n_m}| > n_m G(n_{m-1}) + \frac{(n_{m-1} - n_N) G(n_{m-1})}{(\rho(q) + A(q)) \prod_{i=1}^{q-2} \log^{[i]} n_{m-1}} - O(1) \\
&= n_m \left[\frac{C-\varepsilon}{\log^{[q-2]} n_{m-1}}\right]^{1/(\rho(q)+A(q))} \\
&\quad + \frac{(C-\varepsilon)^{1/(\rho(q)+A(q))} (n_{m-1} - n_N)}{(\rho(q) + A(q)) \left(\log^{[q-2]} n_{m-1}\right)^{1/(\rho(q)+A(q))} \prod_{i=1}^{q-2} \log^{[i]} n_{m-1}} - O(1) \\
&= \left[\frac{(C-\varepsilon)}{\log^{[q-2]} n_{m-1}}\right]^{1/(\rho(q)+A(q))} n_m \left[1 + \frac{n_{m-1}(1 - O(1))}{(\rho(q) + A(q)) \prod_{i=1}^{q-2} \log^{[i]} n_{m-1}}\right] - O(1) \\
&= \left[\frac{(C-\varepsilon)}{\log^{[q-2]} n_{m-1}}\right]^{1/(\rho(q)+A(q))} n_m (1 + o(1) - O(1)) \quad \text{for large } m.
\end{aligned}$$

Since $n_{m-1}/n_m < 1$, so

$$\log^{+}|a_{n_m}| > n_m \left[\frac{C-\varepsilon}{\log^{[q-2]} n_{m-1}}\right]^{1/(\rho(q)+A(q))}$$

Proceeding to limits, we get

$$\liminf_{m\to\infty} \log^{[q-2]} n_{m-1} \left[\frac{\log^{+}|a_{n_m}|}{n_m}\right]^{\rho(q)+A(q)} \geq C.$$

Since $\delta \geq$ left hand side of above expression so, we get $\delta \geq C$. To prove equality in (2.12) first we suppose that $\delta > 0$. By definition of δ, for any $\varepsilon > 0$ and for all r such that $R > r > r_0(\varepsilon)$, we have

$$\log^{[q-2]} v(r) > (\delta-\varepsilon)(R\,r/(R-r))^{\rho(q)+A(q)}.$$

Since $u(n_k)$ forms a strictly increasing function of k for $k > k_0$,

$$v(r) = n_k \quad \text{for} \quad u\left(n_{k-1}\right) \leq r < u\left(u_k\right).$$

Hence, for r in above range and using $\log^{[q-2]} n_{k-1} \sim \log^{[q-2]} n_k$ as $k \to \infty$. After a simple calculation, we obtain

$$\delta \leq \liminf_{k\to\infty} \log^{[q-2]} n_{k-1} \left(\frac{1}{n_k - n_{k-1}} \log^{+}\left|\frac{a_{n_k}}{a_{n_{k-1}}}\right|\right)^{\rho(q)+A(q)}.$$

This inequality is obviously true if $\delta = 0$. Hence the proof is completed. ■

Lemma 2.2. *Let $H \in H_R$, $O < R < \infty$, and $R > R_0$. Then*

$$M(r,H) < |a_0| + \frac{2\sqrt{P(\alpha,\beta)}}{\Gamma(\eta+1)} M(r,h),$$

where

$$h(u) = \sum_{n=1}^{\infty} \left((2n+\alpha+\beta+1)\,P\left(n,\alpha,\beta\right)\right)^{\frac{1}{2}} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)} \times E_{n-1}\left(H,R_0\right)\left(u/R_0\right)^{2n},$$

$$P\left(n,\alpha,\beta\right) = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}, \qquad P\left(\alpha,\beta\right) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Proof. Proof of above lemma follows from Kumar and Kasana [5]. ■

3. Main results

We prove the following theorems.

Theorem 3.1. *Let $H \in H_R$, $O < R < \infty$ ($R_0 < R$) and H have q-growth number $\gamma_q(H)$ and lower q-growth number $\delta_q(H)$. Let*

$$u\left(n_k\right)=\frac{1}{\left(n_{k+1}-n_k\right)} \log^{+} \frac{E_{n_k}(H, R_0)(R/R_0)^{2n_k}}{E_{n_{k+1}}(H, R_0)(R/R_0)^{2n_{k+1}}},$$

$u(n_k) > u(n_{k-1})$ for $k > k_0$ and $\{n_k\}$ has been defined as in Lemma 1. Then

$$\begin{aligned} \gamma_q(H) = \limsup_{k\to\infty} \log^{[q-2]} n_k & \\ \times \left[\frac{1}{(n_k-n_{k-1})} \log^{+} \frac{E_{n_k}(H, R_0)(R/R_0)^{2n_k}}{E_{n_{k+1}}(H, R_0)(R/R_0)^{2n_{k+1}}}\right]^{\rho^*(q)} & \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \delta_q(H) \geq \liminf_{k\to\infty} \log^{[q-2]} n_{k-1} & \\ \times \left[\frac{1}{(n_k-n_{k-1})} \log^{+} \frac{E_{n_k}(H, R_0)(R/R_0)^{2n_k}}{E_{n_{k+1}}(H, R_0)(R/R_0)^{2n_{k+1}}}\right]^{\rho^*(q)} & \end{aligned} \tag{3.12}$$

where $\rho^*(q) = \rho_q(H, R) + A(q, H)$. The equality holds in (3.12) provided $\log^{[q-2]} n_{k-1} \sim \log^{[q-2]} n_k$ as $k \to \infty$.

Proof. First let $\gamma_q(H) < \infty$. By (1.6) for any $\varepsilon > 0$ and for all r such that $R > r > r_0 = r_0(\varepsilon)$, we have

$$\log^{[q-2]} v(r, H) < (\gamma_q(H) + \varepsilon)(R\,r/(R-r))^{\rho^*(q)}.$$

Since $u(n_k)$ forms a strictly increasing function of k for $k > k_0$,

$$v(r, H) = v(r, h) = n_k \quad \text{for} \quad u\left(n_{k-1}\right) \leq r < u\left(n_k\right).$$

Hence for r in above range, we get

$$\log^{[q-2]} n_k < \left(\gamma_q(H) + \varepsilon\right)(R\,r/(R-r))^{\rho^*(q)}.$$

Since this inequality holds for all $r > u(n_{k-1})$, we get

$$\begin{aligned} (\gamma_q(H)+\varepsilon) &> \log^{[q-2]} n_k \left[\frac{R-u\left(n_{k-1}\right)}{R\,u\left(n_{k-1}\right)}\right]^{\rho^*(q)} \\ &\simeq \log^{[q-2]} n_k \left[\frac{1}{(u_{k-1})}\right]^{\rho^*(q)} \quad \text{as} \;\; k \to \infty. \end{aligned}$$

Now $u(n_k) \to R$ as $k \to \infty$, therefore on proceeding to limits, we have

$$\gamma_q(H) = \limsup_{k\to\infty} \log^{[q-2]} n_k \times \left[\log^+ \frac{E_{n_k}(H,R_0)(R/R_0)^{2n_k}}{E_{n_{k+1}}(H,R_0)(R/R_0)^{2n_{k+1}}}\right]^{\rho^*(q)}.$$

To prove the reverse inequality, use Lemma 1 and apply Lemma 2.2 to the function $h(u)$ given by (2.16), (3.12) can be proved in a similar manner on the lines of proof of (2.12). Hence the theorem follows. ■

Theorem 3.2. *Let $H \in H_R$, $0 < R < \infty$, $(R_0 < R)$ and H have q-growth number $\gamma_q(H)$ and lower q-growth number $\delta_q(H)$. Then*

$$\delta_q(H) \le \gamma_q(H) \liminf_{k\to\infty} \frac{\log^{[q-2]} n_k}{\log^{[q-2]} n_{k-1}} \tag{3.13}$$

Proof. Let $\alpha^* = \liminf_{k\to\infty} \frac{\log^{[q-2]} n_k}{\log^{[q-2]} n_{k-1}}$. If $\beta^* > \alpha^*$ there exists a sequence $\{c(m_k)\}$ such that $n_{c(m_k)} < \beta^* n_{c(m_{k+1})}$. Let r_t be a value of r at which $v(r,H)$ jumps from a value less than or equal to $n_{c(t)}$ to a value greater than or equal to $n_{c(t)+1}$. Now, since

$$v\left(r_t - 0, H\right) < n_{c(t)} < \beta^* n_{c(t)+1} \le \beta^* v\left(r_t + 0, H\right),$$

therefore

$$\begin{aligned}\delta_q(H) &\le \limsup_{t\to\infty} \frac{\log^{[q-2]} v\left(r_t - 0, H\right)}{(Rr/(R-r))^{\rho^*(q)}} \\ &\le \beta^* \le \limsup_{t\to\infty} \frac{\log^{[q-2]} v\left(r_t - 0, H\right)}{(R\,r/(R-r))^{\rho^*(q)}} \le \beta^* \gamma_q(H).\end{aligned}$$

Since the last inequality holds for all $\beta^* > \alpha^*$, hence $\delta_q(H) < \alpha^* \gamma_q(H)$, which proves (3.13). ■

Corollary. *If $H \in H_R$, $0 < R < \infty$ $(R_0 < R)$ and H is of perfectly regular growth, then* $\log^{[q-2]} n_k \approx \log^{[q-2]} n_{k-1}$ *as $k \to \infty$.*

The Corollary follows easily from Theorem 3.2 and in view of the fact that $0 < \delta_q(H) = \gamma_q(H) < \infty$ if and only if H is of perfectly regular growth.

References

[1] S.K. Bajpei, G.P. Kapoor, O.P. Juneja, On entire functions of fast Growth, *Trans. Amer. Math. Soc.,* 203(1975), 275–298.

[2] R.P. Gilbert, Intrigal operator methods in bi-axially symmetric potentials theory, *Contrib. Diff. Eqns.,* 2(1963), 441–458.

[3] O.P. Juneja, *Some properties of entire functions,* Dissertation, Indian Institute of Technology, Kanpur, 1965.

[4] G.P. Kapoor, On the proximate order and maximum term of functions analytic in the unit disc, *Rev Roumaine Math. Pures Appl.,* 18(1973), 1207–1215.

[5] D. Kumar, H.S. Kasana, On approximation of generalized bi-axially symmetric potentials, *Soochow J. Math.,* 21(4)(1995), 365–375.

[6] P.A. Mccoy, Approximation of pseudoanalytic functions on the disk, *Complex Variables: Theory Appl.,* 6(1986), 123–133.

[7] P.A. Mccoy, Polynomials approximation of generalized bi-axially symmetric potentials, *J. Approx. Theory,* 25(1979), 153–168.

Devendra Kumar
Department of Mathematics, Faculty of Science
P.O. Box 1176 Addis Ababa University
Addis Ababa, Ethiopia
e-mail: d_kumar001@rediffmail.com

Received on 09.09.2003 and, in revised from, on 17.03.2004