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Yuji Liu

GLOBAL ATTRACTIVITY IN A NON-AUTONOMOUS LOGISTIC TYPE MODEL WITH UNBOUNDED DELAY

Abstract: Consider the non-autonomous logistic model

$$\Delta x_n = p_n x_n \frac{1 - x_{n-k_n}}{1 - \lambda x_{n-k_n}}, \qquad n = 0, 1, 2, \cdots,$$

where $\{p_n\}$ is a sequence of positive real numbers, $\{k_n\}$ a sequence of nonnegative integers satisfying $\lim_{n\to\infty}(n-k_n) = \infty, \lambda \in [0, 1)$. We obtain new sufficient conditions for the attractivity of equilibrium $x = 1$ of the model, which improve and generalize some recent results established by Chen and Yu [4], Zhou and Zhang [6].

Key words: global attractivity, difference equation, oscillation

1. Introduction

The asymptotic behavior of solutions of difference equations with unbounded delay was studied in [1, 5, 6, 9, 10, 11]. In this paper, we consider the non-autonomous logistic model

$$\Delta x_n = p_n x_n \frac{1 - x_{n-k_n}}{1 - \lambda x_{n-k_n}}, \qquad n = 0, 1, 2, \cdots, \tag{1}$$

where $\{p_n\}$ is a sequence of positive real numbers, $\{k_n\}$ a sequence of nonnegative integers satisfying $\lim_{n\to\infty}(n-k_n) = \infty$, $\lambda \in [0, 1), \Delta x_n = x_{n+1} - x_n$. *Eq.*(1) contains the special case

$$\Delta x_n = p_n x_n(1 - x_{n-k_n}), \qquad n = 0, 1, 2, \cdots, \tag{2}$$

Let $r = -\min\{n - k_n, n \geq 0\}$ be a nonnegative integer, $\sigma_0 = \max\{n : n - k_n < 0\} + 1$, $\sigma = \max\{n : n - k_n < \sigma_0\} + 1$. By a solution of *Eq.*(1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -r$, and satisfies (1) for $n \geq 0$. The initial condition of *Eq.*(1) is

$$x_i = a_i, \quad i = -r, -r + 1, \cdots, 0, \tag{3}$$

with $a_i \in (0, 1/\lambda)$, for $i = -r, -r+1, \cdots, 0$. The global attractivity of equilibrium $x = 1$ of $Eq.(2)$ has been well studied by [5, 6]. In most results of these papers, it is supposed that the solution $\{x_n\}$ satisfies $0 < x_n < 1/\lambda$, but we find that this does not always succeed. One may see the examples in paper [10].

Two problems appear naturally, one is under what conditions every solution satisfies $0 < x_n < 1/\lambda$. The other is under what conditions every solution satisfying $0 < x_n < 1/\lambda$ converges to 1. To answer these problems, in a recent paper [10], the authors studied the global attractivity of Eq(1), following result was proved:

Theorem LG1. Suppose that there are $\beta > 0$ and $\theta > 1$ such that

$$\sum_{j=n-k_n}^{n} p_j \le \alpha, \quad n = \sigma_0, \sigma_0 + 1, \cdots; \quad p_n \le \beta, \quad n = 0, 1, 2, \cdots, \sigma,$$

$$\frac{\theta\alpha + 1}{\theta\alpha + \lambda}(1+\beta)^{\sigma} < \frac{1+\alpha}{\lambda + \alpha}.$$

where α is real root of the transcendental equation

$$e^{\alpha}\left(\frac{1-\lambda}{1-\lambda e^{-\alpha}}\right)^{\frac{1-\lambda}{\lambda}} = \frac{\alpha+1}{\alpha+\lambda}, \quad \lambda \in (0,1);$$

or

$$e^{\alpha - 1 + e^{-\alpha}} = 1 + \frac{1}{\alpha}, \quad \lambda = 0.$$

Furthermore, the initial condition satisfies

$$0 < x_i < \frac{\theta\alpha + 1}{\theta\alpha + \lambda}, \quad i = -r, -r+1, \cdots, -1, 0.$$

Then every solution of $Eq.(1)$ with initial condition (9) satisfies

$$0 < x_n \le \frac{\alpha+1}{\alpha+\lambda}, \quad n = 1, 2, \cdots.$$

Theorem LG2. Suppose that $\lambda \in (0,1)$ and there is a constant $\delta > 0$ such that for sufficiently large n

$$\sum_{s=n-k_n}^{n} p_s \le \delta(1-\lambda),$$

holds and

$$\sum_{n=1}^{+\infty} p_n = +\infty$$

$$1-\delta(\delta-\frac{1}{2})e^{(\delta-\frac{1}{2})(1-\lambda)}\geq 0 \tag{4}$$

are valid. Then every solution that satisfies $0 < x_n < 1/\lambda$ tends to 1 as $n \to +\infty$.

The purpose of this paper is to present a different answer to above problems. In section 2, we answer the first problem, and in section 3, the second problem is settled.

By the way, Eq.(1) is the discrete type of the following equation

$$N'(t) = r(t)N(t)\frac{1-N(t-\tau)}{1-\lambda N(t-\tau)}\ , \tag{5}$$

which was studied by many authors, see [2 − 4] and the references cited therein.

2. Every solution satisfies $0 < x_n < 1/\lambda$

Theorem 1. *Suppose that there are $\beta > 0$ and $\theta > 1$ such that*

$$\sum_{j=n-k_n}^{n} p_j \leq \alpha, \quad n=\sigma_0,\sigma_0+1,\cdots; \quad p_n\leq\beta, \quad n=0,1,2,\cdots,\sigma; \tag{6}$$

$$\frac{\theta\alpha+1}{\theta\alpha+\lambda}(1+\beta)^{\sigma} < \frac{1+\alpha}{\lambda+\alpha}.$$

where α is real root of the transcendental equation

$$e^{\alpha} = \frac{\alpha+1}{\alpha+\lambda}, \qquad \lambda\in(0,1); \tag{7}$$

Furthermore, suppose that the initial condition satisfies

$$0 < x_i < \frac{\theta\alpha+1}{\theta\alpha+\lambda}, \qquad i=-r,-r+1,\cdots,-1,0. \tag{8}$$

Then every solution of Eq.(1) with initial condition (8) satisfies

$$0 < x_n \leq \frac{\alpha+1}{\alpha+\lambda}, \qquad n=1,2,\cdots. \tag{9}$$

Proof. By (1), we get

$$x_{n+1} = x_n(1+p_n\frac{1-x_{n-k_n}}{1-\lambda x_{n-k_n}}).$$

Since $0 < x_{-r}, x_{-r+1}, \cdots, x_0 < \frac{\theta\alpha+1}{\theta\alpha+\lambda} < \frac{\alpha+1}{\alpha+\lambda}$, by (6), (7), we find

$$x_1 = x_0(1+p_0\frac{1-x_{-k_0}}{1-\lambda x_{-k_0}}) > x_0(1+p_0\frac{1-\frac{\alpha+1}{\alpha+\lambda}}{1-\lambda\frac{\alpha+1}{\alpha+\lambda}}) \geq x_0(1+\alpha\frac{1-\frac{\alpha+1}{\alpha+\lambda}}{1-\lambda\frac{\alpha+1}{\alpha+\lambda}}) = 0.$$

$$x_1 = x_0(1+p_0\frac{1-x_{-k_0}}{1-\lambda x_{-k_0}}) \leq \frac{\theta\alpha+1}{\theta\alpha+\lambda}(1+\beta) < \frac{\alpha+1}{\alpha+\lambda}.$$

Similarly, one obtains

$$x_2 = x_1(1+p_1\frac{1-x_{1-k_1}}{1-\lambda x_{1-k_1}}) > x_1(1+p_1\frac{1-\frac{\alpha+1}{\alpha+\lambda}}{1-\lambda\frac{\alpha+1}{\alpha+\lambda}}) \geq x_1(1+\alpha\frac{1-\frac{\alpha+1}{\alpha+\lambda}}{1-\lambda\frac{\alpha+1}{\alpha+\lambda}}) = 0.$$

$$x_2 = x_1(1+p_1\frac{1-x_{1-k_1}}{1-\lambda x_{1-k_1}}) \leq \frac{\theta\alpha+1}{\theta\alpha+\lambda}(1+\beta)^2 < \frac{\alpha+1}{\alpha+\lambda}.$$

$$\cdots\cdots$$

Finally, we get

$$\begin{aligned} x_\sigma &= x_{\sigma-1}(1+p_{\sigma-1}\frac{1-x_{\sigma-1-k_{\sigma-1}}}{1-\lambda x_{\sigma-1-k_{\sigma-1}}}) > x_{\sigma-1}(1+p_{\sigma-1}\frac{1-\frac{\alpha+1}{\alpha+\lambda}}{1-\lambda\frac{\alpha+1}{\alpha+\lambda}}) \\ &\geq x_{\sigma-1}(1+\alpha\frac{1-\frac{\alpha+1}{\alpha+\lambda}}{1-\lambda\frac{\alpha+1}{\alpha+\lambda}}) = 0. \\ x_\sigma &= x_{\sigma-1}(1+p_{\sigma-1}\frac{1-x_{\sigma-1-k_{\sigma-1}}}{1-\lambda x_{\sigma-1-k_{\sigma-1}}}) \leq \frac{\theta\alpha+1}{\theta\alpha+\lambda}(1+\beta)^\sigma < \frac{\alpha+1}{\alpha+\lambda}. \end{aligned}$$

Now, it suffices to prove the following results: if $n_0 \geq \sigma$, and

$$0 < x_n \leq \frac{\alpha+1}{\alpha+\lambda}, \quad 0 \leq n \leq n_0, \tag{10}$$

then

$$0 < x_{n_0+1} < \frac{\alpha+1}{\alpha+\lambda}. \tag{11}$$

By (6), (10), and $\lambda \in [0, 1)$, since the function $f(x) = (x-1)/(1-\lambda x)$ is increasing on $(0, 1/\lambda)$, then

$$p_{n_0}\frac{x_{n_0-k_{n_0}}-1}{1-\lambda x_{n_0-k_{n_0}}} < \alpha\frac{\frac{\alpha+1}{\alpha+\lambda}-1}{1-\lambda\frac{\alpha+1}{\alpha+\lambda}} = 1,$$

(1) implies

$$x_{n_0+1} > 0.$$

Next, we prove that

$$x_{n_0+1} < \frac{\alpha+1}{\alpha+\lambda}. \tag{12}$$

If $x_{n_0+1} < \frac{\alpha+1}{\alpha+\lambda}$, then (12) is obvious. If $x_{n_0+1} \geq \frac{\alpha+1}{\alpha+\lambda}$, let $p(t) = p_n$ for $t \in [n, n+1)$, $n = 0, 1, 2, \cdots, n_0$, and

$$x(t) = \begin{cases} x_n, & t = n, \\ x_n(\frac{x_{n+1}}{x_n})^{t-n}, & n \leq t < n+1, \end{cases} \tag{13}$$

then $x(t)$ is positive continuous function on internal $[0, n_0+1]$ and $x(n) = x_n, n \geq 0$, $x(t)$ is monotone on $[n, n+1)$. Let $[.]$ denote the maximum integer function, $x'(t)$ stands for the left derivative of function $x(t)$, then

$$x'(t) = x(t)\ln\{1 + p(t)\frac{1 - x([t - k_{[t]}])}{1 - \lambda x([t - k_{[t]}])}\}, \quad 0 \leq t \leq n_0 + 1. \tag{14}$$

Since $x([t - k_{[t]}]) > 0$, $0 \leq t \leq n_0 + 1$, by $(1-x)/(1-\lambda x) \leq 1, \frac{1}{\lambda} > x \geq 0$, we have

$$x'(t) \leq x(t)\ln(1 + p(t)) \leq p(t)x(t), \quad 0 \leq t < n_0 + 1, \quad a.e. \tag{15}$$

Again by $\Delta x_{n_0} = x_{n_0+1} - x_{n_0} > 0$ and (1), we have $x_{n_0 - k_{n_0}} < 1$. Then there exists $\xi \in [n_0 - k_{n_0}, n_0 + 1)$ such that $x(\xi) = 1$ and $x(t) > 1$ for $t \in (\xi, n_0+1]$. It follows that

$$\ln x(n_0 + 1) < \int_{\xi}^{n_0+1} p(t)dt. \tag{16}$$

Since $\int_{\xi}^{n_0+1} p(t)dt \leq \sum\limits_{j=n_0-k_{n_0}}^{n_0} p_j \leq \alpha$, we get

$$\ln x(n_0 + 1) < \alpha,$$

and

$$x_{n_0+1} = x(n_0 + 1) < e^{\alpha} = \frac{\alpha+1}{\alpha+\lambda}, \tag{17}$$

which contradicts the assumption $x_{n_0+1} \geq \frac{\alpha+1}{\alpha+\lambda}$. This completes the proof. By a similar method, we can get the theorem in the case $\lambda = 0$. ∎

Remark 1. Since $1 < \frac{\theta\alpha+1}{\theta\alpha+\lambda} < \frac{\alpha+1}{\alpha+\lambda} < \frac{1}{\lambda}$, theorem 1 gives the sufficient conditions which guarantee $0 < x_n < \frac{1}{\lambda}$ for each solution $\{x_n\}$ of Eq.(1).

3. Global attractivity

In this section, we give the sufficient condition that guarantees every solution satisfying $0 < x_n < \frac{1}{\lambda}$ to converge to 1 as $n \to \infty$.

Theorem 2. *Suppose that $\lambda \in [0, 1)$ and there is a constant $\delta > 0$ such that $\delta(1 - 2\lambda) + \lambda > 0$ and for sufficiently large n*

$$\sum_{s=n-k_n}^{n} p_s \leq \delta(1 - \lambda), \tag{18}$$

holds and

$$\sum_{n=1}^{+\infty} p_n = +\infty, \tag{19}$$

$$\frac{\delta(1 - 2\lambda) + 1}{\delta(1 - 2\lambda) + \lambda} \geq e^{\frac{\delta^2}{2}} \tag{20}$$

are satisfied. Then every solution satisfying $0 < x_n < 1/\lambda$ tends to 1 as $n \to +\infty$.

In order to prove theorem 2, we need the following lemmas.

Lemma 1. *Suppose* (19) *holds, $\{x_n\}$ is a solution of Eq.(1) that satisfies $0 < x_n < 1/\lambda$. Furthermore, if $\{x_n\}$ is eventually greater than 1 or eventually less than 1, then $\{x_n\}$ tends to 1 as $n \to +\infty$.*

The proof is similar to that of the result in [5] and is omitted.

Lemma 2. *Suppose that* (18), (20) *hold and $\{x_n\}$,$0 < x_n < 1/\lambda$, is a solution of* (1) *oscillating about 1. Then there are $0 < a < b < \frac{1}{\lambda}$ such that $\{x_n\}$ satisfies $a < x_n < b$ for every n.*

Proof. Set $\ln x_n = y_n$, for $n \geq 0$, then $\{y_n\}$ is oscillatory. By (1), we find

$$\Delta y_n = \ln(1 + p_n \frac{1 - e^{y_{n-k_n}}}{1 - \lambda e^{y_{n-k_n}}}), \qquad n = 0, 1, 2, \cdots. \tag{21}$$

Then

$$\Delta y_n \leq \ln(1 + p_n), \qquad n = 0, 1, 2, \cdots. \tag{22}$$

Now, let y_{n_i} be any left maximum term of $\{y_n\}$ with $n_i > \sigma$, $y_{n_i} > 0$ and $y_{n_i} \geq y_{n_i - 1}$, by (21) one gets $y_{n_i - 1 - k_{n_i - 1}} \leq 0$ and then there is n_i^*,

$n_i - 1 - k_{n_i-1} \le n_i^* \le n_i - 1$ such that $y_{n_i^*} \le 0, y_n > 0$ for $n_i^* + 1 \le n \le n_i$. Choose a number $\xi_i \in [0,1)$ such that

$$y_{n_i^*} + \xi_i(y_{n_i^*+1} - y_{n_i^*}) = 0. \tag{23}$$

By the inequality

$$(\prod_{i=1}^{m} a_i^{\alpha_i})^{\frac{1}{\sum_{i=1}^{m}\alpha_i}} \le \frac{\sum_{i=1}^{m}\alpha_i a_i}{\sum_{i=1}^{m}\alpha_i},$$

we get

$$\begin{aligned}
-y_{j-k_j} &= -y_{n_i^*} + \sum_{s=j-k_j}^{n_i^*-1}(y_{s+1} - y_s) \\
&\quad = \xi_i(y_{n_i^*+1} - y_{n_i^*}) + \sum_{s=j-k_j}^{n_i^*-1} \ln(1 + p_s \frac{1 - e^{y_{s-k_s}}}{1 - \lambda e^{y_{s-k_s}}}) \\
&\quad \le \xi_i \ln(1 + \frac{p_{n_i^*}}{1-\lambda}) + \sum_{s=j-k_j}^{n_i^*-1} \ln(1 + \frac{p_s}{1-\lambda}) \\
&\le (n_i^* - j + k_j + \xi_i)\ln[1 + \frac{1}{1-\lambda}\frac{1}{n_i^* - j + k_j + \xi_i}(\xi_i p_{n_i^*} + \sum_{s=j-k_j}^{n_i^*-1} p_s)].
\end{aligned}$$

Then

$$e^{y_{j-k_j}} \ge [1 + \frac{1}{1-\lambda}\frac{1}{n_i^* - j + k_j + \xi_i}(\xi_i p_{n_i^*} + \sum_{s=j-k_j}^{n_i^*-1} p_s)]^{-(n_i^* - j + k_j + \xi_i)}.$$

By $(1 + \frac{x}{n})^{-n} \ge 1 - x$, for $n > 0$, $x \ge 0$, we have

$$e^{y_{j-k_j}} \ge 1 - \frac{1}{1-\lambda}(\xi_i p_{n_i^*} + \sum_{s=j-k_j}^{n_i^*-1} p_s). \tag{24}$$

Thus by (23) and (24), we get

$$\begin{aligned}
y_{n_i} &= y_{n_i^*+1} + \sum_{s=n_i^*+1}^{n_i-1}(y_{s+1} - y_s) \\
&\quad = (1 - \xi_i)(y_{n_i^*+1} - y_{n_i^*}) + \sum_{n=n_i^*+1}^{n_i-1} \ln(1 + p_n \frac{1 - e^{y_{n-k_n}}}{1 - \lambda e^{y_{n-k_n}}})
\end{aligned}$$

$$\leq (1-\xi_i)\ln(1+\frac{p_{n_i^*}}{1-\lambda}(1-e^{y_{n_i^*-k_{n_i^*}}}))+\sum_{n=n_i^*+1}^{n_i-1}\ln(1+\frac{p_n}{1-\lambda}(1-e^{y_{n-k_n}}))$$

$$\leq (1-\xi_i)\ln[1+\frac{p_{n_i^*}}{(1-\lambda)^2}(\xi_i p_{n_i^*}+\sum_{s=n_i^*-k_{n_i^*}}^{n_i^*-1} p_s)]$$

$$+\sum_{n=n_i^*+1}^{n_i-1}\ln[1+\frac{p_n}{(1-\lambda)^2}(\xi_i p_{n_i^*}+\sum_{s=n-k_n}^{n_i^*-1} p_s)].$$

By condition (18), we have

$$y_{n_i}\leq (1-\xi_i)\ln[1+\frac{p_{n_i^*}}{(1-\lambda)^2}(\delta(1-\lambda)-(1-\xi_i)p_{n_i^*})]$$

$$+\sum_{n=n_i^*+1}^{n_i-1}\ln[1+\frac{p_n}{(1-\lambda)^2}(\delta(1-\lambda)-\sum_{s=n_i^*+1}^{n} p_s-(1-\xi_i)p_{n_i^*})]$$

$$\leq (n_i-n_i^*-\xi_i)\ln\{1+\frac{1}{n_i-n_i^*-\xi_i)}\frac{1}{(1-\lambda)^2}[(1-\xi_i)p_{n_i^*}(\delta(1-\lambda)-(1-\xi_i)p_{n_i^*})$$

$$+\sum_{n=n_i^*+1}^{n_i-1}p_n(\delta(1-\lambda)-\sum_{s=n_i^*+1}^{n}p_s-(1-\xi_i)p_{n_i^*})]\}.$$

Suppose $k_n\leq k$, since $n_i-n_i^*-\xi_i\leq k_{n_i-1}+1\leq k+1$, it results in

$$y_{n_i}\leq (k+1)\ln\{1+\frac{1}{k+1}\frac{1}{(1-\lambda)^2}[(1-\xi_i)p_{n_i^*}(\delta(1-\lambda)-(1-\xi_i)p_{n_i^*})$$

$$+\sum_{n=n_i^*+1}^{n_i-1}p_n(\delta(1-\lambda)-\sum_{s=n_i^*+1}^{n}p_s-(1-\xi_i)p_{n_i^*})]\}.$$

Let $d_i=\sum\limits_{n=n_i^*+1}^{n_i-1}p_n+(1-\xi_i)p_{n_i^*}$. Then by the inequality

$$\sum_{i=1}^{m}x_s^2\geq \frac{1}{m}(\sum_{s=1}^{m}x_s)^2,$$

we get

$$y_{n_i}\leq (k+1)\ln\{1+\frac{1}{k+1}\frac{\delta}{1-\lambda}d_i$$

$$-\frac{1}{k+1}\frac{1}{(1-\lambda)^2}[(1-\xi_i)^2p_{n_i^*}^2+(1-\xi_i)p_{n_i^*}\sum_{n=n_i^*-1}^{n_i-1}p_n$$

$$+\sum_{n=n_i^*+1}^{n_i-1} p_n \sum_{s=n_i^*+1}^{n_i-1} p_s]\}$$

$$= (k+1)\ln\{1+\frac{1}{k+1}\frac{\delta}{1-\lambda}d_i - \frac{1}{2(k+1)}\frac{1}{(1-\lambda)^2}d_i^2$$

$$-\frac{1}{2(k+1)(1-\lambda)^2}[\sum_{n=n_i^*+1}^{n_i-1} p_n^2 + (1-\xi_i)^2 p_{n_i^*}^2]\}$$

$$\leq (k+1)\ln\{1+\frac{\delta}{(k+1)(1-\lambda)}d_i - \frac{1}{2(k+1)(1-\lambda)^2}d_i^2$$

$$-\frac{1}{2(k+1)(1-\lambda)^2}\frac{1}{n_i-n_i^*}d_i^2\}$$

$$\leq (k+1)\ln\{1+\frac{\delta}{(k+1)(1-\lambda)}d_i - \frac{k+2}{2(k+1)^2(1-\lambda)^2}d_i^2\}.$$

Since function $\frac{\delta}{1-\lambda}x - \frac{k+2}{2(k+1)(1-\lambda)^2}x^2$ is increasing when $x \leq \frac{k+1}{k+2}\delta(1-\lambda)$, the maximum point of function is $x = \frac{k+1}{k+2}\delta(1-\lambda)$, we get

$$y_{n_i} \leq (k+1)\ln(1+\frac{\delta^2}{2(k+2)}). \tag{25}$$

It is easy to see that function $x\ln(1+\frac{\delta^2}{2(x+1)})$ is increasing on $(0,+\infty)$, hence

$$\limsup_{n\to\infty} y_n \leq (k+1)\ln(1+\frac{\delta^2}{2(k+2)}) \to \frac{\delta^2}{2}, \quad k\to\infty$$

it combines condition (20), we have

$$y_{n_i} \leq \ln\frac{\delta(1-2\lambda)+1}{\delta(1-2\lambda)+\lambda} - \ln\frac{1}{\lambda} + \ln\frac{1}{\lambda} = \ln\frac{\delta(1-2\lambda)\lambda+\lambda}{\delta(1-2\lambda)+\lambda} + \ln\frac{1}{\lambda} < \ln\frac{1}{\lambda}.$$

So $\{x_n\}$ is bounded above away from $\frac{1}{\lambda}$. Now we prove that $\{y_n\}$ is bounded below. Suppose $y_n \leq M < \ln\frac{1}{\lambda}$, then from (21),

$$e^{y_{n+1}-y_n} \geq 1 + p_n\frac{1-e^{y_{n-k_n}}}{1-\lambda e^{y_{n-k_n}}}.$$

Let $y_{n^*} = \max\{1, x_n\}$. Suppose y_{n_i} is a left minimum term of $\{y_n\}$, then from (1), $y_{n_i-1-k_{n_i-1}} > 0$, we get

$$e^{y_{n_i}} \geq \prod_{s=n_i-1-k_{n_i-1}}^{n_i-1}(1+p_s\frac{1-e^{y_{s-k_s}}}{1-\lambda e^{y_{s-k_s}}})$$

$$\geq 1 + \sum_{s=y_{s-k_s}}^{n_i-1} p_s \frac{1-e^{y_{(s-k_s)^*}}}{1-\lambda e^{y_{(s-k_s)^*}}}$$

$$\geq 1 + \delta(1-\lambda)\frac{1-e^M}{1-\lambda e^M} = a > 0.$$

This shows that $\{y_n\}$ is bounded below. The proof is complete. ■

Proof of Theorem 2. By Lemma 1, it suffices to prove that every solution of Eq.(1) satisfying $0 < x_n < \frac{1}{\lambda}$ converges to 1 as n tends to infinity. By Lemma 2, $\{x_n\}$ is bounded above away from $\frac{1}{\lambda}$ and bounded below away from zero. We prove now that $\lim_{n\to+\infty} y_n = 0$. Let

$$\limsup_{n\to+\infty} y_n = u, \qquad \liminf_{n\to+\infty} y_n = v. \tag{26}$$

Then

$$-\infty < v \leq 0 \leq u < \ln\frac{1}{\lambda},$$

and there are two subsequence of $\{y_n\}$, denoted by $\{y_{n_i}\}$ and $\{y_{m_i}\}$ such that

$$y_{n_i} > 0,\ \ y_{n_i} \geq y_{n_i-1},\ \ i = 1, 2, \cdots,\ \ \lim_{i\to\infty} n_i = \infty,\ \ \lim_{i\to\infty} y_{n_i} = u,$$

$$y_{m_i} < 0,\ \ y_{m_i} \geq y_{n_i-1},\ \ i = 1, 2, \cdots,\ \ \lim_{i\to\infty} m_i = \infty,\ \ \lim_{i\to\infty} y_{m_i} = v.$$

For any $\epsilon \in (0, \ln\frac{1}{\lambda} - u)$, there is N_1 such that

$$v_1 = v - \epsilon < y_{n-k_n} < u + \epsilon = u_1, \qquad n = N_1, N_1+1, \cdots, \tag{27}$$

Then by (21), we have

$$\Delta y_n \leq \ln(1 + p_n\frac{1-e^{v_1}}{1-\lambda e^{v_1}}), \tag{28}$$

$$\Delta y_n \geq \ln(1 + p_n\frac{1-e^{u_1}}{1-\lambda e^{u_1}}). \tag{29}$$

Again from (21), (28), by the same method using in the proof of Lemma 2, we obtain

$$y_{n_i} \leq \ln(1 + \frac{\delta^2(1-\lambda)}{2(k+2)}\frac{1-e^{v_1}}{1-\lambda e^{v_1}})^{k+1}, \tag{30}$$

Let

$$y_{n_*} = \max\{1, y_n\}.$$

Again since $\Delta y_{m_i-1} \le 0$, by (21), $y_{m_i-1-k_{m_i-1}} \ge 0$, one sees

$$\begin{aligned} y_{m_i} &= y_{m_i-1-k_{m_i-1}} + \sum_{s=m_i-1-k_{m_i-1}}^{m_i-1} \ln(1+p_s\frac{1-e^{y_{s-k_s}}}{1-\lambda e^{y_{s-k_s}}}) \\ &\ge \sum_{s=m_i-1-k_{m_i-1}}^{m_i-1} \ln(1+p_s\frac{1-e^{y_{(s-k_s)*}}}{1-\lambda e^{y_{(s-k_s)*}}}) \\ &\ge \ln(1+\sum_{s=m_i-1-k_{m_i-1}}^{m_i-1} p_s\frac{1-e^{y_{(s-k_s)*}}}{1-\lambda e^{y_{(s-k_s)*}}}) \\ &\ge \ln(1+\delta(1-\lambda)\frac{1-e^{u_1}}{1-\lambda e^{u_1}}), \end{aligned}$$

and hence

$$e^{y_{m_i}} \ge 1+\delta(1-\lambda)\frac{1-e^{u_1}}{1-\lambda e^{u_1}}. \tag{31}$$

Let $i \to +\infty$, $\epsilon \to 0$, one has

$$u \le \ln(1+\frac{\delta^2(1-\lambda)}{2(k+2)}\frac{1-e^{v}}{1-\lambda e^{v}})^{k+1}, \tag{32}$$

$$e^{v} \ge 1+\delta(1-\lambda)\frac{1-e^{u}}{1-\lambda e^{u}}. \tag{33}$$

If $u \ne 0$, then $u > 0$. By $(32), (33)$, we get

$$u \le \ln(1+\frac{\delta^3(1-\lambda)}{2(k+2)}\frac{e^{u}-1}{1-\delta\lambda-\delta(1-\delta)e^{u}})^{k+1}. \tag{34}$$

From (32),

$$u < \ln(1+\frac{\delta^2}{2(k+2)})^{k+1} = u_0. \tag{35}$$

Let

$$f(u) = u - \ln(1+\frac{\delta^3(1-\lambda)}{2(k+2)}\frac{e^{u}-1}{1-\delta\lambda-\lambda(1-\delta)e^{u}})^{k+1}. \tag{36}$$

Clearly,

$$f(0) = 0, \quad f''(u) \le 0,$$

$f(u)$ has at most two zero points in $[0,+\infty)$ and

$$f(u_0)=\ln(1+\frac{\delta^2}{2(k+2)})^{k+1}-\ln(1+\frac{\delta^3(1-\lambda)}{2(k+2)}\frac{e^{u_0}-1}{1-\delta\lambda-\lambda(1-\delta)e^{u_0}})^{k+1}.$$

By (35), $u_0\le\frac{\delta^2}{2}$, hence $e^{u_0}\le e^{\frac{\delta^2}{2}}$, using (20), we get

$$e^{u_0}\le\frac{\delta(1-2\lambda)+1}{\delta(1-2\lambda)+\lambda}.$$

So

$$\frac{\delta(1-\lambda)(e^{u_0}-1)}{1-\delta\lambda-\lambda(1-\delta)e^{u_0}}\le 1. \tag{37}$$

Thus $f(u_0)\ge 0$, we see $f(u)>0$ for $u\in(0,u_0)$, this contradicts (34), then $u=0$ and $v=0$, which implies $\lim\limits_{n\to+\infty}y_n=0$, this completes the proof. ∎

Corollary 1. *Suppose that* (19) *holds and there is an integer* n_0 *such that for sufficiently large* n

$$\sum_{s=n-k_n}^{n}p_s\le 1-\lambda, n=n_0,n_0+1,\cdots \tag{38}$$

is valid. Then every solution of Eq.(1) *satisfying* $0<x_n<\frac{1}{\lambda}$ *tends to* 1 *as* n *tends to infinity.*

Corollary 2. *Suppose that* (19) *holds and there is an integer* $\delta>0$ *such that for sufficiently large* n

$$\sum_{s=n-k_n}^{n}p_s\le\delta \tag{39}$$

is valid, and

$$(1+\frac{\delta^2}{2(k+2)})^{k+1}\le 1+\frac{1}{\delta}. \tag{40}$$

Then every positive solution of Eq.(2) *tends to* 1 *as* n *tends to infinity.*

The proofs of corollary 3 and 4 come directly from Theorem 2.

Theorem 3. *Assume that* (19) *holds and* $k_n\le k$ *for all* $n=0,1,\cdots$, *furthermore, for sufficiently large* n

$$\sum_{s=n-k_n}^{n}p_s\le\delta(1-\lambda) \tag{41}$$

is valid, and

$$1 - \delta(\delta - \frac{k+2}{2(k+2)})e^{\delta - \frac{k+2}{2(k+1)}} \geq 0, \tag{42}$$

then every solution of Eq.(1) *satisfying* $0 < x_n < \frac{1}{\lambda}$ *tends to* 1 *as* n *tends to infinity.*

Proof. Since

$$\frac{1-e^x}{1-\lambda e^x} \leq -\frac{1}{1-\lambda}x, \quad \ln(1+x) \leq x,$$

by (21), we have

$$\Delta y_n \leq -\frac{p_n}{1-\lambda} y_{n-k_n}.$$

Then (28) implies ,

$$\Delta y_n \leq p_n \frac{1-e^{v_1}}{1-\lambda e^{v_1}}.$$

Using the method in [7], we have

$$u \leq (\delta - \frac{k+2}{2(k+1)})(1-\lambda)\frac{1-e^{v_1}}{1-\lambda e^{v_1}}.$$

By the same method in the proof of Theorem 2, we have (33). Hence we obtain

$$u \leq -\delta(\delta - \frac{k+2}{2(k+1)})(1-\lambda)\frac{1-e^u}{1-\lambda\delta-\lambda(1-\delta)e^u}.$$

We note $\delta \geq 1$, so

$$u \leq -\delta(\delta - \frac{k+2}{2(k+1)})(1-e^u).$$

Let

$$f(u) = u + \delta(\delta - \frac{k+2}{2(k+1)})(1-e^u),$$

then $f(0) = 0$ and

$$f'(u) = 1 - \delta(\delta - \frac{k+2}{2(k+1)})e^u,$$

From the condition (42) and $u \leq \delta - \frac{k+2}{2(k+1)}$, we have $f'(u) > 0$. Then $f(u) > 0$, which is a contradiction. The proof is completed. ■

When $\{k_n\}$ is unbounded, condition (42) can be substituted

$$1 - \delta(\delta - \frac{1}{2})e^{\delta - \frac{1}{2}} \geq 0. \tag{43}$$

Remark 2. Theorem 1 is new. Theorem 2 improves the known results. In [4], Chen and Yu proved that if

$$\limsup_{n\to+\infty}\sum_{s=n-k_n}^{n} p_s < \frac{1}{2}, \tag{44}$$

and (19) hold then all positive solution of Eq.(2) tend to 1 as n tends to infinity.

In [6], Zhou and Zhang proved that if

$$\limsup_{n\to+\infty}\sum_{s=n-k_n}^{n} p_s < \alpha, \tag{45}$$

and (19) hold, then all positive solution of $Eq.$(2) tend to 1 as n tends to infinity, where α satisfies $\frac{1}{x}+1=e^{\frac{x^2}{2}}$. When $\lambda=0$, Corollary 4 improves the results in [6].

Remark 3. In [8], the difference equation

$$x_{n+1}=\frac{\alpha x_n}{1+\beta x_{n-k}},\quad n=0,1,2,\cdots, \tag{46}$$

was considered, where $\alpha>1$, $\beta\in(0,+\infty)$, we can reform (46) into

$$\Delta x_n = x_n\frac{\alpha-1-\beta x_{n-k}}{1+\beta x_{n-k}}. \tag{47}$$

Let $\frac{\beta}{\alpha-1}x_n=y_n$,we have

$$\Delta y_n=(\alpha-1)y_n\frac{1-y_{n-k_n}}{1+(\alpha-1)y_{n-k_n}}. \tag{48}$$

By using Theorem 1,2, when $\alpha\in(0,1)$, $\beta\in(-\infty,+\infty)$, we get similar results which improve the theorems in [8].

Remark 4. Theorem 2 is different from Theorem 5, condition (42) is different from (20), by a simple computation from (20), (40), (42), we find $\delta\geq 1$.

Remark 5. By the method similar to above discussion, we can establish existence result for the positive solutions of the following equation [8]

$$\Delta x_n = p_n x_n(1+bx_{n-k_n}-cx^2_{n-k_n}),\qquad n=0,1,2,\cdots,$$

where $\{p_n\}$ is a sequence of positive real numbers, $\{k_n\}$ a sequence of non-negative integers satisfying $\lim\limits_{n\to\infty}(n-k_n)=\infty, b\in R, c\in(0,+\infty)$, and the global atrractivity result can also be established, we leave the details to the readers.

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Yuji Liu
Department of Mathematics, Hunan Institute of Technology
Yueyang, Hunan 414000, P.R.China
e-mail: liuyuji888@sohu.com

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