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ON THE DISCRETE VERSION OF GENERALIZED KIGURADZE'S LEMMA

Abstract: The Kiguaradze's lemma for quasi-differences of a real sequence is presented. Some examples illustrating the result are included.

Key words: quasi-differences, difference equation, nonoscillatory solution.

In the last few years there has been increasing interest in the study of qualitative behaviour of solutions of difference equations. In many papers (see for example [2-4], [6-11]) the following Kiguradze's Lemma is used to prove the main results.

Let $N = \{0, 1, \dots\}$, $N(a) = \{a, a+1, \dots\}$, where $a \in N$.

Lemma 1. *(see [1, Th.1.8.11]) Let x be defined on $N(a)$, and $x(n) > 0$ with $\Delta^m x(n)$ on constant sign on $N(a)$ and not identically zero. Then, there exists an integer l, $0 \le l \le m$ with $m + l$ odd for $\Delta^m x(n) \le 0$ or $m + l$ even for $\Delta^m x(n) \ge 0$ and such that*

$$\Delta^i x(n) > 0 \; for \; all \; large \; n \in N(a), \quad 1 \le i \le l-1,$$

$$(-1)^{l+i} \Delta^i x(n) > 0 \; for \; all \; n \in N(a), \quad l \le i \le m-1.$$

Let r_i $(i = 1, 2, \dots, m)$ be positive real sequences. For any real sequence x we denote

$$L_0 x(n) = x(n),$$

$$L_i x(n) = r_i(n) \Delta L_{i-1} x(n), \; i = 1, 2, \dots, m, \; n \in N.$$

The sequences $L_i x$ are called quasi-differences of x.

For quasi-differences we can prove similar result, which we formulate as

Lemma 2. *Suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{r_i(n)} = \infty \; for \; all \; i = 1, 2, \dots, m. \tag{1}$$

Let $x : N \longrightarrow R \setminus \{0\}$ be a sequence of constant sign. If $L_m x$ is of constant sign and not identically zero for $n > n_0$ and for some $j \in \{1, 2\}$

$$(-1)^j x(n) L_m x(n) \geq 0 \quad for \quad n \geq n_0,$$

then there exists an integer $l \in \{0, 1, \ldots, m\}$ with $m + l + j$ even such that

$$(2) \qquad \begin{aligned} x(n) L_i x(n) > 0 \quad & for \ \ large \ \ n \ \ and \quad i = 0, 1, \ldots, l \\ (-1)^{l+i} x(n) L_i x(n) > 0 \quad & for \ \ all \ \ n \geq n_0 \quad i = l+1, l+2, \ldots, m. \end{aligned}$$

To prove this lemma we will need following.

Lemma 3. (see [5])*Let (1) holds. If $L_{m-1}x(n) > 0$ and $L_m x(n) > 0$ for all $n \geq n_0$, then*

$$\lim_{n \longrightarrow \infty} L_i x(n) = \infty \quad for \quad all \quad i = 0, 1, ..., m-2.$$

If $L_{m-1}x(n) < 0$ and $L_m x(n) < 0$ for all $n \geq n_0$, then

$$\lim_{n \to \infty} L_i x(n) = -\infty \quad for \quad all \quad i = 0, 1, \ldots, m-2.$$

Proof of Lemma 2. We proof Lemma for a positive sequence x. (For negative sequence the proof is similar). We consider two cases with respect to the sign of $L_m x$.

Case 1. $L_m x(n) \leq 0$ for $n \geq n_0$.

First we shall prove that $L_{m-1}x(n) > 0$ for every $n \geq n_0$. Suppose that there exists some $n_1 \geq n_0$ such that $L_{m-1}x(n_1) \leq 0$. Since $L_{m-1}x$ is nonincreasing and not identically constant for $n \geq n_0$, there exists $n_2 \geq n_1$ such that

$$L_{m-1}x(n) \leq L_{m-1}x(n_2) < L_{m-1}x(n_1) \leq 0 \quad for \quad all \quad n \geq n_2.$$

From Lemma 3 we have $\lim_{n \longrightarrow \infty} x(n) = -\infty$ which is a contradiction to $x(n) > 0$. Thus $L_{m-1}x(n) > 0$ for all $n \geq n_0$ and there exists a smallest integer $l \in \{0, 1, ..., m-1\}$ with $m + l$ odd and

$$(3) \qquad (-1)^{l+i} L_i x(n) > 0 \quad for \quad every \quad n \geq n_0, \quad i = l, l+1, ..., m-1.$$

Let $l > 1$. Now suppose

$$(4) \qquad L_{l-1}x(n) < 0 \quad for \quad n \geq n_0,$$

then once again from Lemma 3 we get

$$(5) \qquad L_{l-2}x(n) > 0 \quad for \quad n \geq n_0.$$

We remark that (3), (4) and (5) can be unified to

$$(-1)^{(l-2)+i} L_i x(n) > 0 \quad for \quad n \geq n_0, \quad i = l-2, l-1, ..., m-1$$

which is a contradiction to the definition of l.
So, (4) fails and $L_{l-1}x(n) > 0$ for all $n \geq n_0$. From (3) we have $L_l x(n) > 0$. Therefore and from Lemma 3 we get $\lim\limits_{n \longrightarrow \infty} L_i x(n) = \infty$ for $i = 0, 1, ..., l-2$. Thus $L_i x(n) > 0$ for all large n and $i = 0, 1, ..., l$.

Case 2. $L_m x(n) \geq 0$ for $n \geq n_0$. We consider two subcases.

1^0 Let $n_3 \geq n_0$ be such that $L_{m-1}x(n) \geq 0$. Since $L_{m-1}x$ is nondecreasing and not identically constant for $n \geq n_0$ then there exist some $n_4 \geq n_3$ such that $L_{m-1}x(n) > 0$ for all $n \geq n_4$. Therefore, by Lemma 3 we have

$$\lim_{n \longrightarrow \infty} L_i x(n) = \infty \quad for \quad i = 0, 1, ..., m-2.$$

So, $L_i x(n) > 0$ for large n and $i = 0, 1, ..., m-1$. This proves the theorem for $l = m$.

2^0 If $L_{m-1}x(n) < 0$ for all $n \geq n_0$ then we find from Lemma 3 that $L_{m-2}x(n) > 0$ for all $n \geq n_0$. The rest of the proof is the same as in the Case 1. ■

Remark 1. *If the assumption* (1) *is not satiesfied then Lemma 2 cannot be true, as the following example shows.*

Example 1. *Let* $x(n) = n$ *and* $r_1(n) = \frac{1}{n}$, $r_2(n) = 1$, $r_3(n) = (n+3)^{(4)}$, $r_4(n) = \frac{1}{n}$, $r_5(n) = 1$. *Then*

$$\begin{aligned}
L_1 x(n) &= \frac{1}{n} > 0 \\
L_2 x(n) &= -\frac{1}{n(n+1)} < 0 \\
L_3 x(n) &= n + 3 > 0 \\
L_4 x(n) &= \frac{1}{n} > 0 \\
L_5 x(n) &= -\frac{1}{n(n+1)} < 0.
\end{aligned}$$

The divergence of the series $\sum\limits_{n=1}^{\infty} \frac{1}{r_i(n)}$, $i = 1, 2, \ldots, m$ plays an important role in study of nonoscillation of difference equation of the form

$$\text{(E)} \qquad L_m x(n) \pm a(n) f(x(n+k)) = 0,$$

where $r_m \equiv 1$, a is a sequence of positive integers, k is an integer and $f : R \longrightarrow R$ with $uf(u) > 0$ for $u \neq 0$. For example if (1) is satisfied, it is possible to classify the nonoscillatory solutions of equation (E) in a very simple way.

Example 2. *Consider the difference equation*

$$\text{(E1)} \qquad \Delta((n+1)\Delta((n+1)\Delta x(n))) = \frac{n+2}{n(n+1)^2}x(n+2),\ n \geq 1$$

Here we have $r_1(n) = r_2(n) = n+1,\ r_3(n) \equiv 1$, *hence (1) is satisfied. For* $x(n) > 0$ *we get* $L_3x(n) = \Delta((n+1)\Delta((n+1)\Delta x(n)))) > 0$. *So, by Lemma 2 every eventually positive solution of equation (E1) is one of the types:*

(I) $x(n) > 0,\ \Delta x(n) > 0,\ \Delta((n+1)\Delta x(n)) > 0$,

(II) $x(n) > 0,\ \Delta x(n) > 0,\ \Delta((n+1)\Delta x(n)) < 0$,

for large n.

Example 3. *Consider the difference equation*

$$\text{(E2)} \qquad \Delta((n+2)^2\Delta(n^2\Delta x(n))) = \frac{1}{n(n+1)(n+2)}\frac{1}{x(n)},\ n \geq 1.$$

Here $L_3x(n) > 0$ *for* $x(n) > 0$ *too, but condition (1) is not satisfied. Any eventually positive solution of equation (E2) is one of the following types:*

(I) $x(n) > 0,\ \Delta x(n) > 0,\ \Delta(n^2\Delta x(n)) > 0$,

(II) $x(n) > 0,\ \Delta x(n) > 0,\ \Delta(n^2\Delta x(n)) < 0$,

(III) $x(n) > 0,\ \Delta x(n) < 0,\ \Delta(n^2\Delta x(n)) > 0$,

(IV) $x(n) > 0,\ \Delta x(n) < 0,\ \Delta(n^2\Delta x(n)) < 0$,

for large n.
The sequence $x_n = \frac{1}{n}$ *is a solution of equation (E2) for which* $L_0x(n) = \frac{1}{n} < 0,\ L_1x(n) = -\frac{n}{n+1} < 0,\ L_2x(n) = -\frac{n+2}{n+1} < 0$, *so it is of type (IV).*

For $k \in N$ we use the usual factorial notation

$$n^{(k)} = n(n-1)\ldots(n-k+1) \text{ with } n^{(0)} = 1.$$

From Lemma 2 we get following

Lemma 4. *Let* $x : N \to R_+$. *Suppose* $L_mx(n) \leq 0$ *and not identically zero and the sequences* r_i $(i = 1, 2, \ldots, m)$ *are nonincreasing. Then, there exists a large* n_0 *such that*

$$\text{(6)} \qquad x(n) \geq \frac{M}{(m-1)!}L_{m-1}x(2^{m-l-1}n)(n-n_0)^{(m-1)} \text{ for all},; n \geq n_0,$$

where $M = \prod_{i=1}^{m-1} \frac{1}{r_i(n_0)}$.

Proof. Since the sequences r_i $(i = 1, \dots, m)$ are positive and nonincreasing, so the condition (1) is satisfied. Then, from Lemma 2 there exists an integer l such that (2) holds. Therefore, $L_{m-1}x(n) > 0$ for all $n \in N$. Summing from n to $2n-1$ the equality

$$L_{m-1}x(n) = r_{m-1}(n)\Delta L_{m-2}x(n)$$

we get

$$\begin{aligned}\sum_{k=n}^{2n-1} L_{m-1}x(k) &\le r_{m-1}(n)\sum_{k=n}^{2n-1}\Delta L_{m-2}x(k)\\ &\le r_{m-1}(n)L_{m-2}(2n) - r_{m-1}(n)L_{m-2}x(n)\\ &\le -r_{m-1}(n)L_{m-2}x(n).\end{aligned}$$

Hence

$$-L_{m-2}x(n) \ge \frac{1}{r_{m-1}(n)}\sum_{k=n}^{2n-1} L_{m-1}x(k) \ge \frac{1}{r_{m-1}(n)}L_{m-1}x(2n)n. \tag{7}$$

Now, since $L_{m-2}x(n) < 0$ and $\Delta L_{m-3}x(n) > 0$ for all $n \in N$ we have

$$\begin{aligned}\sum_{k=n}^{2n-1} L_{m-2}x(k) &\ge r_{m-2}(n)\sum_{k=n}^{2n-1}\Delta L_{m-3}x(k)\\ &\ge r_{m-2}(n)L_{m-3}x(2n) - r_{m-2}(n)L_{m-3}x(n)\\ &\ge -r_{m-2}(n)L_{m-3}x(n).\end{aligned}$$

Therefore, by (7) we get

$$\begin{aligned}L_{m-3}x(n) &\ge \frac{-1}{r_{m-2}}(n)\sum_{k=n}^{2n-1} L_{m-2}x(k)\\ &\ge \frac{1}{r_{m-2}(n)}\sum_{k=n}^{2n-1}\frac{1}{r_{m-1}(k)}L_{m-1}x(2k)k^{(1)}\\ &\ge \prod_{i=1}^{2}\frac{1}{r_{m-i}(n)}L_{m-1}x(2^2n)\sum_{k=n}^{2n-1}(k-n)^{(1)}\\ &\ge \prod_{i=1}^{2}\frac{1}{r_{m-i}(n)}L_{m-1}x(2^2n)\frac{n^{(2)}}{2!}.\end{aligned}$$

After $(m-l-1)$ steps we obtain

$$L_l x(n) \ge \frac{1}{(m-l-1)!}\prod_{i=1}^{m-l-1}\frac{1}{r_{m-i}(n)}L_{m-1}x(2^{m-l-1}n)n^{(m-l-1)}. \tag{8}$$

Next, from (2) we have $L_l x(n) > 0$ for $n \geq n_0$.
Summing the equality

$$L_l x(n) = r_l(n) \Delta L_{l-1} x(n)$$

from n_0 to $n-1$ we get

$$\sum_{k=n_0}^{n-1} L_l x(k) \leq r_l(n_0) \sum_{k=n_0}^{n-1} \Delta L_{l-1} x(k) = r_l(n_0) L_{l-1} x(n) - r_l(n_0) L_{l-1} x(n)$$
$$\leq r_l(n_0) L_{l-1} x(n) \quad for\ n \geq n_0.$$

Hence, and by (8)

$$L_{l-1} x(n) \geq \frac{1}{r_l(n_0)} \sum_{k=n_0}^{n-1} L_l x(k)$$
$$\geq \frac{1}{r_l(n_0)} \frac{1}{(m-l-1)!} \sum_{k=n_0}^{n-1} \prod_{i=1}^{m-l-1} \frac{1}{r_{m-i}(k)} L_{m-1} x(2^{m-l-1} k) k^{(m-l-1)}$$
$$\geq \frac{1}{(m-l-1)!} \prod_{i=1}^{m-l} \frac{1}{r_{m-i}(n_0)} L_{m-1} x(2^{m-l-1} n) \sum_{k=n_0}^{n-1} k^{(m-l-1)}$$
$$\geq \frac{1}{(m-l)!} \prod_{i=1}^{m-l} \frac{1}{r_{m-i}(n_0)} L_{m-1} x(2^{m-l-1} n)(n-n_0)^{(m-l)}.$$

Summing again the above inequality we get

$$L_{l-2} x(n) \geq \frac{1}{(m-l)!} \prod_{i=1}^{m-l+1} \frac{1}{r_{m-i}(n_0)} \sum_{k=n_0}^{n-1} L_{m-1} x(2^{m-l-1} k)(k-n_0)^{(m-l)}$$
$$\geq \frac{1}{(m-l+1)!} \prod_{i=1}^{m-l+1} \frac{1}{r_{m-i(n_0)}} L_{m-1} x(2^{m-l-1} n)(n-n_0)^{(m-l+1)}$$

and after $(l-1)$ summations, we obtain (6). The proof is complete. ■

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