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EXPLICIT BOUNDS ON SOME RETARDED INTEGRAL INEQUALITIES

Abstract: In this paper we establish some new retarded integral inequalities which provide explicit bounds on unknown functions. Applications are also given to illustrate the usefulness of one of our results.

Key words: explicit bounds, retarded integral inequalities, partial derivatives, nondecreasing, estimate on the solution, uniueness of solution, dependency of solution.

1. Introduction

In the development of the theory of differential and integral equations, integral inequalities which provide explict bounds on unknown functions take very important place. In view to widen the scope of such inequalities, in the past few decades many such new inequalities have been discovered to achieve a diversity of desired goals, see [1, 3-8] and the references cited therein. However, in certain situations the available inequalities do not apply directly and it is desirable to find some new inequalities which would be equally important in certain applications. In the present paper, we offer some fundamental integral inequalities which can be used as tools in the study of various classes of differential and integral equations involving several retarded arguments. We also present some basic applications of one of our results to convey the importance of the results to the literate. We believe that, the inequalities given here will have a frofound and eduring influence in the development of the theory of differential and integral equations involving retarded arguments.

2. Statement of results

In what follows, R denote the set of real numbers, $R_+ = [0, \infty)$, $I = [t_0, T)$, $I_1 = [x_0, X)$ and $I_2 = [y_0, Y)$ are the given subsets of R, $\Delta = I_1 \times I_2$ and $'$ denote the derivative. The first order partial derivatives of a function

$z(x,y)$,$x,y \in R$ with respect to x and y are denoted by $D_1 z(x,y)$ and $D_2 z(x,y)$ respectively and $D_2 D_1 z(x,y) = D_1 D_2 z(x,y)$.

Our main results are given in the following theorems.

Theorem 1. *Let $u, a, b_i \in C(I, R_+)$ and $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \le t$ on I for $i = 1, ..., n$ and $k \ge 0$ be a constant.*

(A_1) If

$$u(t) \le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s) u(s) ds, \tag{2.1}$$

for $t \in I$, then

$$u(t) \le k \exp(A(t)), \tag{2.2}$$

for $t \in I$, where

$$A(t) = \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(\sigma) d\sigma, \tag{2.3}$$

for $t \in I$.

(A_2) If $a(t)$ is nondecreasing for $t \in I$ and

$$u(t) \le a(t) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s) u(s) ds, \tag{2.4}$$

for $t \in I$, then

$$u(t) \le a(t) \exp(A(t)), \tag{2.5}$$

for $t \in I$, where $A(t)$ is given by (2.3).

Theorem 2. *Let u, b_i, α_i be as in Theorem 1. Let $k \ge 0, p > 1$ be constants . Let $g \in C(R_+, R_+)$ be nondecreasing function with $g(u) > 0$ for $u > 0$.*

(B_1) If for $t \in I$,

$$u(t) \le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s) g(u(s)) ds, \tag{2.6}$$

then for $t_0 \le t \le t_1$,

$$u(t) \le G^{-1}\left[G(k) + A(t)\right], \tag{2.7}$$

where $A(t)$ *is given by (2.3) and* G^{-1} *is the inverse function of*

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad r > 0, \tag{2.8}$$

$r_0 > 0$ *is arbitrary and* $t_1 \in I$ *is chosen so that*

$$G(k) + A(t) \in Dom\left(G^{-1}\right),$$

for all t *lying in the interval* $[t_0, t_1]$.

(B_2) *If for* $t \in I$,

$$u^p(t) \le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)\, g(u(s))\, ds, \tag{2.9}$$

then for $t_0 \le t \le t_2$,

$$u(t) \le \left\{H^{-1}\left[H(k) + A(t)\right]\right\}^{\frac{1}{p}}, \tag{2.10}$$

where $A(t)$ *is given by (2.3) and* H^{-1} *is the inverse function of*

$$H(r) = \int_{r_0}^{r} \frac{ds}{g\left(s^{\frac{1}{p}}\right)}, \quad r > 0, \tag{2.11}$$

$r_0 > 0$ *is arbitrary and* $t_2 \in I$ *is chosen so that*

$$H(k) + A(t) \in Dom\left(H^{-1}\right),$$

for all t *lying in the interval* $[t_0, t_2]$.

In the following theorems we establish two independent variable versions of Theorems 1 and 2 which can be used as tools in the study of certain hyperbolic partial differential equations.

Theorem 3. *Let* $u, a, b_i \in C(\Delta, R_+)$ *and* $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha_i(x) \le x$ *on* I_1, $\beta_i(y) \le y$ *on* I_2, *for* $i = 1, ..., n$ *and* $k \ge 0$ *be a constant.*

(C_1) *If*

$$u(x,y) \leq k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, u(s,t)\, dtds, \tag{2.12}$$

for $(x,y) \in \Delta,$ *then*

$$u(x,y) \leq k \exp(B(x,y)), \tag{2.13}$$

for $(x,y) \in \Delta,$ *where*

$$B(x,y) = \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, dtds, \tag{2.14}$$

for $(x,y) \in \Delta.$

(C_2) *If* $a(x,y)$ *is nondecreasing for* $(x,y) \in \Delta$ *and*

$$u(x,y) \leq a(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, u(s,t)\, dtds, \tag{2.15}$$

for $(x,y) \in \Delta,$ *then*

$$u(x,y) \leq a(x,y) \exp(B(x,y)), \tag{2.16}$$

for $(x,y) \in \Delta,$ *where* $B(x,y)$ *is given by (2.14).*

Theorem 4. *Let* $u, b_i, \alpha_i,\ \beta_i$ *be as in Theorem 3 and* k, p, g *be as in Theorem 2.*

(D_1) *If for* $(x,y) \in \Delta,$

$$u(x,y) \leq k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\, g(u(s,t))\, dtds, \tag{2.17}$$

then for $x_0 \leq x \leq x_1, y_0 \leq y \leq y_1;\ x, x_1 \in I_1, y, y_1 \in I_2,$

$$u(x,y) \leq G^{-1}\left[G(k) + B(x,y)\right], \tag{2.18}$$

where G, G^{-1} *are as in part* (B_1) *of Theorem 2,* $B(x,y)$ *is given by (2.14) and* $x_1 \in I_1$, $y_1 \in I_2$ *are chosen so that*

$$G(k) + B(x,y) \in Dom\left(G^{-1}\right),$$

for all x and y lying in $[x_0, x_1]$ and $[y_0, y_1]$ respectively.

(D_2) *If for $(x, y) \in \Delta$,*

$$u^p(x, y) \leq k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s, t)\, g(u(s, t))\, dt ds, \tag{2.19}$$

then for $x_0 \leq x \leq x_2, y_0 \leq y \leq y_2$,

$$u(x, y) \leq \left\{H^{-1}\left[H(k) + B(x, y)\right]\right\}^{\frac{1}{p}}, \tag{2.20}$$

where H, H^{-1} are as in part (B_2) of Theorem 2 , $B(x,y)$ is given by (2.14) and $x_2 \in I_1$, $y_2 \in I_2$ are chosen so that

$$H(k) + B(x, y) \in Dom\left(H^{-1}\right),$$

for all x and y lying in $[x_0, x_2]$ and $[y_0, y_2]$ respectively.

3. Proofs of Theorems 1-4

In the proofs, we make use of the following elementary fact.

Lemma. *Let $a, b \in R$ and $a + b \geq 0$. If $a > 0$, then $a + b > 0$.*

Proof. Suppose that the conclusion is not true, then $a + b \leq 0$. The case $a + b = 0$ is trivial. If $a + b < 0$, i.e. $a < -b$, then $0 \leq a + b < -b + b = 0$, a contradidication. Hence the conclusion is true.

Since the proofs of Theorems 1-4 resemble one another, we give the details for $(A_1), (A_2), (C_1)$ only, the proofs of the remaining inequalities can be completed by following the proofs of the above mentioned inequalities and closely looking at the proofs of the similar inequalities given in [6,7].

From the hypotheses on α_i, β_i we observe that $\alpha_i'(t) \geq 0$ for $t \in I$, $\alpha_i'(x) \geq 0$ for $x \in I_1$, $\beta_i'(y) \geq 0$ for $y \in I_2$.

(A_1) Let $k > 0$ and define a function $z(t)$ by the right hand side of (2.1). Then $z(t_0) = k$, $u(t) \leq z(t)$, $z(t) > 0$ by Lemma and

$$z'(t) = \sum_{i=1}^{n} b_i(\alpha_i(t))\, u(\alpha_i(t))\, \alpha_i'(t)$$

$$\leq \sum_{i=1}^{n} b_i(\alpha_i(t))\, z(\alpha_i(t))\, \alpha_i'(t)$$

$$\leq \sum_{i=1}^{n} b_i\left(\alpha_i\left(t\right)\right) z\left(t\right) \alpha_i'\left(t\right)$$

i.e.

$$\frac{z'\left(t\right)}{z\left(t\right)} \leq \sum_{i=1}^{n} b_i\left(\alpha_i\left(t\right)\right) \alpha_i'\left(t\right). \tag{3.1}$$

Integrating (3.1) from t_0 to t, $t \in I$ and then the change of variables yield

$$z\left(t\right) \leq k \exp\left(A\left(t\right)\right), \tag{3.2}$$

for $t \in I$. Using (3.2) in $u\left(t\right) \leq z\left(t\right)$ we get the inequality in (2.2). If $k \geq 0$ we carry out the above procedure with $k + \varepsilon$ instead of k, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass the limit $\varepsilon \to 0$ to obtain (2.2).

(A_2) First we assume that $a(t) > 0$ for $t \in I$. From the hypotheses we observe that for $s \leq \alpha_i\left(t\right) \leq t, a\left(s\right) \leq a\left(\alpha_i\left(t\right)\right) \leq a\left(t\right).$ In view of this, from (2.4) we observe that

$$\frac{u\left(t\right)}{a\left(t\right)} \leq 1 + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i\left(s\right) \frac{u\left(s\right)}{a\left(s\right)} ds. \tag{3.3}$$

Now an application of the inequality in part (A_1) to (3.3) yields the required inequality in (2.5). If $a(t) = 0$, then from (2.4) we observe that

$$u\left(t\right) \leq \varepsilon + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i\left(s\right) u\left(s\right) ds,$$

where $\varepsilon > 0$ is an arbitrary small constant. An application of the inequality in part (A_1) yields

$$u\left(t\right) \leq \varepsilon \exp\left(A\left(t\right)\right). \tag{3.4}$$

Now by letting $\varepsilon \to 0$ in (3.4) we have $u(t) = 0$ and hence (2.5) holds.

(C_1) Let $k > 0$ and define a function $z(x, y)$ by the right hand side of (2.12). Then $z\left(x_0, y\right) = z\left(x, y_0\right) = k$, $u\left(x, y\right) \leq z\left(x, y\right)$, $z\left(x, y\right) > 0$ by Lemma and

$$D_1 z\left(x, y\right) = \sum_{i=1}^{n} \left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i\left(\alpha_i\left(x\right), t\right) u\left(\alpha_i\left(x\right), t\right) dt \right) \alpha_i'\left(x\right)$$

$$\le \sum_{i=1}^{n}\left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i\left(\alpha_i\left(x\right),t\right) z\left(\alpha_i\left(x\right),t\right) dt\right)\alpha_i'\left(x\right)$$

$$\le z\left(x,y\right)\sum_{i=1}^{n}\left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i\left(\alpha_i\left(x\right),t\right) dt\right)\alpha_i'\left(x\right)$$

i.e.

$$\frac{D_1 z\left(x,y\right)}{z\left(x,y\right)} \le \sum_{i=1}^{n}\left(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i\left(\alpha_i\left(x\right),t\right) dt\right)\alpha_i'\left(x\right). \tag{3.5}$$

Keeping y fixed in (3.5), setting $x=\sigma$, and integrating it with respect to σ from x_0 to x, $x\in I_1$, and making the change of variables we get

$$z\left(x,y\right)\le k\exp\left(B\left(x,y\right)\right), \tag{3.6}$$

for $(x,y)\in\Delta$. Using (3.6) in $u\left(x,y\right)\le z\left(x,y\right)$ we get the required inequality in (2.13). The case $k\ge 0$ follows as mentioned in the proof of (A_1). ■

4. Further inequalities

In this section we present the useful variants of the inequalities in Theorem 2.1, part (a_1) and Theorem 2.2, part (b_1) given in [6] which can be used as convenient tools in some applications .

Theorem 5. *Let $u,a,b\in C\left(I,R_+\right)$ and $\alpha\in C^1\left(I,I\right)$ be nondecreasing with $\alpha\left(t\right)\le t$ on I and $k\ge 0$ be a constant. If*

$$u\left(t\right)\le k+\int_{\alpha(t_0)}^{\alpha(t)} a\left(s\right)\left[u\left(s\right)+\int_{\alpha(t_0)}^{s} b\left(\sigma\right)u\left(\sigma\right)d\sigma\right]ds, \tag{4.1}$$

for $t\in I$, then

$$u\left(t\right)\le k\left[1+\int_{\alpha(t_0)}^{\alpha(t)} a\left(s\right)\exp\left(\int_{\alpha(t_0)}^{s}\left[a\left(\sigma\right)+b\left(\sigma\right)\right]d\sigma\right)ds\right], \tag{4.2}$$

for $t\in I$.

Theorem 6. *Let* $u, a, b \in C(\Delta, R_+)$ $\alpha(x) \in C^1(I_1, I_1)$, $\beta(y) \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha(x) \leq x$ *on* I_1, $\beta(y) \leq y$ *on* I_2 *and* $k \geq 0$ *be a constant. If*

$$u(x,y) \leq k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[u(s,t) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(\sigma,\eta) u(\sigma,\eta) \, d\eta d\sigma \right] dtds, \tag{4.3}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq k \left[1 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(m,n) \times \exp\left(\int_{\alpha(x_0)}^{m} \int_{\beta(y_0)}^{n} [a(\sigma,\eta) + b(\sigma,\eta)] \, d\eta d\sigma \right) dndm \right], \tag{4.4}$$

for $(x,y) \in \Delta$.

Proof. Here we give the details of Theorem 6 only, the proof of Theorem 5 can be completed similarly, see also [6,7].

From the hypotheses we observe that $\alpha'(x) \geq 0$ for $x \in I_1$, $\beta'(y) \geq 0$ for $y \in I_2$. Let $k > 0$ and define a function $z(x,y)$ by the right hand side of (4.3). Then $z(x_0,y) = z(x,y_0) = k$, $u(x,y) \leq z(x,y)$, $z(x,y) > 0$ by Lemma and

$$\begin{aligned} D_1 D_2 z(x,y) &= a(\alpha(x), \beta(y)) \\ &\quad \times \left[u(\alpha(x), \beta(y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(\sigma,\eta) u(\sigma,\eta) d\eta d\sigma \right] \\ &\quad \times \beta'(y) \alpha'(x) \leq a(\alpha(x), \beta(y)) \\ &\quad \times \left[z(\alpha(x), \beta(y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(\sigma,\eta) z(\sigma,\eta) d\eta d\sigma \right] \beta'(y) \alpha'(x) \\ &\leq a(\alpha(x), \beta(y)) \left[z(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(\sigma,\eta) z(\sigma,\eta) d\eta d\sigma \right] \beta'(y) \alpha'(x). \end{aligned}$$

Define a function $v(x,y)$ by

$$v\left(x,y\right)=z\left(x,y\right)+\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{\beta(y_0)}^{\beta(y)}b\left(\sigma,\eta\right)z\left(\sigma,\eta\right)d\eta d\sigma. \tag{4.5}$$

Then $v\left(x_0,y\right)=z\left(x_0,y\right)=k,\ v\left(x,y_0\right)=z\left(x,y_0\right)=k,\ z\left(x,y\right)\le v\left(x,y\right)$, $z\left(x,y\right)>0$ by Lemma,

$$D_1D_2z\left(x,y\right)\le a\left(\alpha\left(x\right),\beta\left(y\right)\right)v\left(x,y\right)\beta'\left(y\right)\alpha'\left(x\right), \tag{4.6}$$

and

$$\begin{aligned} D_1D_2v\left(x,y\right)&=D_1D_2z\left(x,y\right)\\ &\quad+b\left(\alpha\left(x\right),\beta\left(y\right)\right)z\left(\alpha\left(x\right),\beta\left(y\right)\right)\beta'\left(y\right)\alpha'\left(x\right)\\ &\le a\left(\alpha\left(x\right),\beta\left(y\right)\right)v\left(x,y\right)\beta'\left(y\right)\alpha'\left(x\right)\\ &\quad+b\left(\alpha\left(x\right),\beta\left(y\right)\right)v\left(\alpha\left(x\right),\beta\left(y\right)\right)\beta'\left(y\right)\alpha'\left(x\right)\\ &\le\left[a\left(\alpha\left(x\right),\beta\left(y\right)\right)+b\left(\alpha\left(x\right),\beta\left(y\right)\right)\right]v\left(x,y\right)\beta'\left(y\right)\alpha'\left(x\right). \end{aligned} \tag{4.7}$$

Now by following the proof of Theorem 4.2.1 given in [4] with suitable changes, from (4.7) we obtain

$$\begin{aligned} v\left(x,y\right)&\le k\\ &\times\exp\left(\int\limits_{x_0}^{x}\int\limits_{y_0}^{y}\left[a\left(\alpha\left(s\right),\beta\left(t\right)\right)+b\left(\alpha\left(s\right),\beta\left(t\right)\right)\right]\beta'\left(t\right)\alpha'\left(s\right)dtds\right). \end{aligned} \tag{4.8}$$

By making the change of variables on the right hand side of (4.8) yields

$$v\left(x,y\right)\le k\exp\left(\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{\beta(y_0)}^{\beta(y)}\left[a\left(\sigma,\eta\right)+b\left(\sigma,\eta\right)\right]d\eta d\sigma\right). \tag{4.9}$$

Using (4.9) in (4.6) we have

$$\begin{aligned} D_1D_2z\left(x,y\right)&\le ka\left(\alpha\left(x\right),\beta\left(y\right)\right)\\ &\times\exp\left(\int\limits_{\alpha(x_0)}^{\alpha(x)}\int\limits_{\beta(y_0)}^{\beta(y)}\left[a\left(\sigma,\eta\right)+b\left(\sigma,\eta\right)\right]d\eta d\sigma\right)\beta'\left(y\right)\alpha'\left(x\right). \end{aligned} \tag{4.10}$$

Keeping x fixed in (4.10), set $y = t$ and integrate with respect to t from y_0 to y, $y \in I_2$, then keeping y fixed in the resulting inequality, set $x = s$ and integrate with respect to s from x_0 to x, $x \in I_1$ to obtain the estimate

$$(4.11) \qquad z(x,y) \leq k \left[1 + \int_{x_0}^{x} \int_{y_0}^{y} a(\alpha(s), \beta(t)) \times \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} [a(\sigma,\eta) + b(\sigma,\eta)] \, d\eta d\sigma \right) \beta'(t) \alpha'(s) \, dt ds \right].$$

By making the change of variables on the right hand side of (4.11) and using the fact that $u(x,y) \leq z(x,y)$ we obtain the desired bound in (4.4). The case $k \geq 0$ follows as mentioned in the proof of (A_1). The proof is complete. ■

5. Some applications

In this section, we present some model applications which display the importance of our results in the analysis of various classes of differential equations. Consider the following differential equation involving several retarded arguments (see [2])

$$(5.1) \qquad x'(t) = f(t, x(t - h_1(t)), ..., x(t - h_n(t))),$$

with the given initial condition

$$(5.2) \qquad x(t_0) = x_0,$$

where $f \in C(I \times R^n, R)$, x_0 is a real constant and $h_i \in C^1(I, I)$ be nonincreasing with $t - h_i(t) \geq 0$, $h_i'(t) < 1$, for $i = 1, ..., n$ and $h_i(t_0) = 0$.

The following theorem deals with the estimate on the solution of (5.1)-(5.2).

Theorem 7. *Suppose that*

$$(5.3) \qquad |f(t, u_1, ..., u_n)| \leq \sum_{i=1}^{n} b_i(t) |u_i|,$$

where $b_i(t)$ are as in Theorem 1, and let

$$(5.4) \qquad M_i = \max_{t \in I} \frac{1}{1 - h_i'(t)}, \qquad i = 1, ..., n.$$

If $x(t)$ is any solution of (5.1)-(5.2), then

$$|x(t)| \le |x_0| \exp\left(\sum_{i=1}^{n} \int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma) d\sigma \right), \tag{5.5}$$

for $t \in I$, where $\bar{b}_i(\sigma) = M_i b_i(\sigma + h_i(s))$, $\sigma, s \in I$.

Proof. The solution $x(t)$ of (5.1)-(5.2) can be written as

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s - h_1(s)), ..., x(s - h_n(s))) ds. \tag{5.6}$$

Using (5.3) in (5.6) and making the change of variables, then using (5.4) we have

$$|x(t)| \le |x_0| + \sum_{i=1}^{n} \int_{t_0}^{t} b_i(s) |x(s - h_i(s))| ds \tag{5.7}$$

$$\le |x_0| + \sum_{i=1}^{n} \int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma) |x(\sigma)| d\sigma,$$

for $t \in I$. Now a suitable application of the inequality (A_1) given in Theorem 1 to (5.7) yields the required estimate in (5.5). ■

The next theorem deals with the uniqueness of solutions of (5.1)-(5.2).

Theorem 8. *Suppose that the function f in (3.5) satisfies the condition*

$$|f(t, u_1, ..., u_n) - f(t, v_1, ..., v_n)| \le \sum_{i=1}^{n} b_i(t) |u_i - v_i|, \tag{5.8}$$

where $b_i(t)$ are as in Theorem 1. Let M_i and $\bar{b}_i(\sigma)$ be as in Theorem 7. Then (5.1)-(5.2) has at most one solution on I.

Proof. Let $x(t)$ and $y(t)$ be two solutions of (5.1)-(5.2) on I, then we have

$$x(t) - y(t) = \int_{t_0}^{t} \{f(s, x(s - h_1(s)), ..., x(s - h_n(s))) \tag{5.9}$$

$$- f(s, y(s - h_1(s)), ..., y(s - h_n(s)))\} ds.$$

Using (5.8) in (5.9) and making the change of variables and using (5.4) we have

$$|x(t) - y(t)| \le \sum_{i=1}^{n} \int_{t_0}^{t} b_i(s)\,|x(s - h_i(s)) - y(s - h_i(s))|ds \tag{5.10}$$

$$\le \sum_{i=1}^{n} \int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma)\,|x(\sigma) - y(\sigma)|d\sigma,$$

for $t \in I$. A suitable application of the inequality (A_1) given in Theorem 1 yields $|x(t) - y(t)| \le 0$. Therefore $x(t) = y(t)$ i.e. there is at most one solution of (5.1)-(5.2). ■

The following theorem shows the dependency of solutions of (5.1)-(5.2) on initial values.

Theorem 9. *Let $x(t)$ and $y(t)$ be the solutions of (5.1) with the given initial conditions*

$$x(t_0) = x_0 \tag{5.11}$$

and

$$y(t_0) = y_0 \tag{5.12}$$

respectively, where x_0, y_0 are real constants. Suppose that the function f satisfies the condition (5.8) in Theorem 8. Let $M_i, \bar{b}_i(\sigma)$ be as in Theorem 7. Then the solutions of (5.1)-(5.2) depends continuously on the initial values and

$$|x(t) - y(t)| \le |x_0 - y_0| \exp\left(\sum_{i=1}^{n} \int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma) d\sigma \right), \tag{5.13}$$

for $t \in I$.

Proof. Since $x(t)$ and $y(t)$ are the solutions of (5.1)-(5.11) and (5.1)-(5.12) respectively, we have

$$x(t) - y(t) = x_0 - y_0 + \int_{t_0}^{t} \{f(s, x(s - h_1(s)), ..., x(s - h_n(s))) \tag{5.14}$$

$$- f(s, y(s - h_1(s)), ..., y(s - h_n(s)))\}\, ds.$$

Using (5.8) in (5.14) and by making the change of variables and using (5.4) we have

$$(5.15)\qquad |x(t)-y(t)| \le |x_0-y_0| + \sum_{i=1}^{n} \int_{t_0}^{t} b_i(s)\,|x(s-h_i(s))-y(s-h_i(s))|\,ds$$
$$\le |x_0-y_0| + \sum_{i=1}^{n} \int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma)\,|x(\sigma)-y(\sigma)|\,d\sigma$$

for $t \in I$. Now a suitable application of the inequality (A_1) given in Theorem 1 to (5.15) yields the required estimate in (5.13) which shows the dependency of solutions of (5.1)-(5.2) on initial values.

We note that the inequality in (C_1) given in Theorem 3 can be used to study the similar properties as in Theorems 7-9 for the solutions of hyperbolic partial differential equation with many retarded arguments of the form

$$(5.16)\qquad D_1D_2z(x,y) = f(x,y,z(x-h_1(x),y-g_1(y)),\ldots, z(x-h_n(x),y-g_n(y))),$$

with the given initial boundary conditions

$$(5.17)\qquad z(x_0,y) = a_1(x),\, z(x,y_0) = a_2(y),\, a_1(x_0) = a_2(y_0),$$

under some suitable conditions on the functions involved in (5.16)-(5.17). Various other applications of the inequalities given here is left to another work. ■

In conclusion, we note that the results given in Theorems 3, 4 and 6 can be extended very easily to functions involving many indepensent variables. Here we omit the details.

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