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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF SOLUTIONS OF A FIRST ORDER FORCED NONLINEAR DIFFERENCE EQUATION WITH SEVERAL DELAYS

Abstract: In this paper necessary and sufficient conditions have been obtained so that every solution of the Neutral Delay Difference Equation (NDDE)

$$\Delta\left(y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j}\right) + q_n\, G\,(y_{n-r}) = f_n$$

oscillates or tends to zero as $n \to \infty$ for different ranges of $\sum_{j=1}^{k} p_j$. This paper improves and generalizes some recent work [2, 6, 8]. The results of this paper hold for linear, sublinear and superlinear equations and also for homogeneous equations, i.e. when $f_n \equiv 0$.

Key words: oscillation, non-oscillation, asymptotic behaviour, neutral difference equations.

1. Introduction

In this paper, the authors have obtained necessary and sufficient conditions so that every solution $y = \{y_n\}$ of the first order difference equation

$$\text{(E)} \qquad \Delta\left(y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j}\right) + q_n\, G\,(y_{n-r}) = f_n$$

oscillates or tends to zero as $n \to \infty$ for various ranges of $\sum_{j=1}^{k} P_j$, where each p_j is a scalar, Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $G \in C(R, R)$, G is non-decreasing and $x\, G(x) > 0$ for $x \neq 0$, $q_n \geq 0$, f_n are real numbers $(n = 0, 1, 2, ...)$, m_j for $j = 1, 2, ..., k$ and r are non negative integers. We further assume the following conditions for its use in the sequel.

(H1) There exists a sequence $\{F_n\}$ such that and $\lim_{n\to\infty} F_n = 0$ and $\Delta F_n = f_n$.

(H2) Let G satisfy Lipschitz condition in the intervals of the type $[a, b]$, $0 < a < b$.

(H3) $\sum_{n=0}^{\infty} q_n = \infty$.

The following ranges for p_j $(j = 1,\ 2,\ \ldots k)$ are considered in this work.

(A1) $-1 < \sum_{j=1}^{k} p_j \le 0$, where each $p_j \le 0$

(A2) $0 \le \sum_{j=1}^{k} p_j < 1$, where each $p_j \ge 0$

(A3) $\sum_{j=1}^{k} p_j > 1$, where each $p_j > 0$ and $p_i > 1 + \sum_{j\neq 1} p_j$ for some $i \in \{1,\ 2,\ 3,\ \ldots,\ k\}$

(A4) $\sum_{j=1}^{k} p_j < -1$, where each $p_j < 0$ and $p_i < -1 + \sum_{j\neq 1} p_j$ for some $i \in \{1,\ 2,\ 3,\ \ldots,\ k\}$

Our results also hold for the equation

(1)
$$\Delta \left(y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j} \right) + \sum_{j=1}^{s} q_n^j\, G\left(y_{n-r_j}\right) = f_n$$

under the assumption

(2)
$$\sum_{n=0}^{\infty} \left(\sum_{j=1}^{k} q_n^j \right) = \infty$$

in place of (H3), where r_j are nonnegative integers for $j = 1, 2, \ldots, s$, $\{q_n^j\}$ are infinite sequences of positive real numbers for each j and other symbols are as defined in (E).

In recent years many authors (see [6], [8], [9]) have taken active interest in studying oscillatory behaviour of solutions of NDDEs. But in the literature few results are available regarding the oscillation criteria for solutions of neutral difference equations with several delays. In a recent paper [6] the authors have obtained the oscillation criteria for the solutions of

(3)
$$\Delta\left(y_n + p\, y_{n-m}\right) + q_n\, G\left(y_{n-r}\right) = f_n.$$

One may easily verify that, the technique used in [6] also works if we attempt to study the same problem for the equation

$$\Delta\left(y_n + p\, y_{n-m}\right) + \sum_{j=1}^{s} q_n^j\, G\left(y_{n-r_j}\right) = f_n, \tag{4}$$

under primary assumptions (2) in place of (H_3). But interestingly, the technique adopted in [6], fails, when one attempts to study the same problem for (E). Thus the bottom line is if the several delay terms are taken under the difference operator, then one may face the real difficulty. This motivated the author for the present work. Further Lemma 2.1 was repeatedly used to get the results in [6], which is the discrete analogue of Lemma 1.5.1 in [3]. The notes 1.8 given in [3, p.31] suggests to extend the Lemma 1.5.1 for application to neutral equations with several delays. But it seems difficult to extend the Lemmas as suggested in [3]. Hence the author became more interested to study the present problem for (E).

We may view (E) as the discrete analogue of the neutral delay differential equation

$$\left(y(t) + \sum_{j=1}^{k} p_j\, y\left(t - \tau_j\right)\right)' + q(t)\, G(y(t-\sigma)) = f(t) \tag{F}$$

about which not much is found in the literature. It is well known that oscillation behaviour of delay differential equation and their discrete analogues can be quite different ([11]). In [10] Yu and Wang have discussed oscillation of solutions of first order neutral difference equation

$$\Delta\left(y_n - p\, y_{n-m}\right) + q_n\, y_{n-k} = 0, \qquad n = 0, 1, 2, ... \tag{5}$$

for $p = 1$ and $p = -1$. In [2], Georgiou et al have studied the oscillatory and asymptotic behaviour of solutions of Eq.(5).

The results of this paper generalizes many work done for delay difference equations and NDDEs because our G can be linear, sublinear or superlinear and f_n could be zero (see Corollary 2.3 and Remark 2).

Let $\ell = \max(r, m_1, m_2, ..., m_k)$. By a solution of Eq.(E) on $[0,\infty)$ we mean a sequence $\{y_n\}$ of real numbers which is defined for $n \geq -\ell$ and which satisfies (E) (for $n = 0, 1, 2, ...$). A solution $\{y_n\}$ of (E) on $[0,\infty)$ is said to be oscillatory if for every positive integer $N > 0$ there exists $n \geq N$ such that $y_n\, y_{n+1} \leq 0$, otherwise, $\{y_n\}$ is said to be nonoscillatory.

2. Main results

In all our results in this section though we do not mention exclusively, we assume $G \in C(R,R)$, G is nondecreasing and $x\,G(x) > 0$ for $x \neq 0$.

Theorem 2.1. *Suppose that* (H_1) *holds and that* $\sum_{j=1}^{k} p_j$ *be in the range* (A_1). *Then every solution of (E) is oscillatory or tends to zero as* $n \to \infty$ *if and only if* (H_3) *holds.*

Proof. Let us first prove that (H_3) is sufficient. Suppose that (H_3) holds and $\{y_n\}$ be a non-oscillatory solution of (E) for $n \geq N_0$. Setting

$$z_n = y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j} \tag{6}$$

and

$$w_n = z_n - F_n \tag{7}$$

we obtain from (E)

$$\Delta w_n = -q_n\, G\left(y_{n-r}\right). \tag{8}$$

If $y_n > 0$ for $n \geq N_0$ then $\Delta w_n \leq 0$, which implies $w_n > 0$ or $w_n < 0$ for $n > N_1 \geq N_0 + \ell$. In both the cases we claim that $\{y_n\}$ is bounded. If not then there exists a subsequence $\{y_{n_i}\}$ such that $n_i \to \infty$ and $y_{n_i} \to \infty$ as $i \to \infty$ and $y_{n_i} = \max\{y_n : N_1 \leq n \leq n_i\}$. We may choose n_i large enough such that $n_i - \ell > N_1$. Hence

$$w_{n_i} = y_{n_i} + \sum_{j=1}^{k} p_j\, y_{n_i-m_j} - F_{n_i} \geq \left(1 + \sum_{j=1}^{k} p_j\right) y_{n_i} - F_{n_i}$$

implies that $w_{n_i} \to \infty$ as $i \to \infty$ by (H_1), a contradiction because $\{w_n\}$ is non increasing. Hence our claim holds and as a consequence $\liminf_{n\to\infty} y_n$, $\limsup_{n\to\infty} y_n$ and $\lim_{n\to\infty} w_n$ and exists. If $\liminf_{n\to\infty} y_n > 0$, then $y_n > \beta > 0$ for $n > N_2 > N_1$. Hence

$$\sum_{n=N_2+r}^{\infty} q_n\, G\left(y_{n-r}\right) > G(\beta) \sum_{n=N_2+r}^{\infty} q_n = \infty \tag{9}$$

and from (8) we obtain

$$\sum_{n=N_2+r}^{M-1} q_n\, G\left(y_{n-r}\right) = -\sum_{n=N_2+r}^{M-1} \Delta w_n = w_{N_2+r} - w_M.$$

If we take limit $M \to \infty$, it follows that

$$\sum_{n=N_2+r}^{\infty} q_n \, G\,(y_{n-r}) < \infty\,, \tag{10}$$

a contradiction to (9). Thus $\liminf_{n\to\infty} y_n = 0$. Since $\lim_{n\to\infty} w_n$ exists then from (H_1) and (7) it follows that there exists $\delta \in R$ such that $\lim_{n\to\infty} z_n = \delta$. If $\delta > 0$, then

$$\begin{aligned}
0 < \delta = \lim_{n\to\infty} z_n &= \liminf_{n\to\infty} \left(y_n + \sum_{j=1}^{k} p_j \, y_{n-m_j} \right) \\
&\le \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} \left(\sum_{j=1}^{k} p_j \, y_{n-m_j} \right) \\
&\le \sum_{j=1}^{k} \limsup_{n\to\infty} p_j \, y_{n-m_j} = \sum_{j=1}^{k} p_j \liminf_{n\to\infty} y_{n-m_j} = 0,
\end{aligned}$$

a contradiction. If $\delta \le 0$ then

$$\begin{aligned}
0 \ge \delta = \lim_{n\to\infty} z_n &= \limsup_{n\to\infty} \left(y_n + \sum_{j=1}^{k} p_j \, y_{n-m_j} \right) \\
&\ge \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} \left(\sum_{j=1}^{k} p_j \, y_{n-m_j} \right) \\
&\ge \limsup_{n\to\infty} y_n + \sum_{j=1}^{k} \liminf_{n\to\infty} p_j \, y_{n-m_j} \\
&\ge \left(\limsup_{n\to\infty} y_n \right) \left(1 + \sum_{j=1}^{k} p_j \right).
\end{aligned}$$

Hence $\limsup_{n\to\infty} y_n \le 0$ by (A_1), which gives $\lim_{n\to\infty} y_n = 0$. If $y_n < 0$ for large n, then we proceed as above and prove $\lim_{n\to\infty} y_n = 0$. Thus the sufficiency part of the theorem is proved.

In order to prove that the condition (H_3) is necessary we assume that

$$\sum_{n=0}^{\infty} q_n < \infty\,, \tag{11}$$

and show that (E) admits a positive solution which does not tend to zero as $n \to \infty$, when the limit exists. It is possible to choose an integer $N > 0$ such that for $n \geq N$,

$$G(1)\sum_{j=n}^{\infty} q_j < \frac{1}{10}\left(1+\sum_{j=1}^{k} p_j\right) \quad \text{for} \quad |F_n| < \frac{1}{10}\left(1+\sum_{j=1}^{k} p_j\right). \tag{12}$$

Let $X = \ell_\infty^N$, the set of all real bounded sequences $x = \{x_n\}$, $n \geq N$. For $x \in X$, we define $\|x\| = \sup\{|x_n| : n \geq N\}$. Clearly X is a Banach space with respect to the above norm. Let $K = \{x \in X : x_n \geq 0 \text{ for } n \geq N\}$. For $x, y \in X$, we define $x \leq y$ if and only if $y - x \in K$. Thus X is a partially ordered Banach space.

Let

$$W = \left\{ x \in X : \frac{1}{10}\left(1+\sum_{j=1}^{k} p_j\right) \leq x_n \leq 1, \;\; n \geq N \right\}.$$

If $x^0 = \{x_n^0\}$ where $x_n^0 = \frac{1}{10}\left(1+\sum_{j=1}^{k} p_j\right)$ for $n \geq N$, then $x^0 \in W$ and $x^0 = \inf W$. Further, let $\phi \subset W^* \subset W$. Then

$$W^* = \left\{ x \in X : \alpha \leq x_n \leq \beta, \; n \geq N, \; \alpha \geq \frac{1}{10}\left(1+\sum_{j=1}^{k} p_j\right) \text{ and } \beta \leq 1 \right\}.$$

If $x^* = \{x_n^*\}$, where $x_n^* = 1$ for $n \geq N$, then $x^* = \sup W^*$ and $x^* \in W$. For $y \in W$, define

$$(Ty)_n = \begin{cases} (Ty)_{N+\ell}\,, & N \leq n \leq N+\ell\,, \\ -\sum\limits_{j=1}^{k} p_j\, y_{n-m_j} + \sum\limits_{j=n}^{\infty} q_j\, G\,(y_{j-r}) & \\ \qquad\qquad + F_n + \frac{1}{5}\left(1+\sum\limits_{j=1}^{k} p_j\right), & n \geq N+\ell\,. \end{cases}$$

Hence for $n \geq N$,

$$\begin{aligned} (Ty)_n &\leq -\sum_{j=1}^{k} p_j + \frac{1}{5}\left(1+\sum_{j=1}^{k} p_j\right) + \frac{1}{5}\left(1+\sum_{j=1}^{k} p_j\right) \\ &= \frac{1}{5}\left(2 - 3\sum_{j=1}^{k} p_j\right) < 1 \end{aligned}$$

and

$$(Ty)_n \leq -\frac{1}{10}\left(1+\sum_{j=1}^{k} p_j\right) + \frac{1}{5}\left(1+\sum_{j=1}^{k} p_j\right) = \frac{1}{10}\left(1+\sum_{j=1}^{k} p_j\right)$$

by (12). Hence $T : W \to W$. Further, for $x, y \in W$ with $x \leq y$, we have $Tx \leq Ty$. Hence T has a fixed point in W by Knaster-Tarski fixed point theorem (see [3, p.30]), which is the required positive solution of (E). Hence the theorem is completely proved. ■

Remark 1. $\quad (\mathrm{H}_1) \Longleftrightarrow \left| \sum_{n=0}^{\infty} f_n \right| < \infty\,.$

Example 1. From Theorem 1. it follows that every non-oscillatory solutions of

$$\Delta\left(y_n - \frac{1}{2}y_{n-1} - \frac{1}{3}y_{n-2}\right) + (y_{n-1})^3 = e^{-3n+3} + \frac{1}{e^{n+1}}\left(1 + \frac{e^3}{3} + \frac{e^2}{6} - \frac{3e}{2}\right)$$

tends to zero as $n \to \infty$. In particular $y = \{e^{-n}\}$ is such a solution of this equation.

Example 2. The equation

$$\Delta\left(y_n + \sum_{j=1}^{3} 2^{-j}\, y_{n-3_j}\right) + \frac{1}{n^2}\, G\left(y_{n-2}\right) = \frac{G(1)}{n^2}$$

has a positive solution $y = \{y_n\}$ where $y_n = 1$ for every n. Here $q_n = \frac{1}{n^2}$ and $f_n = \frac{G(1)}{n^2}$ satisfies the conditions of the theorem. This illustrates the necessary part of Theorem 1.

Theorem 2.2. *Let $\sum_{j=1}^{k} p_j$ be in the range (A_2). Suppose that (H_1) holds. Then (i). (H_3) holds implies every solution of (E) oscillates or tends to zero as $n \to \infty$. (ii). Every solution of (E) oscillates or tends to zero as $n \to \infty$ such that (H_2) holds, implies (H_3) holds.*

Proof. Let us prove (i). Suppose (H_3) holds and $\{y_n\}$ be an ultimately positive solution of (E) for large n. Then setting z_n and w_n as in (6) and (7) we get (8), which implies $w_n > 0$ or $w_n < 0$ for $n \geq N_1$. We prove $\{y_n\}$ is bounded and $\liminf_{n\to\infty}$ as in the proof of Theorem 1. If $w_n > 0$ then $\lim_{n\to\infty} w_n$ exists. Hence from (H_1) and (7) it follows that $\lim_{n\to\infty} z_n = \delta \in R$. As $z_n \geq 0$, so $\delta \geq 0$. We claim $\delta = 0$, if not then $\delta > 0$, which implies

$$\delta = \lim_{n\to\infty}\ z_n = \liminf_{n\to\infty}\left(y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j}\right)$$

$$\leq \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} \sum_{j=1}^{k}\left(p_j\, y_{n-m_j}\right)$$

$$\leq \sum_{j=1}^{k} p_j \limsup_{n\to\infty} y_{n-m_j} = \left(\sum_{j=1}^{k} p_j\right) \alpha\,,$$

where $\alpha = \limsup\limits_{n\to\infty} y_n$. Hence we get

$$\alpha \geq \frac{\delta}{\sum\limits_{j=1}^{k} p_j} > \delta. \tag{13}$$

Again

$$\begin{aligned}
\delta &= \limsup_{n\to\infty} z_n = \limsup_{n\to\infty} \left(y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j} \right) \\
&\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} \left(\sum_{j=1}^{k} p_j\, y_{n-m_j} \right) \\
&\geq \alpha + \sum_{j=1}^{k} p_j \liminf_{n\to\infty} \left(p_j\, y_{n-m_j}\right) \geq \alpha + \sum_{j=1}^{k} p_j \left(\liminf_{n\to\infty} y_{n-m_j} \right) = \alpha\,,
\end{aligned}$$

a contradiction due to the inequality (13). Hence we conclude $\delta = 0$ and from $z_n > y_n$, it follows that $\limsup_{n\to\infty} y_n \leq 0$. Hence $\lim\limits_{n\to\infty} y_n = 0$. Further if $w_n < 0$ then either $\lim\limits_{n\to\infty} w_n = -\infty$ or $\lim\limits_{n\to\infty} w_n = \delta < 0$. But in both the cases $\lim\limits_{n\to\infty} z_n = \delta < 0$, which is a contradiction because $z_n > 0$ for large n. The proof for the case $y_n < 0$ for large n is similar. Hence (i) is proved.

Next let us prove (ii). Suppose to the contrary that (11) holds. From this and (H_1), we can find a large positive integer N such that for $n \geq N$

$$L \sum_{j=n}^{\infty} q_j < \frac{1}{5}\left(1 - \sum_{j=1}^{k} p_j\right) \quad \text{and} \quad |F_n| < \frac{1}{10}\left(1 - \sum_{j=1}^{k} p_j\right)$$

where $L = \max\{G(1), L_1\}$, L_1 is the Lipschitz constant of G in $\frac{1}{10}[(1 - \sum\limits_{j=1}^{k} p_j), 1]$. Let $X = \ell_\infty^N$ and $S = \{x \in X : \frac{1}{10}(1 - \sum\limits_{j=1}^{k} p_j) \leq x_n \leq 1,\, n \geq N\}$, S is a complete metric space, where the metric is induced by the norm in X. For $y \in S$, define

$$(Ty)_n = \begin{cases} (Ty)_{N+\ell}\,, & N \leq n \leq N+\ell\,, \\ -\sum\limits_{j=1}^{k} p_j y_{n-m_j} + \sum\limits_{j=n}^{\infty} q_j\, G\left(y_{j-r}\right) & \\ \qquad + F_n + \frac{1}{5}\left(1 + 4\sum\limits_{j=1}^{k} p_j\right), & n \geq N+\ell\,. \end{cases}$$

Hence for $n \geq N$,

$$\begin{aligned}(Ty)_n &\leq \frac{1}{5}\left(1-\sum_{j=1}^{k} p_j\right)+\frac{1}{10}\left(1-\sum_{j=1}^{k} p_j\right)+\frac{1}{5}\left(1+4\sum_{j=1}^{k} p_j\right)\\ &= \frac{1}{2}\left(1+\sum_{j=1}^{k} p_j\right)<1.\end{aligned}$$

$$\begin{aligned}(Ty)_n &\geq -\sum_{j=1}^{k} p_j+\frac{1}{10}\left(1-\sum_{j=1}^{k} p_j\right)+\frac{1}{5}\left(1+4\sum_{j=1}^{k} p_j\right)\\ &= \frac{1}{10}\left(1-\sum_{j=1}^{k} p_j\right).\end{aligned}$$

Hence $T : S \to S$. Further for $u, \nu \in S$

$$\begin{aligned}\|T\,u - T\,\nu\| &= \sup\left\{|(Tu)_n-(T\nu)_n| \,:\, n \geq N\right\}\\ &\leq \left(\sum_{j=1}^{k} p_j+\frac{1}{5}\left(1-\sum_{j=1}^{k} p_j\right)\right)\|u-v\|\end{aligned}$$

This shows T is a contraction. Hence T has a unique fixed point $y = \{y_n\}$ in S, which is the required positive solution of (E) on $[N+\ell, \infty)$ by Banach contraction principle. Hence the theorem is completely proved. ■

Corollary 2.3. *Theorem 2.1 and 2.2 are true when $p_j = 0$ for every j. Hence if (H_1) and (H_3) holds then every solution of the delay difference equation*

(D) $$\Delta y_n + q_n\, G\,(y_{n-r}) = f_n$$

oscillates or tends to zero as $n \to \infty$.

Theorem 2.4. *Let $\sum\limits_{j=1}^{k} p_j$ be in the range (A_3). Suppose that (H_1) holds. Then*

(i) *(H_3) holds implies that every solution of (E) oscillates or tends to zero as $n \to \infty$,*

(ii) *Every solution of (E) oscillates or tends to zero as $n \to \infty$ such that (H_2) holds implies (H_3) holds.*

Proof. Let us prove (i) Suppose that (H_3) holds and $y = \{y_n\}$ be an ultimately positive solution of (E). Then setting z_n and w_n as in (6) and (7), we obtain (8), which implies $w_n > 0$ or $w_n < 0$ for $n \geq N_1$. Suppose $w_n > 0$ for $n \geq N_1$ which implies $\lim_{n\to\infty} w_n = \delta \in R$. Using (H_1), we obtain $\lim_{n\to\infty} z_n = \delta \geq 0$. We prove $\{y_n\}$ is bounded and $\liminf_{n\to\infty} y_n = 0$ as in the proof of Theorem 1. Then

$$
\begin{aligned}
(14) \qquad \delta = \lim_{n\to\infty} z_n &= \liminf_{n\to\infty} \left(y_n + \sum_{j=1}^{k} p_j \, y_{n-m_j} \right) \\
&\leq \limsup_{n\to\infty} \left(y_n + \sum_{j\neq 1} p_j \, y_{n-m_j} \right) + \liminf_{n\to\infty} p_i \, y_{n-m_i} \\
&= \left(1 + \sum_{j\neq 1} p_j \right) \limsup_{n\to\infty} y_n
\end{aligned}
$$

and

$$
\begin{aligned}
(15) \qquad \delta = \lim_{n\to\infty} z_n &= \limsup_{n\to\infty} \left(y_n + \sum_{j\neq 1}^{k} p_j \, y_{n-m_j} \right) \\
&\geq \liminf_{n\to\infty} y_n + \limsup_{n\to\infty} \left(\sum_{j=1}^{k} p_j \, y_{n-m_j} \right) \\
&\geq \limsup_{n\to\infty} p_i \, y_{n-m_i} + \liminf_{n\to\infty} \sum_{j\neq i} p_j \, y_{n-m_j} \\
&\geq p_i \limsup_{n\to\infty} y_n + \sum_{j\neq i} \liminf_{n\to\infty} p_j \, y_{n-m_j} \\
&\geq p_i \limsup_{n\to\infty} y_n .
\end{aligned}
$$

From (14) and (15) we obtain

$$
\left(1 + \sum_{j\neq i} p_j \right) \limsup_{n\to\infty} \, y_n \geq p_i \limsup_{n\to\infty} \, y_n
$$

which implies

$$
\left(\left(1 + \sum_{j\neq i} p_j \right) - p_i \right) \limsup_{n\to\infty} \, y_n \geq 0 \, .
$$

Hence by (A_3), we obtain $\lim_{n\to\infty} y_n \leq 0$. Thus we have $\lim_{n\to\infty} y_n = 0$. If $w_n < 0$ then $\lim_{n\to\infty} w_n = -\infty$ or $\lim_{n\to\infty} w_n = \ell < 0$ exists. In both the cases $z_n < 0$ for large n, a contradiction. We can hold similar arguments for the case $y_n < 0$ for large n. Hence (i) is proved.

Next let us prove (ii) and choose

$$0 < a < p_i - 1 - \sum_{j \neq i} p_j$$

and

$$\lambda > \frac{a\left(1 + \sum_{j=1}^{k} p_j\right)}{\left(p_i - 1 - \sum_{j \neq i} p_j\right)} > 0\,.$$

Set

$$H = \frac{(\lambda + a)}{p_i} \quad \text{and} \quad h = \frac{1}{p_i^2}\left(-(\lambda + a)\left(1 + \sum_{j \neq i} p_j\right) - p_i(a - \lambda)\right)$$

we claim $h > 0$, otherwise

$$-(\lambda + a)\left(1 + \sum_{j \neq i} p_j\right) \leq p_i(a - \lambda) \quad \Leftrightarrow \quad \lambda \leq \frac{a\left(1 + \sum_{j=1}^{k} p_j\right)}{\left(p_i - 1 - \sum_{j \neq i} p_j\right)}$$

a contradiction. Hence $h > 0$. Further if $H \leq h$ then

$$\frac{(\lambda + a)}{p_i} \leq \frac{1}{p_i^2}\left\{-(\lambda + a)\left(1 + \sum_{j \neq i} p_j\right) - p_i(a - \lambda)\right\}$$
$$\Leftrightarrow 2p_i\, a \leq -(\lambda + a)\left(1 + \sum_{j \neq i} p_j\right)\,,$$

a contradiction. Hence $H > h > 0$. We may complete the proof by proceeding as in the proof of Theorem 2. and with the following changes

$$L \sum_{j=n}^{\infty} q_j < \frac{a}{2} \quad \text{and} \quad |F_n| < \frac{a}{2} \quad \text{for} \quad n \geq N,$$

where $L = \max\{L_1, G(H)\}$, L_1 is the Lipschitz constant of G in $[h, H]$.

Let $X = \ell_\infty^N$ and $S = \{x \in X : h \le x_n \le H\}$. For $y \in S$, we define

$$(Ty)_n = \begin{cases} -\frac{1}{p_i} y_{n+m_i} - \frac{1}{p_i} \sum\limits_{j \neq i} p_j \, y_{n-m_j+m_i} + \frac{1}{p_i} \sum\limits_{j=n+m_i}^{\infty} q_j \, G(y_{j-r}) \times \\ \qquad\qquad \times \frac{\lambda}{p_i} + \frac{1}{p_i} F_{n+m_i} \quad \text{for} \quad n \ge N+\ell \,, \\ (Ty)_{N+\ell} \,, \qquad N \le n \le N+\ell \,. \end{cases}$$

For $y \in S$,

$$(Ty)_n \le \frac{\left(L \sum\limits_{j+n+m_j}^{\infty} q_j \right)}{p_i + (\lambda/p_i) + (a/2p_i)} \le \frac{(a+\lambda)}{p_i} = H \,,$$

and

$$\begin{aligned} (Ty)_n &\ge -\frac{H}{p_i} - \left(\frac{H \sum\limits_{j\neq i} p_j}{p_i} \right) + \frac{\lambda}{p_i} - \frac{a}{2} p_i \\ &= \frac{1}{p_i^2} \left\{ -(\lambda + a) \left(1 + \sum_{j \neq i} p_j \right) + p_i(\lambda - a) \right\} = h \,. \end{aligned}$$

Hence $T : S \to S$ and for $u, \nu \in S$

$$\|Tu - T\nu\| \le \mu \, \|u - \nu\| \,, \quad \text{where} \quad 0 < \mu = \frac{1}{p_i} \left(1 + \frac{a}{2} + \sum_{j \neq i} p_j \right) < 1 \,.$$

Hence by Banach contraction principle T has a fixed point, which is the required positive solution. ■

Example 3. We may note that $y_n = e^n + e^{-n}$ is an unbounded solution of the equation

$$\Delta \left(y_n - (1+e) y_{n-1} \right) + (e-1) y_{n-1} = \left(2e^2 + e^{-1} - 2 - e \right) e^{-n}, \quad n \ge 0 \,.$$

Here $p = -(1+e) < -1$ and $\sum f_n < \infty$.

The above example is a source of motivation for the next theorem.

Theorem 5. *Let $\sum\limits_{j=1}^{k} p_j$ be in the range (A_4) and suppose that (H_1) holds. Then (i) (H_3) holds implies every bounded solution of (E) oscillates or tends to zero as $n \to \infty$ and (ii) every bounded solution of (E) oscillates or tends to zero as $n \to \infty$ such that (H_2) holds implies (H_3) holds.*

Proof. Let if possible $y = \{y_n\}$ be an ultimately positive bounded solution of (E) for $n \geq N_1$. Then setting z_n and w_n as in (6) and (7) we obtain (8), which implies $w_n > 0$ or $w_n < 0$ for $n \geq N_2 \geq N_1$. Then $\lim_{n\to\infty} w_n = \delta \in R$ and by (H$_1$) $\lim_{n\to\infty} z_n = \delta$. We prove $\liminf_{n\to\infty} y_n = 0$ as in Theorem 2.1. Let $\limsup_{n\to\infty} y_n = \beta$. If $\delta \geq 0$ then

$$\begin{aligned}0 \leq \delta &= \liminf_{n\to\infty} \left(y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j} \right) \\ &\leq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} \left(\sum_{j=i}^{k} p_j\, y_{n-m_j} \right) \\ &\leq \beta + \liminf_{n\to\infty}\ p_i\, y_{n-m_i} + \limsup_{n\to\infty}\ \sum_{j\neq i} p_j\, y_{n-m_j} \\ &\leq \beta + p_i \limsup_{n\to\infty}\ y_{n-m_i} + \sum_{j\neq i} \limsup_{n\to\infty}\ p_j\, y_{n-m_j} \leq (1+p_i)\,\beta.\end{aligned}$$

Hence $\beta = 0$ as $p_i < 1$, which implies $\lim_{n\to\infty} y_n = 0$. If $\delta < 0$ then we get

$$\begin{aligned}\delta &= \lim_{n\to\infty} z_n = \liminf_{n\to\infty} \left(y_n + \sum_{j=1}^{k} p_j\, y_{n-m_j} \right) \\ &\leq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} \left(\sum_{j=i}^{k} p_j\, y_{n-m_j} \right) \\ &\leq (1+p_i)\,\beta + \sum_{j\neq i} p_j\ \liminf_{n\to\infty}\ y_{n-m_j} = (1+p_i)\,\beta\end{aligned} \tag{16}$$

and

$$\begin{aligned}\delta &= \limsup_{n\to\infty} z_n \geq \liminf_{n\to\infty} y + \limsup_{n\to\infty} \sum_{j=1}^{k} p_j\, y_{n-m_j} \\ &\geq \limsup_{n\to\infty} p_i\, y_{n-m_i} + \liminf_{n\to\infty} \sum_{j\neq i} p_j\, y_{n-m_j} \geq \left(\sum_{j\neq i} p_j \right) \beta.\end{aligned} \tag{17}$$

From (16) and (17) we obtain

$$\left((1+p_i) - \sum_{j\neq i} p_i \right) \beta \geq 0\,.$$

Hence from (A_4) we conclude $\beta \le 0$. Then $\lim_{n\to\infty} y_n = 0$. Thus (i) is proved. The proof of (ii) is similar to that of Theorem 2.4 (ii), hence is omitted. ■

Remark 2. In view of Remark 1 all the results in [6] follow as particular cases of our results. It may be noted that the technique used in our results is different from that of [6]. Also the work of this paper generalize some of the results of [2], [8].

Remark 3. The condition imposed on p_i in (A_3) and (A_4) are required for both the necessary and sufficient part of the Theorem 2.4 and Theorem 2.5.

Note. One may easily find example to illustrate all other theorems as we have done for Theorem 2.1.

3. Conclusion

In this section we mainly discuss some of the questions un answered in this paper. If we assume that $\{y_n\}$ is unbounded in Theorem 2.5., then it appears very hard to prove: either all solutions of (E) oscillate or tend to $\pm\infty$. Again for the range where $\sum_{j=1}^{k} p_j = \pm 1$, we do not have any result. We dont find any such result in literature as well. It would also be interesting, if one drops (H_2) and still find a positive solution as in Theorems 2.2, 2.4, 2.5. One may easily find that our results can be extended to neutral difference equations with variable coefficients

$$\Delta\left(y_n + \sum_{j=1}^{k} p_n^j\, y_{n-m_j}\right) + \sum_{j=1}^{s} q_n^j\, G\left(y_{n-r_j}\right) = f_n$$

under the primary assumption (2), to generalize the work in [9].

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