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STRONGLY (V_σ, θ, q) – SUMMABLE SEQUENCES DEFINED BY ORLICZ FUNCTIONS

Abstract: The purpose of this paper is to introduce the space of sequences those are strongly (V_σ,θ,q)–summable with respect to an Orlicz function. We give some relations related to these sequence spaces. We also show that the spaces $[V_\sigma,\theta,M,q]_1 \cap \ell_\infty(q)$ may be represented as a $S_\theta^* \cap \ell_\infty(q)$ space.

Key words: sequence spaces, seminorm, statistical convergence, Orlicz function.

1. Introduction

The notion of statistical convergence was introduced by Fast [4] and Schoenberg [25], independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Šalát [21], Connor [2], Maddox [12], Savas and Nuray [22], Rath and Tripathy[19] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of Stone-Cech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

Orlicz [17] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [8] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \le p < \infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary [18], Nuray and Gülcü [16], Bhardwaj and Singh [1] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref. [7].

The main purpose of this paper is to introduce and study some sequence spaces by using the concept of an Orlicz function. We examine some topological properties of these spaces and establish elementary connections on these spaces. In section 2 we give a brief information about statistical convergence, invariant means, Orlicz functions and lacunary sequences. In section 3 we prove the main results of this paper. The results which we give in this paper are more general than those of Nuray and Gülcü [16], Bhardwaj and Singh [1] Savaş and Nuray [22] and Savaş [23].

2. Definitions and Preliminaries

Let l_∞ and c denote the Banach spaces of real bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_n |x_n|$, respectively.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma\left(\sigma^{m-1}(n)\right)$, $m = 1, 2, \dots$. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ-mean if and only if
i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
iii) $\varphi\left(\left\{x_{\sigma(n)}\right\}\right) = \varphi(\{x_n\})$ for all $x \in l_\infty$.

For certain kinds of mappings σ every invariant mean φ extends the limit functional on the space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. The set of all $\sigma-$convergent sequences will be denote by V_σ.

If $x = (x_n)$, set $Tx = (Tx_n) = \left(x_{\sigma(n)}\right)$. It can be shown [24] that

$$V_\sigma = \left\{x \in l_\infty : \lim_m t_{mn}(x) = \ell e \text{ uniformly in } n,\ \ell = \sigma - \lim x\right\}, \tag{1}$$

where $t_{mn}(x) = (x_n + Tx_n + \dots + T^m x_n) / (m+1)$.

The special case of (1) in which $\sigma(n) = n + 1$ was given by Lorentz [9]. Several authors including Schaefer [24], Mursaleen [14], Savaş [23] and many others have studied invariant convergent sequences.

A bounded sequence $x = (x_n)$ is said to be strongly $\sigma-$convergent to a number ℓ if and only if $(|x_n - \ell|) \in V_\sigma$ with $\sigma-$limit zero (see [13]). By $[V_\sigma]$, we denote the set of all strongly $\sigma-$convergent sequences. It is known that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \dots$, where $k_0 = 0$, we shall mean an increasing sequence of non negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by s_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman *et al* [5] as follows :

$$N_\theta = \left\{x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - \ell| = 0, \text{ for some } \ell\right\}.$$

Let $\|x\|_\theta = \sup_r \left(h_r^{-1} \sum_{i \in I_r} |x_i| \right)$, whenever $x \in N_\theta$. Then $(N_\theta, \|.\|_\theta)$ is a BK- space. There is a strong connection between N_θ and the sequence space $|\sigma_1|$,which is defined by

$$|\sigma_1| = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell| = 0, \text{ for some } \ell \right\}.$$

In the special case $\theta = (2^r)$, we have $N_\theta = |\sigma_1|$.

Later on lacunary sequences have been studied by Bhardwaj and Singh [1], Das and Patel [3], Waszak [26] and others.

The definitions of statistical convergence and strong $p-$Cesaro or w_p $(0 < p < \infty)$ summability of a sequence were introduced in the literature independently of one another and have followed different lines of development since their first appearence. It turns out, however, that the two definitions can be simply related to one another in general and are equivalent for bounded sequences. The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

A sequence $x = (x_k)$ is called statistically convergent to a number L, if for every $\varepsilon > 0$, $\delta\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0$ (see [4], [6]). In this case we write $S - \lim x_k = \ell$.

Recall ([7], [17]) that an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M can always be represented in the following integral form: $M(x) = \int\limits_0^x q(t)\, dt$, where q known as the kernel of M, is right differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is nondecreasing and $q(t) \to \infty$ as $t \to \infty$.

If the convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called modulus function, defined and discussed by Ruckle [20] and Maddox [11].

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Let (Ω, Σ, μ) be a finite measure space. We denote by $E(\mu)$ the space of all (equivalence classes of) $\Sigma-$measurable functions x from Ω into $[0, \infty)$. Given an Orlicz function M, we define on $E(\mu)$ a convex functional I_M by

$$I_M(x) = \int_\Omega M(x(t))\, d\mu,$$

and an Orlicz space $L^M(\mu)$ by $L^M(\mu) = \{x \in E(\mu) : I_M(\lambda x) < +\infty \text{ for some } \lambda > 0\}$, (For detail see [7], [17]).

The sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

and this space is called an Orlicz sequence space. For $M(t) = t^p$, $1 \leq p < \infty$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = G$, and let $D = \max\left(1, 2^{G-1}\right)$. Then for $a_k, b_k \in \mathbb{C}$, the set of complex numbers, for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}, \quad [10] \tag{2}$$

Now we can give our new definition.

Definition 1. *Let M be an Orlicz function, X be a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous seminorms q and $p = (p_k)$ be a sequence of positive real numbers. $w(X)$ denotes the space of all sequences $x = (x_k)$, where $x_k \in X$. We define the following sequence spaces:*

$$[V_\sigma, \theta, M, p, q]_1 = \left\{ \begin{array}{r} x \in w(X) : \lim\limits_{r \to \infty} \frac{1}{h_r} \sum\limits_{k \in I_r} \left[M\left(q\left(\frac{x_{\sigma^k(m)} - \ell}{\rho} \right) \right) \right]^{p_k} = 0, \\ \text{uniformly in } m, \text{ for some } \rho > 0 \text{ and } \ell > X \end{array} \right\},$$

$$[V_\sigma, \theta, M, p, q]_0 = \left\{ \begin{array}{r} x \in w(X) : \lim\limits_{r \to \infty} \frac{1}{h_r} \sum\limits_{k \in I_r} \left[M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho} \right) \right) \right]^{p_k} = 0, \\ \text{uniformly in } m, \text{ for some } \rho > 0 \end{array} \right\},$$

$$[V_\sigma, \theta, M, p, q]_\infty = \left\{ \begin{array}{r} x \in w(X) : \sup\limits_{r,m} \frac{1}{h_r} \sum\limits_{k \in I_r} \left[M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho} \right) \right) \right]^{p_k} < \infty, \\ \text{for some } \rho > 0 \end{array} \right\}.$$

Throughout the paper Z will be denote any one of the notation $0, 1$ or ∞.

In this case $\theta = (2^r)$ and $M(x) = x$, $p_k = 1$ for all $k \in \mathbb{N}$ we shall write $[V_\sigma, M, p, q]_Z$ and $[V_\sigma, \theta, q]_Z$ instead of $[V_\sigma, \theta, M, p, q]_Z$.

Lemma 1. ([5], Lemma 2.1) *In order for* $|\sigma_1| \subseteq N_\theta$ *it is necessary and sufficient that* $\liminf_r s_r > 1$.

Lemma 2. ([5], Lemma 2.2) *In order for* $N_\theta \subseteq |\sigma_1|$ *it is necessary and sufficient that* $\limsup_r s_r < \infty$.

Lemma 3. ([5], Theorem 2.1) *Let* θ *be a lacunary sequence, then* $N_\theta = |\sigma_1|$ *if and only if* $1 < \liminf_r s_r \leq \limsup_r s_r < \infty$.

3. Main Results

In this section we examine some topological properties of $[V_\sigma, \theta, M, p, q]_Z$ spaces and investigate some inclusion relations between these spaces.

Theorem 1. *Let the sequence* (p_k) *be bounded, then* $[V_\sigma, \theta, M, p, q]_Z$ *are linear spaces over the set of complex numbers.*

Proof. We shall prove the theorem only for the space $[V_\sigma, \theta, M, p, q]_0$. The others can be proved by the same way. Let $x, y \in [V_\sigma, \theta, M, p, q]_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho_1} \right) \right) \right]^{p_k} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(q\left(\frac{y_{\sigma^k(m)}}{\rho_2} \right) \right) \right]^{p_k} = 0 \quad \text{uniformly in } m.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is nondecreasing and convex, q is a seminorm by (2) we have

$$\begin{aligned} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(q\left(\frac{\alpha x_{\sigma^k(m)} + \beta y_{\sigma^k(m)}}{\rho_3} \right) \right) \right]^{p_k} &\leq \\ \leq \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(q\left(\frac{\alpha x_{\sigma^k(m)}}{\rho_3} \right) + q\left(\frac{\beta y_{\sigma^k(m)}}{\rho_3} \right) \right) \right]^{p_k} & \\ \leq D\frac{1}{h_r} \sum_{k\in I_r} \left[M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho_1} \right) \right) \right]^{p_k} & \\ + D\frac{1}{h_r} \sum_{k\in I_r} \left[M\left(q\left(\frac{y_{\sigma^k(m)}}{\rho_2} \right) \right) \right]^{p_k} \to 0 & \end{aligned}$$

as $r \to \infty$ uniformly in m. This proves that $[V_\sigma, \theta, M, p, q]_0$ is linear. ■

Theorem 2. *The spaces $[V_\sigma, \theta, M, p, q]_Z$ are paranormed space (not necessarily totally paranormed), paranormed by*

$$g(x) = \inf\left\{\rho^{p_n/H} : \sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho}\right)\right) \leq 1,\ \rho > 0,\ \textit{uniformly in } m\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. Consider the space $[V_\sigma, \theta, M, p, q]_\infty$. Clearly $g(x) = g(-x)$ and $g(\bar{\theta}) = 0$, where $\bar{\theta}$ is the zero sequence of X. Let $(x_k), (y_k) \in [V_\sigma, \theta, M, p, q]_\infty$. Then there exist ρ_1, ρ_2 such that

$$\sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho_1}\right)\right) \leq 1, \quad \text{uniformly in } m$$

and

$$\sup_{k\geq 1} M\left(q\left(\frac{y_{\sigma^k(m)}}{\rho_2}\right)\right) \leq 1, \quad \text{uniformly in } m.$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\begin{aligned}\sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)} + y_{\sigma^k(m)}}{\rho}\right)\right) &\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho_1}\right)\right) \\ &+ \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{k\geq 1} M\left(q\left(\frac{y_{\sigma^k(m)}}{\rho_2}\right)\right) \leq 1, \quad \text{uniformly in } m.\end{aligned}$$

Hence

$$\begin{aligned} g(x+y) &= \inf\left\{\begin{array}{r} (\rho_1+\rho_2)^{p_n/H} : \sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)}+y_{\sigma^k(m)}}{\rho}\right)\right) \leq 1, \\ \rho > 0, \quad \text{uniformlyin } m \end{array}\right\} \\ &\leq \inf\left\{\begin{array}{r} (\rho_1)^{p_n/H} : \sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho_1}\right)\right) \leq 1, \\ \rho_1 > 0, \quad \text{uniformly in } m \end{array}\right\} \\ &\quad + \inf\left\{\begin{array}{r} (\rho_2)^{p_n/H} : \sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho_2}\right)\right) \leq 1, \quad \rho_2 > 0, \\ \text{uniformly in } m \end{array}\right\} \\ &= g(x) + g(y). \end{aligned}$$

Hence g satisfies the inequality.

The continuity of product follows from the following equality:

$$\begin{aligned} g(\lambda x) &= \inf\left\{\begin{array}{r} \rho^{p_n/H} : \sup_{k\geq 1} M\left(q\left(\frac{\lambda x_{\sigma^k(m)}}{\rho}\right)\right) \leq 1, \\ \rho > 0, \quad \text{uniformly in } m \end{array}\right\} \\ &= \inf\left\{\begin{array}{r} (|\lambda|\, t)^{p_n/H} : \sup_{k\geq 1} M\left(q\left(\frac{x_{\sigma^k(m)}}{t}\right)\right) \leq 1, \\ t > 0, \quad \text{uniformly in } m \end{array}\right\}, \end{aligned}$$

where $t=\rho/|\lambda|$.

The proof of the following result is easy and thus omitted. ■

Theorem 3. *Let M_1, M_2 be Orlicz function. Then we have* $[V_\sigma,\theta,M_1,p,q]_Z\cap[V_\sigma,\theta,M_2,p,q]_Z\subseteq[V_\sigma,\theta,M_1+M_2,p,q]_Z$.

Propertion 1. *For any two sequences $p=(p_k)$ and $t=(t_k)$ of positive real numbers and for any two seminorms q_1 and q_2 on X, we have* $[V_\sigma,\theta,M,p,q_1]_Z\cap[V_\sigma,\theta,M,p,q_2]_Z\neq\emptyset$.

Proof. Since the zero element belongs to each of the above classes of sequences, thus the intersection is nonempty. ■

Propertion 2. *Let M be an Orlicz function and q_1 and q_2 be two seminorms on X. Then*

i) *If q_1 is stronger than q_2, then* $[V_\sigma,\theta,M,p,q_1]_Z\subset[V_\sigma,\theta,M,p,q_2]_Z$,

ii) $[V_\sigma,\theta,M_1,p,q_1]_Z\cap[V_\sigma,\theta,M_1,p,q_2]_Z\subset[V_\sigma,\theta,M_1,p,q_1+q_2]_Z$.

Proof. Omitted. ■

Theorem 4. *Let M be an Orlicz function. Then* $[V_\sigma,\theta,M,p,q]_0\subset[V_\sigma,\theta,M,p,q]_1\subset[V_\sigma,\theta,M,p,q]_\infty$.

Proof. The inclusion $[V_\sigma,\theta,M,p,q]_0\subset[V_\sigma,\theta,M,p,q]_1$ is obvious. Now let $x\in[V_\sigma,\theta,M,p,q]_1$. Then we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho}\right)\right)\right]^{p_k}&\leq\\ \leq\frac{D}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{x_{\sigma^k(m)}-\ell}{\rho}\right)\right)\right]^{p_k}&+\frac{D}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{\ell}{\rho}\right)\right)\right]^{p_k}\\ \leq\frac{D}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{x_{\sigma^k(m)}-\ell}{\rho}\right)\right)\right]^{p_k}&+D\max\left\{1,\left[M\left(q\left(\frac{\ell}{\rho}\right)\right)\right]^{G}\right\}\end{aligned}$$

Thus $x\in[V_\sigma,\theta,M,p,q]_\infty$. ■

Taking $y_k=\left[M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho}\right)\right)\right]^{p_k}$ for all $k\in\mathbb{N}$, we have the following results those follow from the Lemmas listed in section 1.

Propertion 3. *Let $\theta=(k_r)$ be a lacunary sequence with* $\liminf_r s_r>1$. *Then for any Orlicz function M,* $[V_\sigma,M,p,q]_Z\subseteq[V_\sigma,\theta,M,p,q]_Z$.

Propertion 4. *Let $\theta=(k_r)$ be a lacunary sequence with* $\limsup_r s_r<\infty$. *Then for any Orlicz function M,* $[V_\sigma,\theta,M,p,q]_Z\subseteq[V_\sigma,M,p,q]_Z$.

The next result follows from Proposition 3 and Proposition 4

Theorem 5. *Let* $\theta = (k_r)$ *be a lacunary sequence with* $1 < \liminf_r s_r \leq \limsup_r s_r < \infty$. *Then for any Orlicz function* M, $[V_\sigma, M, p, q]_Z = [V_\sigma, \theta, M, p, q]_Z$.

Theorem 6. *Let* $0 \leq p_k \leq t_k$ *and* $\left(\frac{t_k}{p_k}\right)$ *be bounded. Then* $[V_\sigma, \theta, M, t, q]_Z \subset [V_\sigma, \theta, M, p, q]_Z$.

Proof. If we take $w_{km} = \left[M\left(q\left(\frac{x_{\sigma^k(m)}}{\rho}\right)\right)\right]^{p_k}$ for all k and m, then using the same technique of Theorem 2 of Nanda [15]. ■

4. S_θ^*-Statistical Convergence

In this section we introduce the concept of S_θ^*- statistical convergence and give some inclusion relations related to this concept. We also show that the spaces $[V_\sigma, \theta, M, q]_1 \cap \ell_\infty(q)$ may be represented as a $S_\theta^* \cap \ell_\infty(q)$ space.

Definition 2. *A sequence* $x = (x_k)$ *is said to be* S_θ^**–statistically convergent to* $\ell \in X$ *if for all* $q \in Q$ *and any* $\varepsilon > 0$ *such that*

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in I_r : q\left(x_{\sigma^k(m)} - \ell\right) \geq \varepsilon\right\}\right| = 0, \quad \text{uniformly in } m = 1, 2, \ldots .$$

In this case we write $S_\theta^* - \lim x = \ell$ *or* $x_k \to \ell\,(S_\theta^*)$ *and we define*

$$S_\theta^* = \{x = (x_k) : S_\theta^* - \lim x = \ell, \text{ for some } \ell\}.$$

Theorem 7. *Let* $\theta = (k_r)$ *be a lacunary sequence, then*

i) $x_k \to \ell\,[V_\sigma, \theta, q]_1$ *implies* $x_k \to \ell\,(S_\theta^*)$,

ii) If $x \in l_\infty(q)$ *and* $x_k \to \ell\,(S_\theta^*)$ *imply then* $x_k \to \ell\,[V_\sigma, \theta, q]_1$,

where $l_\infty(q)$ *denotes the set of* $q-$*bounded sequences, that is* $\ell_\infty(q) = \{x \in w(X) : \sup_k q(x) < \infty\}$.

Proof. i) Let $\varepsilon > 0$ and $x_k \to \ell\,[V_\sigma, \theta, q]$. We can write

$$\begin{aligned}\sum_{k\in I_r} q\left(x_{\sigma^k(m)} - \ell\right) &\geq \sum_{\substack{k\in I_r \\ q\left(x_{\sigma^k(m)}-\ell\right)\geq\varepsilon}} q\left(x_{\sigma^k(m)} - \ell\right) \\ &\geq \varepsilon\left|\left\{k \in I_r : q\left(x_{\sigma^k(m)} - \ell\right) \geq \varepsilon\right\}\right|.\end{aligned}$$

Hence $x_k \to \ell\,(S_\theta^*)$.

ii) Suppose that $x_k \to \ell\,(S_\theta^*)$ and $x \in l_\infty(q)$, say $q\left(x_{\sigma^k(m)} - \ell\right) \leq M$ for all k and m. Given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} q\left(x_{\sigma^k(m)}-\ell\right) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q\left(x_{\sigma^k(m)}-\ell\right)\geq\varepsilon}} q\left(x_{\sigma^k(m)}-\ell\right)\\ &\quad+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ q\left(x_{\sigma^k(m)}-\ell\right)<\varepsilon}} q\left(x_{\sigma^k(m)}-\ell\right)\\ &\leq \frac{M}{h_r}\left|\left\{k\in I_r : q\left(x_{\sigma^k(m)}-\ell\right)\geq\varepsilon\right\}\right|+\varepsilon\end{aligned}$$

which implies that $x_k\to\ell\,[V_\sigma,\theta,q]_1$.

In (ii), $q-$ boundedness condition cannot be omitted. For this consider the following example. ∎

Example. Let $q(x)=|x|$, and θ be given. We define x_k to be $1,2,...,[\sqrt{h_r}]$ for $k=\sigma^n(m)$, $n=k_{r-1}+1,k_{r-1}+2,...,k_{r-1}+\left[\sqrt{h_r}\right]$; $m\geq 1$, and $x_k=0$ otherwise (where [ ] denotes the greatest integer function). Note that x is not $q-$ bounded, $x\to 0\,(S_\theta^*)$ and $x\notin[V_\sigma,\theta,p,q]$.

Theorem 8. *Let M be an Orlicz function. Then $[V_\sigma,\theta,M,p,q]_1\subset S_\theta^*$.*

Proof. Let $x\in[V_\sigma,\theta,M,p,q]_1$. Then there exists a number $\rho>0$ such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{x_{\sigma^k(m)}-\ell}{\rho}\right)\right)\right]^{p_k}\to 0,\text{ uniformly in } m.$$

Then

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{x_{\sigma^k(m)}-\ell}{\rho}\right)\right)\right]^{p_k} &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q\left(x_{\sigma^k(m)}-\ell\right)\geq\varepsilon}}\left[M\left(q\left(\frac{x_{\sigma^k(m)}-\ell}{\rho}\right)\right)\right]^{p_k}\\ &\quad+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ q\left(x_{\sigma^k(m)}-\ell\right)<\varepsilon}}\left[M\left(q\left(\frac{x_{\sigma^k(m)}-\ell}{\rho}\right)\right)\right]^{p_k}\\ &\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q\left(x_{\sigma^k(m)}-\ell\right)\geq\varepsilon}}\left[M\left(q\left(\frac{x_{\sigma^k(m)}-\ell}{\rho}\right)\right)\right]^{p_k}\\ &\geq \frac{1}{h_r}\left|\left\{k\in I_r : q\left(x_{\sigma^k(m)}-\ell\right)\geq\varepsilon\right\}\right|\min\left\{[M(\varepsilon)]^{\inf p_k},[M(\varepsilon)]^{G}\right\}.\end{aligned}$$

Hence $x\in S_\theta^*$. ∎

Theorem 9. $S_\theta^* \cap \ell_\infty(q) = [V_\sigma, \theta, M, q]_1 \cap \ell_\infty(q)$.

Proof. By Theorem 8, we need only show that $S_\theta^* \cap \ell_\infty(q) \subset [V_\sigma, \theta, M, q]_1 \cap \ell_\infty(q)$. For each $m \geq 1$, let $\sigma_{km} = x_{\sigma^k(m)} - \ell \to 0\,(S_\theta)$. Since $x \in \ell_\infty(q)$, there exists $K > 0$ such that

$$M\left[q\left(\frac{\sigma_{km}}{\rho}\right)\right] \leq K.$$

Then for a given $\varepsilon > 0$ and for each $m \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{\sigma_{km}}{\rho}\right)\right)\right] &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ q\left(x_{\sigma^k(m)}-\ell\right)\geq\varepsilon}}\left[M\left(q\left(\frac{\sigma_{km}}{\rho}\right)\right)\right]\\ &\quad+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ q\left(x_{\sigma^k(m)}-\ell\right)<\varepsilon}}\left[M\left(q\left(\frac{\sigma_{km}}{\rho}\right)\right)\right]\\ &\leq \frac{K}{h_r}\left|\left\{k\in I_r : q(\sigma_{km})\geq\varepsilon\right\}\right| + M\left(\frac{\varepsilon}{\rho}\right).\end{aligned}$$

Hence $x \in [V_\sigma, \theta, M, q]_1 \cap \ell_\infty(q)$. ■

5. Conclusion

Giving particular values to M, θ, X and q we obtain some sequence spaces which are the special cases of the sequence spaces that we have defined, for example

i) If $X = \mathbb{C}$, $M(x) = x$, $\theta = (2^r)$ and $q(x) = |x|$, then $[V_\sigma, \theta, M, p, q]_Z = [V_\sigma]^Z_{(p_k)}$, (see [23]).

ii) If $X = \mathbb{C}$, $q(x) = |x|$ and $\theta = (2^r)$, then $[V_\sigma, \theta, M, p, q]_Z = [V_\sigma, M]^Z_{(p_k)}$, (see [16]).

The most of the results proved in the previous sections will be true for these spaces.

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