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ON CLOSED INVERSE IMAGES OF MESOCOMPACT SPACES

Abstract: In this paper, we prove that mesocompactness is inversely preserved under paracompact mappings with regular images. As an application of this result, we prove that mesocompactness is inversely preserved under closed Lindelöf mappings with regular domains and images, which answers a question on mesocompactness posed by S. Lin. We also give a counterexample to show that the regularity of domains in this application can not be omitted.

Key words: mesocompact space, paracompact mapping, closed Lindelöf mapping, regular.

1. Introduction and Basic Notions

Mesocompactness, was introduced by J.R.Boone [2], is one of the important covering properties in General Topology, which lies between paracompactness and metacompactness. V.J. Mancuso proved that mesocompactness is inversely preserved under perfect mappings ([8]). Note that paracompactness and metacompactness are inversely preserved under closed Lindelöf mappings with regular domains ([4]). Naturally, we are interested in that whether the analogous result on mesocompactness is true. So S.Lin raised following question in a surveys on "spaces and mappings" ([7]).

Question 1. ([7]). (1) *If domains and images are regular, is mesocompactness inversely preserved under closed Lindelöf mappings?*

(2) *Furthermore, can the regularity in the above* (1) *be omitted?*

In this paper, we investigate the above Question 1. We prove that mesocompactness is inversely preserved under paracompact mappings with regular images. As an application of this result, we prove that mesocompactness is inversely preserved under closed Lindelöf mappings with regular domains and images. In addition, we give a counterexample to show that the regularity of domains in this application can not be omitted, but we do not know whether the regularity of images can be omitted.

Throughout this paper, all spaces are assumed to be Hausdorff and all mappings are assumed to be continuous and onto. N denotes the set of all natural numbers. Let $\mathcal{U}$ and $\mathcal{V}$ be two families of (open) subsets of a space X. We say that $\mathcal{V}$ is a partial (open) refinement of $\mathcal{U}$, if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$; moreover, we say that $\mathcal{V}$ is a (open) refinement of $\mathcal{U}$, if in addition $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ is also satisfied. Let $\mathcal{U}$ be a family of subsets of a space X. $\bigcup \mathcal{U}$ and $\bigcap \mathcal{U}$ denote the union $\bigcup\{U : U \in \mathcal{U}\}$ and the intersection $\bigcap\{U : U \in \mathcal{U}\}$ respectively. Let $A \subset X$. $\mathcal{U} \bigwedge A$ and $(\mathcal{U})_A$ denote the family $\{U \bigcap A : U \in \mathcal{U}\}$ and the family $\{U \in \mathcal{U} : U \bigcap A \neq \emptyset\}$ respectively. For $x \in X$, $(\mathcal{U})_{\{x\}}$ is replaced by $(\mathcal{U})_x$ and the cardinality of $(\mathcal{U})_x$ is denoted by $ord(x,\mathcal{U})$. Let $f : X \longrightarrow Y$ be a mapping, and let $\mathcal{U}$ and $\mathcal{V}$ are two families of subsets of X and Y respectively, then $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$. One may refer to [5] for undefined notations and terminology.

2. The Main Results

Definition 1. ([8]). *A family $\mathcal{U}$ of subset of a space X is called compact finite if for every compact subset $K \subset X$, $(\mathcal{U})_K$ is finite. A space X is called mesocompact if every open cover of X has a compact finite open refinement.*

Definition 2. ([9]). *Let $\mathcal{U}$ be a family of subsets of a space X and let X_0 be a subset of X.*

(1) *$\mathcal{U}$ is called locally finite at X_0 if for each $x \in X_0$, there is an open neighborhood U of x such that $(\mathcal{U})_U$ if finite.*

(2) *$\mathcal{U}$ is called compact finite at X_0 if $(\mathcal{U})_K$ is finite for every compact subset $K \subset X_0$.*

Definition 3. ([3]). *A mapping $f : X \longrightarrow Y$ is called paracompact, if for every $y \in Y$ and every family $\mathcal{U}$ of open subsets of X satisfying $f^{-1}(y) \subset \bigcup \mathcal{U}$, there exists a neighborhood V_y of y such that $f^{-1}(V_y)$ is covered by $\mathcal{U}$ and $\mathcal{U} \wedge f^{-1}(V_y)$ has a open refinement $\mathcal{V}_y$ such that $\mathcal{V}_y$ is locally finite at $f^{-1}(y)$.*

Remark 1. ([3]). (1) Every paracompact mapping is closed.

(2) A mapping $f : X \longrightarrow Y$ is paracompact if and only if for every $y \in Y$ and every family $\mathcal{U}$ of open subsets of X satisfying $f^{-1}(y) \subset \bigcup \mathcal{U}$, there exists an open neighborhood V_y of y and a partial open refinement $\mathcal{V}_y$ of $\mathcal{U}$ such that $f^{-1}(V_y) \subset \bigcup \mathcal{V}_y$ and $\mathcal{V}_y$ is locally finite at $f^{-1}(V_y)$.

Definition 4. *A closed mapping $f : X \longrightarrow Y$ is called perfect (closed Lindelöf), if $f^{-1}(y)$ is a compact subset (Lindelöf subset) of X for every $y \in Y$.*

Lemma 1. ([5]). *A mapping $f : X \longrightarrow Y$ is closed if and only if for every $y \in Y$ and every open subset U in X which contains $f^{-1}(y)$, there exists an open neighborhood V of y such that $f^{-1}(V) \subset U$.*

Lemma 2. *Let $f : X \longrightarrow Y$ be a paracompact mapping. Then for every $y \in Y$ and every family $\mathcal{U}$ of open subsets of X satisfying $f^{-1}(y) \subset \bigcup \mathcal{U}$, there exists an open neighborhood V_y of y and a partial open refinement $\mathcal{V}_y$ of $\mathcal{U}$ such that $f^{-1}(V) \subset \bigcup \mathcal{V}_y$ and $\mathcal{V}_y$ is compact finite at $f^{-1}(V_y)$.*

Proof. Let $y \in Y$ and $\mathcal{U}$ be a family of open subsets of X such that $f^{-1}(y) \subset \bigcup \mathcal{U}$. Since f is paracompact, by Remark 1(2), there exists an open neighborhood V_y of y and a partial open refinement $\mathcal{V}_y$ of $\mathcal{U}$ such that $f^{-1}(V) \subset \bigcup \mathcal{V}_y$ and $\mathcal{V}_y$ is locally finite at $f^{-1}(V_y)$. Let $K \subset f^{-1}(V_y)$ be a compact subset. For every $x \in K$, there exists an open neighborhood U_x of x such that $(\mathcal{V}_y)_{U_x}$ is finite. Note that $\{U_x : x \in K\}$ covers K. There exists a finite subset K' of K such that $\{U_x : x \in K'\}$ covers K. Put $U_K = \bigcup\{U_x : x \in K'\}$, then $(\mathcal{V}_y)_{U_K}$ is finite, so $(\mathcal{V}_y)_K$ is finite. Consequently, $\mathcal{V}_y$ is compact finite at $f^{-1}(V_y)$. ■

Theorem 1. *Let $f : X \longrightarrow Y$ be a paracompact mapping with a regular image. If Y is mesocompact, then X is mesocompact.*

Proof. Suppose Y is a mesocompact space. Let $\mathcal{U}$ be an open cover of X. By Lemma 2, for every $y \in Y$, there exists an open neighborhood V_y of y and a partial open refinement $\mathcal{V}_y$ of $\mathcal{U}$ such that $f^{-1}(V_y) \subset \bigcup \mathcal{V}_y$ and $\mathcal{V}_y$ is compact finite at $f^{-1}(V_y)$. By the regularity of Y, there exists an open neighborhood O_y of y such that $\overline{O_y} \subset V_y$. Since Y is mesocompact, the open cover $\{O_y : y \in Y\}$ of Y has a compact finite open refinement $\mathcal{W}$. Without loss of generality, we may assume $\mathcal{W} = \{W_y : y \in Y\}$, where $W_y \subset O_y$ for every $y \in Y$. Put $\mathcal{F}_y = \mathcal{V}_y \bigwedge f^{-1}(W_y)$ for every $y \in Y$ and put $\mathcal{F} = \bigcup\{\mathcal{F}_y : y \in Y\}$. It is obvious that $\mathcal{F}$ is an open refinement of $\mathcal{U}$. It suffices to prove that $\mathcal{F}$ is compact finite.

Let K be a compact subset of X. Note that $\mathcal{W}$ is compact finite in Y. It is easy to see that $f^{-1}(\mathcal{W})$ is compact finite in X. Thus there exists a finite $Y_0 \subset Y$ such that for every $y \in Y - Y_0$, K misses all elements of $\mathcal{F}_y$. Now we only need to prove that $(\mathcal{F}_y)_K$ is finite for every $y \in Y_0$.

Let $y \in Y_0$. Then $(\mathcal{F}_y)_K = \{V \bigcap f^{-1}(W_y) : (V \bigcap f^{-1}(W_y)) \bigcap K \neq \emptyset,\ V \in \mathcal{V}_y\} = \{V \bigcap f^{-1}(W_y) : V \bigcap (f^{-1}(W_y) \bigcap K) \neq \emptyset,\ V \in \mathcal{V}_y\} \subset \{V \bigcap f^{-1}(W_y) : V \bigcap (\overline{f^{-1}(W_y)} \bigcap K) \neq \emptyset,\ V \in \mathcal{V}_y\}$. Since $\overline{f^{-1}(W_y)} \bigcap K \subset \overline{f^{-1}(W_y)} \subset f^{-1}(\overline{W_y}) \subset f^{-1}(\overline{O_y}) \subset f^{-1}(V_y)$, $\overline{f^{-1}(W_y)} \bigcap K$ is a compact subset of $f^{-1}(V_y)$. Note that $\mathcal{V}_y$ is compact finite at $f^{-1}(V_y)$. $\{V \bigcap f^{-1}(W_y) : V \bigcap (\overline{f^{-1}(W_y)} \bigcap K) \neq \emptyset,\ V \in \mathcal{V}_y\}$ is finite. This proves that $(\mathcal{F}_y)_K$ is finite. ■

Lemma 3. *If $f : X \longrightarrow Y$ is a closed Lindelöf mapping with a regular domain, then f is paracompact.*

Proof. Let $y \in Y$ and let $\mathcal{U}$ be a family of open subsets of X which covers $f^{-1}(y)$. Since f is Lindelöf, there exists a countable subfamily $\mathcal{U}' = \{U_n : n \in N\}$ of $\mathcal{U}$ which covers $f^{-1}(y)$. By the regularity of X, there exists a family $\mathcal{W} = \{W_n : n \in N\}$ of open subsets of X such that $f^{-1}(y) \subset \bigcup \mathcal{W}$ and $\overline{W}_n \subset U_n$ for every $n \in N$.

Put $V_1 = U_1 \bigcap (\bigcup \mathcal{W})$ and $V_n = (U_n - \bigcup\{\overline{W_i} : i < n\}) \bigcap (\bigcup \mathcal{W})$ for every $n \geq 2$. Put $\mathcal{V}_y = \{V_n : n \in N\}$. It is clear that $\mathcal{V}_y$ is a partial open refinement of $\mathcal{U}$ and $\bigcup \mathcal{V}_y \subset \bigcup \mathcal{W}$.

Claim 1. $f^{-1}(y) \subset \bigcup \mathcal{V}_y$.

Let $x \in f^{-1}(y)$. Put $n = min\{i \in N : x \in \overline{W_i}\}$, then $x \in V_n \in \mathcal{V}$. So $f^{-1}(y) \subset \bigcup \mathcal{V}_y$.

By Lemma 1, there is an open neighborhood V_y of y such that $f^{-1}(V_y) \subset \bigcup \mathcal{V}_y$.

Claim 2. $\mathcal{V}_y$ is locally finite at $f^{-1}(V_y)$.

Let $x \in f^{-1}(V_y)$. Since $f^{-1}(V_y) \subset \bigcup \mathcal{V}_y \subset \bigcup \mathcal{W}$, there is $i \in N$ such that $x \in W_i$, thus W_i is an open neighborhood of x which misses V_n for every $n > i$, that is, $(\mathcal{U})_{W_i}$ is finite. So $\mathcal{V}_y$ is locally finite at $f^{-1}(V_y)$.

By Remark 1(2), f is paracompact. ■

Now we give the main theorem in this paper, which is obtained from Theorem 1 and Lemma 3.

Theorem 2. *Let f be a closed Lindelöf mapping from a regular space X onto a regular space Y. If Y is mesocompact, then X is mesocompact.*

3. The Counterexample

Now we give an example to show that the regularity of the domain in Theorem 2 can not be omitted. Recall a space X is said to be (countable) θ-refinable ([6]), if for every (countable) open cover of X, there is a sequence $\{\mathcal{U}_n : n \in N\}$ of open refinements such that for every $x \in X$, there is some $n \in N$ with $ord(x,\mathcal{U}_n) < \infty$; is said to be strongly paracompact ([4]) if every open cover of X has a star-finite open refinement. Note that strong paracompactness $\Longrightarrow$ mesocompactness $\Longrightarrow$ θ-refinability. It suffices to give an example to show that the closed Lindelöf inverse image of a normal strongly paracompact space even need not be θ-refinability.

Construction. *Let X, Q and I be the set of all real numbers, the set of all rational numbers and the set of all irrational numbers respectively. Define a base $\mathcal{B}$ of X as follows.*

$\mathcal{B} = \{\{x\} : x \in I\} \bigcup \{G(x,n) : x \in Q, n \in N\}$, *here* $G(x,n) = \{y \in I : -1/n < y - x < 1/n\} \bigcup \{x\}$.

That is, X is a Bennett and Lutzer's space([1]). Define an equivalence relation R on X as follows: xRy if and only if either $x, y \in Q$ or $x = y$. Put Y is the quotient space X/R and put $f : X \longrightarrow Y$ is a natural mapping, that is, $f(x) = [x]_R$ for every $x \in X$.

Claim 3. f is a closed *Lindelöf* mapping.

Proof. It is clear. ■

Claim 4. Y is Hausdorff, strongly paracompact, hence it is normal.

Proof. It is clear that Y is Hausdorff. Let $\mathcal{U}$ be any open cover of Y. Pick $x_0 \in Q$ and put $y_0 = f(x_0)$. Pick $U \in \mathcal{U}$ such that $y_0 \in U$. Then $\{U\} \bigcup \{\{y\} : y \in Y - U\}$ is a discrete (hence star-finite) open refinement of $\mathcal{U}$. Thus Y is strongly paracompact. ■

Claim 5.([1]). X is Hausdorff, but X is neither regular nor θ-refinable.

Remark 2. In fact, X is not countably θ-refinable.

Proof. Assume X is countably θ-refinable. Let $\mathcal{U}$ be an open cover of X. Then there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ which cover Q. Put $W = \bigcup \mathcal{V}$. Then W is both open and closed in X, and $\mathcal{V}$ is a countable open cover of W. Notice that countable θ-refinability is hereditary to closed subspace, W is countably θ-refinable, so there exists a sequence of open refinements $\{\mathcal{V}_n : n \in N\}$ of $\mathcal{V}$ such that for every $x \in W$ there exists $n \in N$ such that $ord(x, \mathcal{V}_n) < \infty$. Put $\mathcal{U}_n = \mathcal{V}_n \bigcup \{\{x\} : x \in X - W\}$ for every $n \in N$. Then $\{\mathcal{U}_n\}$ is a sequence of open refinements of $\mathcal{U}$. For every $x \in X$, if $x \in W$, there exists $n \in N$ such that $ord(x, \mathcal{V}_n) < \infty$, hence $ord(x, \mathcal{U}_n) = ord(x, \mathcal{V}_n) < \infty$; if $x \in X - W$, then $ord(x, \mathcal{U}_n) = 1 < \infty$ for every $n \in N$. Thus X is θ-refinable. This is a contradiction, as X is not θ-refinable from Claim 5. ■

Remark 3. By the above, all covering properties which are between strong paracompactness and countable θ-refinability are not inversely preserved under closed *Lindelöf* mappings if without requiring the regularity of domains involved.

Unfortunately, we do not know whether the regularity of the Y in Theorem 2 can be omitted. So we raise the following question.

Question 2. *Is mesocompactness inversely preserved under closed Lindelöf mappings with regular domains?*

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References

[1] Bennett H.R., Lutzer D.J., A note on weak θ-refinability, *Gen. Top. Appl.,* 2(1972), 49–54.

[2] Boone J.R., Some characterizations of paracompactness in k-spaces, *Fund. Math.,* 72(1971), 145–155.

[3] Buhagiar D., Miwa T., Covering properties on maps, *Questions Answers in General Topology,* 16(1998), 53–66.

[4] Burke D.K., Covering properties, *Handbook of Set-Theoretic Topology,* (K. Kunen and J. Vanghan, eds.), North-Holland, Amsterdam 1984, 347–422.

[5] Engelking R., General Topology, *Sigma Series in Pure Mathematics 6, (Heldermann, Berlin, revised ed.),* 1989.

[6] Junilla H.J.K., Three covering properties, Surveys in General Topology, *Academic Press,* 1980, 195–245.

[7] Lin S., On spaces and mappings, *J. of Suzhou Univ.,* 5(1989), 313–326.

[8] Mancuso V.J., Mesocompactness and related properties, *Pacific J. Math.,* 33(1970), 345–355.

[9] Qu Z., Yasui Y., Relatively subparacompact spaces, *Scientiae Mathematicae Japonicae,* 54(2001), 281–287.

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