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# A NOTE ON THE CORE TOPOLOGY AND THREE OTHER ONES

Abstract: In the paper there are considered the topologies defined in real linear spaces: the core topology, the topology generated by the family of directionally continuous functions, and the topology defined by Klee in [7]. The notion of the last one is extended to infinite dimensional case by means of the finite topology investigated by Kakutani and Klee [5]. Some properties of the finite topology are proved. The main result says that every one of considered topologies contain essentially the next one in the order listed above.

Key words: topology in real linear spaces.

## 1. Introduction and Basic Notions

In the paper we deal with some topological spaces defined in real linear spaces. We establish the inclusions between them. The strongest of them is the core topology [6, p.446]. It is stronger than topology generated by the family of all directionally continuous functions. The weakest of them is the topology defined by Klee [7, p.27-28] in finite dimensional case. Taking use of the concept of the finite topology [5, p.55-56] we extend it for infinite dimensional spaces. To do it we have to state some properties of the finite topology. Before we give definitions of all these topologies, we establish the notation and terminology. We base on the monography [1], the page references are to its Polish edition.

The sets of natural, real and nonnegative numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The letter $X$ always stands for a real linear space, and its zero element is written as 0. The closed line segment between points $a, b \in X$ is designated as

$$\langle a, b\rangle = \{\lambda a + (1-\lambda)b : 0 \leqslant \lambda \leqslant 1\}\,,$$

analogical denotations are used for open and semiopen intervals, e.g. $\langle a, b) = \langle a, b\rangle \setminus \{b\}$. For any sets $S \subset \mathbb{R}$ and $A, B \subset X$ and for any $s \in S$ and $x \in X$ we write

$$SA \,=\, \{sa : s \in S, a \in A\}\,,\;\; sA \,=\, \{s\}A,$$

$$A + B = \{a + b : a \in A, b \in B\}, \quad x + B = \{x\} + B.$$

If $A$ and $B$ are sets contained in linear subspaces $L$ and $M$, respectively, and $L \cap M = \{0\}$, then $A + B$ is called, as usually, a direct sum and denoted by $A \oplus B$.

We write $\sum_{t \in T} a_t$ when almost all summing elements $a_t$ are equal to 0.

The linear space spanned by the set $A \subset X$ is defined as the set

$$\mathrm{Lin}(A) = \left\{ \sum_k \alpha_k u_k : \alpha_k \in \mathbb{R},\ u_k \in A \right\}.$$

The restriction of the function $f$ to the set $A$ contained in the domain of $f$ is denoted by $f|A$, and $f^{-1}(B) = \{a : f(a) \in B\}$ is the inverse-image of the function $f$ assuming values in the set $B$.

$\sup\{f_k : k = 1, 2, \ldots, n\}$ is the function $\varphi$ defined by the formula

$$\varphi(x) = \sup\{f_k(x) : k = 1, 2, \ldots, n\}.$$

The cardinal of $\mathbb{N}$ is denoted by $\aleph_0$.

The strongest topology in $X$ such that in every finite dimensional subspace $L$ of $X$ it induces the Euclidean topology, is called a **finite topology** (see [5, p.55]) and is denoted by $\tau(X)$. We refer to it as to a **natural**, or **usual**, **topology** in $X$.

$\mathrm{Int}\, A, \mathrm{Cl}\, A, \mathrm{Fr}\, A$ denote the interior, the closure and the boundary set of the set $A \subset X$ (in $X$, in the natural topology in $X$), respectively. Sometimes, to make our consideration more legible, we write $\mathrm{Int}_X A, \mathrm{Cl}_X A, \mathrm{Fr}_X A$.

Given a subset $L$ of $X$, the **topology induced** in $L$ by the topology $\eta$ in $X$ is denoted by $\eta|L$. Sometimes it is called a **subspace topology**. The interior, the closure and the boundary set of the set $A$ in the topology induced in $L$ are denoted by $\mathrm{Int}_L A, \mathrm{Cl}_L A, \mathrm{Fr}_L A$, respectively.

Let $A$ be any subset of $X$. The **core** of $A$ with respect to $X$, denoted by $\mathrm{Cor}_X A$, is defined to be the subset of $A$ such that $a \in \mathrm{Cor}_X A$ if and only if for every $x \in X \setminus \{a\}$ there exists an element $y$ in the segment $(a, x)$ such that $\langle a, y \rangle \subset A$. We will write $\mathrm{Cor}\, A$ instead of $\mathrm{Cor}_X A$ if it is clear which space $X$ is considered. Following [4] we call a set $A$ a **core set** if $A = \mathrm{Cor}\, A$. In [9] and [8] it is called an algebraically open set. Obviously, the empty set $\emptyset$ is core set. Examples of sets, for which

$$\mathrm{Cor}\,\mathrm{Cor} A \neq \mathrm{Cor} A$$

are given in [4] and [9].

It is obvious that the family of all core sets is a topology. This topology is called a **core topology**, and it is denoted by $\tau_1(X)$, or $\tau_1$ if it is clear in which space $X$ it is considered. It was first defined by V.L.Klee in

1951 [6]. The core topology is an initial point to define such topologies as approximate-core topology, core-almost everywhere topology, Hashimoto-core topology etc, see e.g. [2], [11], [12], [3].

The investigation of core topologies has been continued by Klee in [7], where he also dealt with an other topology (however only in finite dimensional spaces) which in this paper is to be denoted by $\tau_3(X)$, or $\tau_3$. To define this topology let us introduce the notion of a Klee pair. A pair $(U, F)$ of subsets $U, F \subset X$ is called a **Klee pair** for a point $x \in X$ if $U$ is open in $\tau(X), F \subset U, \{x\} \cup F$ is closed and $x \in \mathrm{Cor}(\{x\} \cup F)$. Now, the **Klee topology** in $X$ is called the topology, the base of which is the family of all open sets in $\tau(X)$ and all sets of the form $\{x\} \cup U$, where $x \in X$, and $U$ is open in $\tau(X)$ and has a subset $F$ of $X$ such that $(U, F)$ is the Klee pair for $x$.

Another topology we deal with in our paper with is to be denoted by $\tau_2(X)$, or $\tau_2$, and called a **directional topology**. By definition, it is the topology generated in $X$ by the family of all directionally continuous functions on $X$. Let's recall that a real-valued function $f$ on $X$ is called **directionally continuous** if for each line $L \subset X$ its restriction $f|L$ is continuous on $L$.

For any $i \in \{1, 2, 3\}$, the interior, the closure and the frontier of a set $A$ with respect to the $i$-th topology $\tau_i(X)$ are denoted $\mathrm{Int}_i A, \mathrm{Cl}_i A$ and $\mathrm{Fr}_i A$, respectively. A set open in the topology $\tau_i(X)$ is called $i$-open set. Analogous symbolism is applied to other notions related to topologies $\tau_i(X)$, so we have $i$-curves, $i$-compactness etc. When we write that a property holds for the index $j$ we mean that it holds for all three topologies $\tau_j(X)$, i.e. for $j = 1, 2, 3$. If a property holds also for the topology $\tau$, we embrace the index $j$ in parethesis, so we have e.g. $(j)$-open sets. We point out that the notions such as open set, closure, component, which are not preceded by the index $j$, relate to the natural topology, unless it is expressively indicated otherwise.

## 2. Some Remarks on Investigated Topologies

**Theorem 1.** *Let $L$ be a linear variety in $X$. The topology induced in $L$ by the topology $\tau(X)$ or $\tau_j(X)$ is identical with the topology $\tau(L)$ and $\tau_j(L)$, respectively. In case $X = \mathbb{R}$ all four topologies $\tau(X)$ and $\tau_j(X)$ are identical.*

**Proof.** We start with case $X = \mathbb{R}$. First, $\tau_1(\mathbb{R}) = \tau(\mathbb{R})$ because every core subset $A$ of the space $\mathbb{R}$ is open. Next, $\tau_2(\mathbb{R}) = \tau(\mathbb{R})$ because any function $f : \mathbb{R} \to \mathbb{R}$ directionally continuous is continuous. To show that $\tau_3(\mathbb{R}) = \tau(\mathbb{R})$ we take arbitrary open set $B \subset \mathbb{R}$, a number $h \in \mathbb{R}$ and a set $F$ such that $(B, F)$ is the Klee pair for $h$. Then $h$ belongs to an interval $(a, b) \subset \{h\} \cup F$. Therefore $\{h\} \cup B$ is open.

Now we go to investigate spaces $X$ of dimension greater than 1. The equality $\tau(L) = \tau(X)|L$ holds true because the variety $L \cap M$ is finite dimensional for arbitrary finite dimensional subspace $M$ of $X$. The identity $\tau_1(L) = \tau_1(X)|L$ is proved in [4, p.240]. To see that $\tau_2(L) = \tau_2(X)|L$, first we notice that for any function $f : X \to \mathbb{R}$ and any set $A \subset \mathbb{R}$ there holds the equality $\varphi^{-1}(A) = f^{-1}(A) \cap L$, where $\varphi = f|L$. So, the restriction $\varphi$ of a directionally continuous function $f$ on $X$ is directionally continuous on $L$. Let $X = L \oplus M$ and $\psi(l + m) = f(l)$ for all $l \in L$ and $m \in M$. Then $\psi^{-1}(A) = f^{-1}(A) \oplus M$ for every directionally continuous function $f$ on $L$. Consequently, for every directionally continuous function $f$ on $X$ and $g$ on $L$ there exist directionally continuous functions $\varphi$ on $L$ and $\xi$ on $X$ such that $\varphi = f|L$ and $\xi|L = g$. Therefore

$$f^{-1}(U) \cap L = \varphi^{-1}(U) \quad \text{and} \quad \xi^{-1}(U) \cap L = g^{-1}(U)$$

for every open set $U \subset \mathbb{R}$. It says that a set $G$ is 2-open in $L$ iff there exists a 2-open set $G_1$ in $X$ such that $G = G_1 \cap L$. This states the equality of topologies $\tau_2(L)$ and that induced in $L$ by $\tau_2(X)$.

To prove the last equality, i.e. $\tau_3(L) = \tau_3(X)|L$, we start with the observation that a set $F$ is closed in $\tau(X)$ iff for every finite dimensional subspace $M$ of $X$ the intersection $F \cap M$ is closed in $M$. Thus, if $(G, F)$ is the Klee pair for $h \in X$, then $(G \cap L, F \cap L)$ is also the Klee pair for $h$, provided $h \in L$. It implies that the topology induced in $L$ by $\tau_3(X)$ is weaker than $\tau_3(L)$. There is no loss of generality when in the proof of the inverse implication we assume that $L$ is a subspace of $X$. Let's take a set $G = G_1 \cup \{h\}$ such that $(G_1, F_1)$ is the Klee pair in $L$ for the point $h \in L$. Let $M$ denote a complementary space to $L$ in $X$, and let $U$ be an open set in $X$ such that $M \oplus \{h\} \setminus \{h\} \subset U$ and $U \cap L = \emptyset$. Let's define $G_1' = (G_1 \cup U) \oplus M$ and $F_1' = (F_1 \cup \{h\}) \oplus M \setminus \{h\}$. Then $(G_1', F_1')$ is the Klee pair in $X$ for the point $h$. This means that $G_1' \cup \{h\}$ is 3-open in $X$. This gives that the topology induced by $\tau_3(X)$ in $L$ is equal to the topology $\tau_3(L)$. And this also closes the proof. ■

Taking into account Theorem 1 we will abbreviate the denotations in $\tau_j(X)$ and $\tau_j(L)$, to $\tau_j$ in every case it does not lead to missunderstanding.

From the proof of Theorem 1, as well as from [4,p.240], it follows

**Corollary 1.** *Let $L$ be a linear variety in the space $X$. If $G \subset X$ is an open or $i$-open set, where $i \in \{1, 2\}$, and $M$ is the subspace which complements $L$ to $X$, then the set $G \oplus M$ is open or $i$-open, respectively.*

**Corollary 2.** *Every linear variety in the space $X$ is $(j)$-closed.*

**Proof.** Obviously, if $L$ is a subspace of $X$ and $\operatorname{codim}(L) = 1$, then $L$ is closed set in the topology $\tau$. Analogous property holds in topologies $\tau_1$ and

$\tau_3$. In the topology $\tau_2$ it is enough to assume that $L \oplus \mathbb{R} = X$ and define the function $f$ by the formula $f(l+r) = r$ for all $l \in L$ and $r \in \mathbb{R}$. Naturally, the function $f$ is directionally continuous in $X$. Thus the subspace $L$ is a closed set.

Now it follows that the intersection of any family of such subspaces is $(j)$-closed set. Since every subspace is an intersection of subspaces of codimension equal to 1, so every subspace is $(j)$-closed. Obviously, if $A$ is $(j)$-closed subset of $X$, then for arbitrary element $x \in X$ the set $x + A$ is $(j)$-closed. Therefore every linear variety is $(j)$-closed. This completes the proof. ■

Now we are going to prove our main result, namely that the topology $\tau_2$ is essentially weaker than $\tau_1$ and it is essentially stronger than $\tau_3$. To state these inclusions we have to exhibit some properties of considered topologies. The first one is related to the result proved in [4, p.245]: 1-closure of a non-empty 1-open subset of $\mathbb{R}^n$ contains a non-empty open set. In view of Corollary 2 this result may be formulated as

**Theorem 2.** *Let $L$ be a finite dimensional subspace of $X$, and $G$ be an 1-open set in $X$ having at least one element common with $L$. Then there exists a non-empty open set $U \subset L$ such that $U \subset \mathrm{Cl}_1 G$.*

**Corollary 3.** *If $G_1$ and $G_2$ are disjoint 1-open non-empty sets in $X$, then*

$$\mathrm{Int}_L\,(\mathrm{Cl}\,(G_1 \cap L)) \,\cap\, \mathrm{Int}_L\,(\mathrm{Cl}\,(G_2 \cap L)) \;=\; \emptyset$$

*for every finite dimensional subspace $L$ of $X$.*

**Proof.** Let's suppose that the above intersection of the interiors, denoted here by $U$, is non-empty. Then the sets $G_1 \cap U$ and $G_2 \cap U$ are dense in $U$. Since each open set is 1-open, the set $G_1 \cap U$ is 1-open in $L$. Therefore, by Theorem 2, in $L$ there exists a non-empty open set $V$ contained in $\mathrm{Cl}_1(G_1 \cap U)$. So $V \subset U$ and $G_2 \cap V$ is dense in $V$. Therefore, by Theorem 2, in $L$ there exists a non-empty open set $W$ contained in $\mathrm{Cl}_1(G_2 \cap V)$. $G_1$ and $G_2$ are disjoint, so $W \cap G_1 = \emptyset$. Resuming we see that $W \subset U \subset \mathrm{Cl}(G_1 \cap U)$. Just obtained relation $W \subset \mathrm{Cl}(G_1 \cap U)$ and the relation $W \cap G_1 = \emptyset$ are contradictory. This way the proof is completed. ■

Corollary 3 does not hold in the infinite dimensional case, and it is to be shown in paper in preparation. Here we show that in Theorem 2 the assumption on the finite dimensionality of the space $X$ can not be suspended. It is verified below, where we construct an 1-open set such that its closure has the empty interior in the natural topology $\tau$.

**Example 1.** Let $X$ be a countable dimensional linear space. We can identify it with the space of all real sequences having finite number of non-zero elements. We can write

$$X = \bigcup_{n=1}^{\infty} \mathbb{R}^n,$$

where $\mathbb{R}^m \subset \mathbb{R}^n$ for $m < n$, i.e. we identify a point $x = (x_1, x_2, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}^n$ with the point $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. Let the set $\{a_n : n \in \mathbb{N}\}$ be dense in $\mathbb{R}$ and let differentiable functions $f_n, g_n$, where $n \in \mathbb{N}$, on $\mathbb{R}$ be such that

(1) $f_n, g_n$ are odd,
(2) $0 < f_n(r) < g_n(r)$ for all $r > 0$,
(3) $g_n(r) - f_n(r) \leq 2^{-2n} r$ for all $r > 0$,
(4) their derivatives $f_n', g_n'$ vanish at 0,
(5) $f_n, g_n$ are convex in an interval $(0, r\rangle$ with an $r > 0$.

For every natural $n \geq 2$ we define the sets $S_n \subset X$ as follows: an element $x = (x_k) \in S_n$ iff

$$\begin{array}{lllll} & 1^\circ & x_n > 0 & \text{and} & f_n(x_n) \leq x_1 - a_n \leq g_n(x_n) \\ \text{or} & 2^\circ & x_n < 0 & \text{and} & g_n(x_n) \leq x_1 - a_n \leq f_n(x_n). \end{array}$$

It's obvious that the sharp inequalities in above relations produce sets $\operatorname{Int} S_n$, and $\operatorname{Cl} S_n = S_n \cup P_n$, where $P_n$ is the set of elements $(x_k) \in X$ such that $x_1 = a_n, x_n = 0$ and other $x_l$'s are arbitrary real numbers.

By the definition of the set $S_n$, the $n$-th element $x_n$ in the sequence $(x_k) \in S_n$ is non-zero, so $\mathbb{R}^m \cap S_n = \emptyset$ for $m < n$. We define $S = \cup_{n=2}^{\infty} S_n$ and we wish to show that for arbitrary $m < n$ the set $\mathbb{R}^n \cap (X \setminus S) \setminus \mathbb{R}^m$ is non-empty. We will show even more: $(I^n \setminus S) \setminus \mathbb{R}^m \neq \emptyset$ where $I = \langle -r, r\rangle$ and $r$ is an arbitrary positive number. Since $\mathbb{R}^n \cap S_k = \emptyset$ for $n < k$, so $I^n \cap S_k = \emptyset$. Therefore

$$I^n \cap (X \setminus S) = I^n \setminus \cup_{k=1}^{n} S_k.$$

Taking into account that for $k = 2, 3, \ldots, n$ there holds

$$\mu(I^n \cap S_k) \leq (2r)^n 2^{-2n-1},$$

where $\mu$ is the usual measure in $\mathbb{R}^n$, we have

$$\mu(I^n \setminus S) \geq (2r)^n - \sum_{k=2}^{n} \mu(I^n \cap S_k) > 0.$$

Since $\mu(\mathbb{R}^m) = 0$, so $\mu\left((I^n \setminus S) \setminus \mathbb{R}^m\right) > 0$, and it proves that $(I^n \setminus S) \setminus \mathbb{R}^m$ is not empty.

Now we are going to show that the set $X \setminus S$ is 1-open. In this aim we take an arbitrary $x = (x_k) \in X \setminus S$ and arbitrary $y = (y_k) \in X \setminus \{x\}$. Obviously, there exists a natural number $m$ such that both $x, y \in \mathbb{R}^m$. We will show that in the interval $(x, y)$ there exists an element $z$ such that the interval $\langle x, z\rangle \subset X \setminus S$. We will show that for every $k = 2, 3 \ldots, m$ there exists element $z_k \in (x, y)$ such that $\langle x, z_k\rangle \subset \mathbb{R}^m \setminus S_k$.

We have to consider two cases

$$\begin{array}{ll} (\alpha) & x \in \mathbb{R}^m \setminus \mathrm{Cl}\, S_k \\ (\beta) & x \in P_k. \end{array}$$

In case $(\alpha)$ we have $x \in \mathrm{Int}_{\mathbb{R}^m}(\mathbb{R}^m \setminus S_k)$, and it proves that there exists an element $z_k$ we looked for.

Let's now consider the case $(\beta)$. Let's take a Hamel base of the space $X$ composed of elements $b_k = (\delta_{k,l})$, where $\delta_{k,l}$ is the Kronecker delta. If $y \in P_k$, then $\langle x, y\rangle \subset \mathbb{R}^m \setminus S_k$ because $P_k \cap S_k = \emptyset$. It proves the existense of a desired $z_k$. If $y \notin P_k$, by $p$ we denote the projection from $\mathbb{R}^m$ onto $L_k = \mathrm{Lin}\left(\{b_1, b_k\}\right)$, so $p(x) = (x_1, x_k)$, $p(y) = (y_1, y_k)$. From conditions (1) and (5) it follows that there exists $u \in (p(x), p(y))$ such that $\langle p(x), u\rangle \subset L_k \setminus S_k$. Let $M_k = \mathrm{Lin}\left(\{b_2, b_3, \ldots, b_n\} \setminus \{b_k\}\right)$. Then $\langle p(x), u\rangle \oplus M_k \subset \mathbb{R}^m \setminus S_k$ and $(p(x), u) \oplus M_k \subset (p(x), p(y)) \oplus M_k$. Therefore in the set $\{u\} \oplus M_k$ there exists an element $z_k$ we looked for. Hence for every $k = 2, 3, \ldots, m$ there exists $z_k \in (x, y)$ such that $\langle x, z_k\rangle \subset \mathbb{R}^m \setminus S_k$. In consequence, $\langle x, z\rangle = \bigcap_{k=2}^m \langle x, z_k\rangle \subset \mathbb{R}^m \setminus \bigcup_{k=2}^m S_k$. Since $S_n \cap \mathbb{R}^m = \emptyset$ for $n > m$, so $\langle x, z\rangle \subset \mathbb{R}^m \setminus \bigcup_{k=2}^\infty S_k = \mathbb{R}^m \setminus S \subset X \setminus S$.

Now we are going to prove that $\mathrm{Int}\,(X \setminus \mathrm{Int} S) = \emptyset$.

Suppose that there exists an open non-empty set $U \subset X \setminus \mathrm{Int} S$.

Then

$$U \subset X \setminus \mathrm{Cl}\,(\mathrm{Int} S) \subset X \setminus \bigcup_{n=2}^{\infty} \mathrm{Cl} S_n.$$

Let's fix $m \in \mathbb{N}$. Since $S_n \cap \mathbb{R}^m = \emptyset$ for $n > m$ and $\mathrm{Cl} S_n = S_n \cup P_n$, so $\mathrm{Cl}\,(S_n) \cap \mathbb{R}^m = P_n \cap \mathbb{R}^m = \{a_n\} \oplus \mathbb{R}^{m-1}$

Therefore

$$\mathbb{R}^m \cap U \subset \mathbb{R}^m \setminus \bigcup_{n=2}^{\infty} \mathrm{Cl} S_n \subset \mathbb{R}^m \setminus \bigcup_{n=m+1}^{\infty} (P_n \cap \mathbb{R}^m) = \mathbb{R}^m \setminus Z,$$

where $Z = \bigcup\limits_{n=m+1}^{\infty} \{a_n\} \oplus \mathbb{R}^{m-1}$.

Since the set $\{a_n : n > m\}$ is dense in $\mathbb{R}$, so $Z$ is dense in $\mathbb{R}^m$. Since $\mathbb{R}^m \cap U$ is open, so the inclusion $\mathbb{R}^m \cap U \subset \mathbb{R}^m \setminus Z$ cannot hold. This contradiction shows that $\operatorname{Int}(X \setminus \operatorname{Int} S) = \emptyset$. Therefore $\operatorname{Int}(\operatorname{Cl}(X \setminus S)) = \emptyset$. So the constructed set $S$ fulfills requirements: $X \setminus S$ is 1-open and its closure has the empty interior.

Below we give a simple example of the set $A$ in the infinite dimensional space $X$ such that $\operatorname{Int}_X A = \emptyset$ and for every $n \in \mathbb{N}$ one can take a $n$-dimensional subspace $L \subset X$ such that $\operatorname{Int}_L(A \cap L) \neq \emptyset$.

**Example 2.** Let $B = \{b_t : t \in T\}$ be a Hamel base of $X$, and $A = \operatorname{Conv}(B \cup \{0\})$. Let's take a finite number of elements $b_1, b_2, \ldots, b_m$ of $B$ and let's denote $L = \operatorname{Lin}(\{b_1, b_2, \ldots, b_m\})$. Then for any element $x \in \operatorname{Int}_L(A \cap L)$ we have $x = \sum_{i=1}^m \lambda_i b_i$, where $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i > 0$ for $i = 1, 2, \ldots, m$. At last, let's take $b \in B \setminus L$ and define $M = L + \mathbb{R}b$. It's clear that $A \cap M = \operatorname{Conv}(B \cup \{b\} \cup \{0\})$. Therefore $x \notin \operatorname{Int}_M(A \cap M)$, so $x \notin \operatorname{Int}_X A$.

A more advanced case is presented in [5, p55]. There it is shown that there exists a set $A$ in the infinite dimensional space $X$ such that $0 \in \operatorname{Int}_L A$ for every finite dimensional space $L$ and $0 \notin \operatorname{Int}_X A$.

Otherwise than in the above examples, in the next part we state that under some assumptions a set is open if its intersection with some subspaces is open. Precisely, it holds true

**Lemma 1.** *Let $\mathcal{L}$ be a family of subspaces of $X$ which are linearly ordered by the inclusion, and $\cup\mathcal{L} = X$. For every $L \in \mathcal{L}$ let $G_L$ be open set in $L$, and let $G_L \subset G_M$ if $L \subset M$. Then the set $G = \cup\{G_L : L \in \mathcal{L}\}$ is open in $X$.*

**Proof.** We can choose a well-ordered subfamily $\mathcal{M} \subset \mathcal{L}$ such that $\cup\mathcal{M} = X$. This way we have $\mathcal{M} = \{L_\alpha : \alpha < \gamma\}$, where $\alpha, \gamma$ are ordinals. Let's take an element $x \in G$. Obviously, there exists $\beta_0 < \gamma$ such that $x \in G_{\beta_0}$, where $G_\beta = G_{L_\beta}$. We will show that in $G$ there exists an open set $U$ containing $x$. In this aim we construct the family $\{U_\beta : \beta < \gamma\}$ such that for every ordinal number $\beta$, where $\beta_0 \leq \beta < \gamma$, the following conditions hold

(i) $U_\beta$ is open in $L_\beta$
(ii) $U_\beta \subset G_\beta$
(iii) $U_\alpha = U_\beta \cap L_\alpha$ for $\alpha < \beta$.

Inductively, we start with $U_{\beta_0} = G_{\beta_0}$ and we assume that for some ordinary number $\varphi$ there are already defined, for all $\beta < \varphi$, the sets $U_\beta$ satisfying conditions (i)-(iii), and we define $U_\varphi$. There are two cases possible: or $\varphi$ has its immediate preceding numer $\delta$, or $\varphi$ is a limit ordinal, i.e. it is not a successor ordinal.

If $\varphi = \delta + 1$, we take the subspace $M$ complementary to $L_\delta$ in the space $L_\varphi$, and we set $U_\varphi = (U_\delta \oplus M) \cap G_\varphi$. It is easy to see that $U_\varphi$ satisfies conditions

(i)-(iii).
If $\varphi$ is a limit ordinal, we set

$$U_\varphi = \bigcup_{\beta<\varphi} U_\beta.$$

For every $\beta < \varphi$ there holds $L_\beta \cap U_\varphi = U_\beta$, so $L_\beta \cap U_\varphi$ is open. Indeed, if $L$ is finite dimensional subspace in $L_\varphi$, then the open set $L \cap U_\varphi = L \cap U_\alpha$ for each ordinary number $\alpha < \varphi$ such that $L \subset L_\alpha$. Now it is clear that $U_\varphi$ satisfies conditions (i)-(iii). Now we set

$$U = \bigcup_{\beta<\gamma} U_\beta$$

and we argue as above to conclude that the set $U$ is contained in $G$, is open in $X$ and contains $x$. Thus $G$ is open, and it makes the proof complete. ■

**Theorem 3.** *The topological space $(X, \tau)$ is hereditarily normal.*

**Proof.** Let $\gamma$ denote the limit ordinal such that card $\gamma = \dim X$, let $\{b_\alpha : \alpha < \gamma\}$ be a Hamel base of the space $X$ (with elements identified by ordinal $\alpha \geq 1$), and let $X_\beta = \mathrm{Lin}\,(\{b_\alpha : \alpha < \beta\})$, where $\beta \geq 2$. The proof is by the transfinite induction, so we will assume that for some ordinary number $\delta$ (where $2 \leq \delta \leq \gamma$) the space $(X_\beta, \tau)$ is hereditarily normal if $\beta < \delta$ and we will show that $(X_\delta, \tau)$ is also hereditarily normal.
Obviously, the space $(X_\beta, \tau)$ is hereditarily normal if $\beta \in \mathbb{N}$, so it is enough to consider ordinals $\delta \geq \aleph_0$. If $\delta$ is not an initial ordinal, there exists an ordinal $\delta' < \delta$ such that card $\delta' = \mathrm{card}\,\delta$. Therefore $\mathrm{card}\,\delta = \mathrm{card}\,\delta' = \dim X_{\delta'} = \dim X_\delta$ and, consequently, $(X_\delta, \tau)$ is hereditarily normal. In case $\delta$ is an initial ordinal, we consider two separated sets $A$, $B$, i.e. sets satisfying conditions $A \cap \mathrm{Cl}(B) = \mathrm{Cl}(A) \cap B = \emptyset$ both in $X_\delta$. For every ordinal number $\beta$ such that $2 \leq \beta$ and $\beta < \delta$ we will define sets $G_\beta$, $H_\beta$ such that

(1) $G_\beta, H_\beta$, are open in $X_\beta$,
(2) $A \cap X_\beta \subset G_\beta$ and $B \cap X_\beta \subset H_\beta$,
(3) $G_{\beta'} \subset G_{\beta''}$ and $H_{\beta'} \subset H_{\beta''}$ for all $\beta' < \beta''$,
(4) $G_\beta \cap H_\beta = \emptyset$,
(5) $G_\beta \subset X_\delta \setminus \mathrm{Cl}\, B$ and $H_\beta \subset X_\delta \setminus \mathrm{Cl}\, A$.

For $\beta = 2$ the hereditary normalness of the space $(\mathbb{R}^2, \tau)$ guarantees that in $\mathbb{R}^2$ there exist sets $G'$ and $H'$ satisfying conditions (1), (2) and (4). Since the sets $A$ and $B$ are separated, so there exist open sets $G''$ and $H''$ such that $A \subset G'' \subset X_\delta \setminus B$ and $B \subset H'' \subset X_\delta \setminus A$. It is easy to see now that the intersections

$$G_2 = G' \cap G'' \text{ and } H_2 = H' \cap H''$$

satisfies conditions (1)-(5), the condition (3) being satisfied by definition (for $\beta < 2$ there do not exist sets $G_\beta$ and $H_\beta$).

Now we suppose that for any ordinal $\varphi$, where $2 \le \varphi \le \delta$, and every ordinal $\beta < \varphi$ there are already defined sets $G_\beta$ and $H_\beta$ satisfying conditions (1)-(5), and we are going to construct analogous sets $G_\varphi$ and $H_\varphi$.

There are two cases to be considered: $\varphi$ has its predecessor or $\varphi$ is a limit ordinal.

We investigate the first case, i.e. when there exists an ordinal $\psi$ such that $\varphi = \psi + 1$. We start with showing that the sets

$$A_\psi = (A \cap X_\varphi) \cup G_\psi \text{ and } B_\psi = (B \cap X_\varphi) \cup H_\psi$$

are separated. We do it by the verification that each component, $A \cap X_\varphi$ and $G_\psi$, of the set $A_\psi$ is separated with every component, $B \cap X_\varphi$ and $H_\psi$, of the set $B_\psi$. Conditions (1) and (4) imply that $G_\psi$ and $H_\psi$ are separated. The sets $A \cap X_\varphi$ and $B \cap X_\varphi$ are separated because $A$ and $B$ are separated. By the condition (5) it is $\mathrm{Cl}\,(A \cap X_\varphi) \cap H_\psi \subset \mathrm{Cl}\,(A \cap X_\delta) \cap (X_\delta \setminus \mathrm{Cl}\, A) \subset \mathrm{Cl}\, A \cap (X_\delta \setminus \mathrm{Cl}\, A) = \emptyset$.

At last, we go to state that

$$(A \cap X_\varphi) \cap \mathrm{Cl}\, H_\psi = \emptyset = (B \cap X_\varphi) \cap \mathrm{Cl}\, G_\psi.$$

By Corollary 2 the subspace $X_\psi$ is closed, so $\mathrm{Cl}\, H_\psi \subset X_\psi$. Hence

$$(A \cap X_\varphi) \cap \mathrm{Cl}\, H_\psi \subset (A \cap X_\psi) \cap \mathrm{Cl}\, H_\psi \subset G_\psi \cap \mathrm{Cl}\, H_\psi,$$

where the last inclusion is implied by (2). Since $G_\psi$ is open in $X_\psi$, so by (4) the intersection $G_\psi \cap \mathrm{Cl}\, H_\psi$ is empty, and it proves that $(A \cap X_\varphi) \cap \mathrm{Cl}\, H_\psi = \emptyset$. Similarly, $(B \cap X_\varphi) \cap \mathrm{Cl}\, G_\psi = \emptyset$. Now, all four possibilities examined, we conclude that $A_\psi$ and $B_\psi$ are separated.
For $(X_\varphi, \tau)$ is hereditarily normal, so there exist disjoint sets $C$ and $D$ which are open in $X_\varphi$ and contain sets $A_\psi$ and $B_\psi$, respectively. Since $A$ and $B$ are separated, so

$$A \cap X_\varphi \subset X \setminus \mathrm{Cl}\, B \text{ and } B \cap X_\varphi \subset X \setminus \mathrm{Cl}\, A.$$

Futhermore, by (5) we have

$$A_\psi \subset X \setminus \mathrm{Cl}\, B \text{ and } B_\psi \subset X \setminus \mathrm{Cl}\, A.$$

Now it is not difficult to check that the sets

$$G_\varphi = C \cap (X \setminus \mathrm{Cl}\, B) \text{ and } H_\varphi = D \cap (X \setminus \mathrm{Cl}\, A)$$

satisfy conditions (1)-(5), and this concludes the proof in the first case. In the second case, i.e. when $\varphi$ is a limit ordinal, we set $G_\varphi = \bigcup_{\psi<\varphi} G_\psi$ and $H_\varphi = \bigcup_{\psi<\varphi} H_\psi$. By Lemma 1, both these sets are open in $X_\varphi$. They contain sets $A \cap X_\varphi$ and $B \cap X_\varphi$, respectively. Since sets $G_\psi$ and $H_\psi$ are disjoint for all $\psi < \varphi$, so $G_\varphi$ and $H_\varphi$ are disjoint, too. Hence $G_\varphi$ and $H_\varphi$ satisfy conditions (1), (2) and (4). It is not difficult to verify that they also satisfy conditions (3) and (5). So, for every $\beta$ such that $2 \le \beta$ and $\beta < \delta$ there exist the families of sets $G_\beta$ and $H_\beta$ satisfying conditions (1)-(5). Therefore the unions $G = \bigcup_{\beta<\delta} G_\beta$ and $H = \bigcup_{\beta<\delta} H_\beta$ are open in $X_\delta$, are disjoint, and contain $A$ and $B$, respectively. By Theorem 2.1.7 in [1, p.96], the space $(X_\delta, \tau)$ is hereditarily normal.

We worked with ordinal $\delta \le \gamma$, so in virtue of the transfinite induction the space $(X_\gamma, \tau)$ is hereditarily normal. In view of the equality $X_\gamma = X$ this completes the proof. ∎

## 3. Theorems on Inclusions between Topologies

In this part of the paper we will prove the inclusions between topologies $\tau_1$, $\tau_2$ and $\tau_3$. But first we show

**Theorem 4.** *The basis of the topology $\tau_2$ is the family of all sets of the form $f^{-1}((0,\infty))$, where $f$'s are directionally continuous functions.*

**Proof.** Let $\mathcal{F}$ denote the set of all directionally continuous functions on $X$. For arbitrary $f \in \mathcal{F}$ and any $r, s \in \mathbb{R}$ such that $r < s$ the equality

$$f^{-1}((r,s)) = f^{-1}((-\infty,s)) \cap f^{-1}((r,\infty))$$

holds. Therefore

$$f^{-1}((r,s)) = g^{-1}((0,\infty)) \cap h^{-1}((0,\infty)), \tag{1}$$

where $g = -f + s, h = f - r$, both functions in $\mathcal{F}$.

Let $\mathcal{P}$ be the family of the subsets in $X$ of the form $f^{-1}((r,s))$, where $f \in \mathcal{F}$ and $r < s$. Obviously, the family $\mathcal{P}$ is a subbase of the topology $\tau_2$.

Let $\mathcal{B}$ be the base of the topology generated by $\mathcal{P}$, i.e. $\mathcal{B} = \bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n$, where

$$\mathcal{B}_n = \left\{ \bigcap_{k=1}^{n} A_k : A_1, A_2, \ldots, A_n \in \mathcal{P} \right\}.$$

By (1) the base $\mathcal{B}$ is contained in the family of all finite intersections of the family

$$\mathcal{B}' = \left\{ f^{-1}((0,\infty)) : f \in \mathcal{F} \right\}$$

It's clear that $f^{-1}((0,\infty))$ is 2-open if $f \in \mathcal{F}$. Therefore the family $\mathcal{B}'$ is also a subbase of the topology $\tau_2$.
It is obvious that

$$f^{-1}\left((0,\infty)\right) = \bigcap_{k=1}^{m} f_k^{-1}\left((0,\infty)\right)$$

where $f_1, f_2, \ldots, f_m \in \mathcal{F}$ and $f = \sup\{f_k : k = 1, 2, \ldots, m\}$. Hence the family $\mathcal{B}'$ is a base of $\tau_2$. ■

Now, at last, we are ready to prove

**Theorem 5.** *In every real linear space of dimension greater than 1 the topology $\tau_1$ is essentially stronger then the topology $\tau_2$.*

**Proof.** Let $f$ be a directionally continuous function on $X$, and $r < s$. Let's denote $U = f^{-1}((r,s))$ and let's take an arbitrary $x \in U$. Let $P$ be a straight line passing through $x$. It's obvious that the restriction $f_P = f|P$ is continuous on $P$ and the set $U \cap P$ is open. So there exist elements $y, z \in P$ such that $x \in (y,z) \subset f_P^{-1}((r,s)) \subset U$. Therefore $x \in \operatorname{Cor} U$. Since $x$ is taken arbitrarily in $U$, so $\operatorname{Cor} U = U$, and it says that every 2-open set is 1-open.

Now we will show that the inclusion $\tau_2 \subset \tau_1$ is sharp in $X = \mathbb{R}^2$. It means we will find a set which is 1-open and is not 2-open. Let $U$ be an open set and $G$ be an 1-open set such that $G \cap U \neq \emptyset$ and $U \setminus G$ is dense in $U$; such a set exists by virtue of Lemma 1 in [4, p.241]. From Theorem 2 it follows that there exists a non-empty open set $V$ such that $V \subset \mathrm{Cl}_1\,(U \cap G)$. The set $V \cap G$ is not 2-open and we prove it by reductio ad absurdum. So we suppose that $V \cap G$ is 2-open. By Theorem 4, for every $x \in V \cap G$ there exists a directionally continuous function $f$ such that $x \in f^{-1}((0,\infty)) \subset V \cap G$. Therefore $f^{-1}\left((-\infty,0\rangle\right) \supset V \setminus G$. Then we take into consideration two sets:

$$A = f^{-1}\left(\left(\frac{r}{2},\infty\right)\right) \text{ and } B = f^{-1}\left(\left(-\infty,\frac{r}{2}\right)\right),$$

where $r = f(x)$. These 2-open sets are 1-open and disjoint. Moreover, $x \in A \subset V \cap G$. Obviously,

$$V \setminus G \subset f^{-1}\left(\left(-\infty,\frac{r}{2}\right\rangle\right).$$

Since $f$ is directionally continuous, so $\mathrm{Cl}_2 B = f^{-1}((-\infty,\frac{r}{2}\rangle))$. Hence $V \setminus G \subset \mathrm{Cl}_2\, B \subset \mathrm{Cl}\, B$. Since $V \setminus G$ is dense in $V$, so $V \subset \mathrm{Cl}\, B$.

On the other hand, $A$ is 1-open, so, by Theorem 2, there exists a non-empty open set $W$ such that $W \subset \mathrm{Cl}_1 A \subset \mathrm{Cl}\, A$. It implies that $W \subset \mathrm{Cl}\, A \subset \mathrm{Cl}\,(V \cap G) \subset \mathrm{Cl}\, V$. Further, since $W$ and $V$ are open, so $W \subset J$, where $J = \mathrm{Int}\,(\mathrm{Cl}\, A) \cap \mathrm{Int}\,(\mathrm{Cl}\, B)$. It contradicts Corollary 3, where it is stated that

$J = \emptyset$. This proves that the set $G \cap V$ is not 2-open. Taking into account that $V$ is open (and, in consequence, it is 2-open), we see that $G$ cannot be 2-open.

So we know that in $\mathbb{R}^2$ there exists a set which is 1-open and is not 2-open. By Theorem 1 it follows that such a set exists in each space $X$ of dimension greater than 2. ■

Arguing as in the second part of the proof of Theorem 5 for any finite dimensional space (it is enough to replace $\mathbb{R}^2$ by $\mathbb{R}^n$) one can show

**Corollary 4.** *Let $G$ be an 1-open set. If there exists a finite dimensional subspace $L$ and an open set $U \subset L$ such that $U \cap G \neq \emptyset$ and $L \setminus G$ is dense in $U$, then $G$ is not 2-open.*

Note that Corollary 4 is slightly stronger than the result given in [8, p.31], where Theorem 3 states that every directionally continuous function is core continuous, and it is mentioned that this does not hold in the case of pointwise continuity.

A different problem concerning core continuous functions is considered by R.J.Pawlak in [10]. He asks do exist core continuous and bounded real-valued functions on $\mathbb{R}^2$ such that they are not almost continuous (a function $f$ mapping a topological space $S$ in a topological space $Y$ is called almost continuous if, for every open set $U \subset S \times Y$ containing the graph of $f$, $U$ contains the graph of some continuous function from $S$ in $Y$). He answers positively, see [10, p.466].

**Theorem 6.** *In every real linear space of dimension greater than 1 the topology $\tau_2$ is essentially stronger than the topology $\tau_3$.*

**Proof.** In the proof we adopt the proof of Theorem 6 in [7, p.29], stating that the topological space $(\mathbb{R}^n, \tau_3)$ is completely regular. The key point of this proof is the normality of the space $(\mathbb{R}^n \setminus \{h\}, \tau)$ with arbitrary $h \in \mathbb{R}^n$. From Theorem 3 it follows that the space $(X \setminus \{h\}, \tau)$ is normal in case $\dim X \geq \aleph_0$, too.

Let $G$ be a set belonging to the base of $\tau_3$.

First we consider the case when $G$ is an open set. Let $x \in G$. It's clear that then there exists a continuous function $f$ on $X$ such that $x \in f^{-1}((0, \infty)) \subset G$. Obviously, $f$ is directionally continuous. Hence for every $x \in G$ there exists a 2-open set contained in $G$. It says that $G$ is open.

Let now $(G_1, F_1)$ be a Klee pair for $h \in X$ and $G = G_1 \cup \{h\}$. Let $F = F_1 \cup \{h\}$. Thanks to the normality of $X \setminus \{h\}$ we can define a function $f : X \to \langle 0, 1 \rangle$ which is continuous on $X \setminus \{h\}$ in the topology $\tau$, and its restrictions to sets $F$ and $X \setminus G_1$ are $f|F = 1$ and $f|X \setminus G_1 = 0$, respectively.

By the construction, $f$ is directionally continuous. Thus, by Theorem 4, the set $G_1 = f^{-1}((0,\infty))$ is 2-open. It proves that $\tau_3 \subset \tau_2$.

We will show that this inclusion is sharp. First we investigate the case $X = \mathbb{R}^2$. Here we find a 2-open set $G$ which is not 3-open one. This set will be symmetric with respect to both axes of the rectangular coordinate system, i.e. a point $(r_1, r_2) \in G$ iff $(|r_1|, |r_2|) \in G$, so we need to define $G$ only on the quarter $\mathbb{R}^2_+$.

Let $(s_n)$ be a decreasing sequence of positive real numbers tending to 0, and let $\gamma, \varphi, \chi$ and $\psi$ be real-valued functions defined on $\mathbb{R}_+$ and such that:
(1) $\gamma$ is strictly convex and differentiate function,
(2) $\varphi, \chi$ and $\psi$ are continuous functions,
(3) $\varphi(0) = \chi(0) = \psi(0) = \gamma(0) = 0$ and the derivative $\gamma'(0) = 0$,
(4) $0 < \varphi(r) < \chi(r) < \psi(r) < \gamma(r)$ if $r \in S = \mathbb{R}_+ \setminus \{s_n : n \in \mathbb{N}\}$,
(5) $\varphi(s_n) = \chi(s_n) = \psi(s_n)$ for each $n \in \mathbb{N}$,
(6) graphs of functions $\varphi, \chi$ and $\psi$ are tangent to the lines $\{(s_n, r) : r \in \mathbb{R}\}$ at points $(s_n, 0), n \in N$.

Now we define the set $G$ in $\mathbb{R}^2_+$, symmetric with respect to both axes, by the equality

$$\begin{aligned} G \cap \mathbb{R}^2_+ = \{(r,t) : 0 \le t < \chi(r) \ \text{ and } \ r \in S\} \\ \cup\{(r,t) : \psi(t) < t < \gamma(t) \ \text{ and } \ r \in (0,\infty) \\ \cup\{(r,0) : r \in \mathbb{R}_+\} \\ \cup\{(r,t) : t > 2\gamma(r) \ \text{ and } \ r \in (0,\infty)\}. \end{aligned}$$

We define the set $F$, also symmetric with respect to both axes, by the formula

$$\begin{aligned} F \cap \mathbb{R}^2_+ = \{(r,t) : 0 \le t \le \varphi(r) \ \text{ and } \ r \in (0,\infty)\} \\ \cup \bigcup_{n=1}^{\infty} \left\{ (s_n, t) : 0 \le t \le \frac{\gamma(s_n)}{2} \right\} \\ \cup\{(r,t) : t \ge 3\gamma(r) \text{ and } r \in \mathbb{R}_+\} \end{aligned}$$

We see that $F$ is closed and $F \subset G$. Let's define another set, $Y = \mathbb{R}^2 \setminus \{(s,0) : s = 0 \text{ or } |s| = s_n \text{ with some } n \in \mathbb{N}\}$.
By Theorem 3, the topological space $(Y, \tau)$ is normal. Thus the sets $Y \setminus G$ and $F \cap Y$ are disjoint and closed in $Y$. Therefore there exists a continuous function $f : Y \to \langle 0, 1 \rangle$ such that $f|Y \setminus G = 0$ and $f|F = 1$. We extend the function $f$ over $\mathbb{R}^2$ by putting $f(x) = 1$ for each $x \in \mathbb{R}^2 \setminus Y$. It is obvious that if $x \in \mathbb{R}^2 \setminus Y$, then $x \in \operatorname{Cor} F$. Therefore $f$ is directionally continuous and $G = f^{-1}((0,\infty))$. This means that the set $G$ is 2-open. It is also obvious that if $x \in \mathbb{R}^2 \setminus Y$, then there does not exist its neighbourhood contained in $G$. So there is no $s > 0$ such that the segment $\{(r, 0) : -s < r < s\}$ is

contained in $\operatorname{Int} G$. In consequence, there does not exists a closed set $E$ such that $(G, E \setminus \{0\})$ is a Klee pair for 0. It implies that the set $G$ is not 3-open in $\mathbb{R}^2$.

By Theorem 1 this result is valid in any space $X$ of dimension equal at least 2. ■

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