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Marian Liskowski

APPROXIMATION BY FUNCTIONS IN $C_0^{\infty}(\Omega)$ IN ORLICZ - SOBOLEV SPACES

Abstract: The results presented in this paper concern the identity of spaces ${W_0}^{k,M}(\Omega)$ and $W^{k,M}(\Omega)$, generated by φ-functions M with parameter for some class domains $\Omega \subset R^n$ and they are the extension of analogous results for clasical Sobolev spaces.

The problem of approximation of elements in $W^{k,M}(\Omega)$ by smooth functions on various domains $\Omega \subset R^n$ were investigated by different authors for classic Sobolev spaces with integer values of k as well as for some generalization of Sobolev space to the case of noninteger values k (see e.g. N. Meyers and J. Serrin [11] in the case $M(u) = u^p,\ p > 1$; T. K. Donaldson and N. S. Trudinger [2], when M is arbitrary N-function; H. Hudzik [3], [4], [5], [6], [7], when M is N-function which depends on parameter; M. Liskowski [9], [10] for some family of generalized Orlicz-Sobolev space, when k is noninteger and M is N-function with parameter).

Key words: Orlicz-Sobolev space, dual of Orlicz-Sobolev space, approximation by smooth function in generalized Sobolev spaces, polar set.

1. Introduction

Let Ω be an open and nonempty set in R^n. A real-valued function $M\colon \Omega \times [0,\infty) \to [0,\infty)$ which satisfies the conditions:

1. $M(t, 0) = 0$ for a.e. $t \in \Omega$,
2. M is convex and continous at zero with respect to second variable for a.e. $t \in \Omega$,
3. $M(t, u)$ is ameasurable function of t for every fixed $u \geq 0$

is called a φ-function of the variable u with parameter t.

A φ-function M is called an N-function if satisfies the following condition

4. $\frac{M(t,u)}{u} \to 0$ as $u \to 0$ and $\frac{M(t,u)}{u} \to \infty$ as $u \to \infty$ for a.e. $t \in \Omega$.

The following conditions will be of importance:

5. there exists a constant $u_0 > 0$ such that $\int_B M(t,u)dt < \infty$ for every compact set $B \subset \Omega$ and for all $0 \le u \le u_0$;
6. for every compact set $B \subset \Omega$ there exists a constant $c > 0$ and nonnegative function $g \in L^1(B)$ such that $u \le c\, M(t,u) + g(t)$ for all $u \ge 0$ and for a.e. $t \in \Omega$.

A function M satisfies the condition Δ_2 if the following inequality holds

$$M(t, 2u) \le K\ M(t,u) + h(t)$$

for all $u \ge 0$ and almost every $t \in \Omega$, where h is a nonnegative integrable function in Ω and K is a positive constant.

Let us denote by X the real space of all complex-valued and locally integrable functions defined on Ω, with equality almost everywhere on Ω. For any fixed integer number $k > 0$ and any φ-function M we define on X a functional

$$I(f) = \sum_{|\alpha| \le k} \int_\Omega M\left(t, |\, D^\alpha f(t)|\,\right) dt,$$

where $D^\alpha f$ is the distributional derivative of f. The functional I is a convex modular on X.

The Orlicz-Sobolev space is defined in the following manner (see e.g. [5])

$$W^{k,M}(\Omega) = \{\, f \in X \, : I(af) < \infty \text{ for some } a > 0 \,\}.$$

If a φ-function M satisfies additionally (5) and (6), then the space $W^{k,M}(\Omega)$ is a Banach space with respect to the Luxemburg norm $\|\ \|_{W^{k,M}}$ (or briefly $\|\ \|_{k,M}$) generated by the convex modular I (see [12]).

The Orlicz-Sobolev space $W^{k,M}(\Omega)$ is a vector subspace of the Orlicz space

$$L^M(\Omega) = \left\{ f \in X \colon \int_\Omega M(t, c|f(t)|)dt < \infty \text{ for some } c > 0 \right\}.$$

In the sequel $L^M(\Omega)$ will be considered with Luxemburg norm $\|\ \|_{L^M}$ generated by a convex modular

$$I_0(f) = \int_\Omega M\left(t, |f(t)|\right) dt.$$

Let ${C_0}^\infty(\Omega)$ be the set of all functions defined on Ω having derivatives of any order on Ω whose supports are compact subset of Ω. If a φ-function M satisfies (5), the inclusion

$${C_0}^\infty(\Omega) \subset W^{k,M}(\Omega)$$

holds for every nonnegative and integer k. We denote by $W_0{}^{k,M}(\Omega)$ the closure in $W^{k,M}(\Omega)$ of the set $C_0{}^\infty(\Omega)$ with respect to the norm $\|\ \|_{W^{k,M}}$.

For a φ-function M satisfying (6) we have

$$L^M(\Omega) \subset L^1{}_{loc}(\Omega).$$

The condition (6) is sufficient and necessary for this inclusion (see [8]). Thus, if (6) is satisfied, then for every function $f \in L^M(\Omega)$ the functional T_f defined by

$$T_f(\varphi) = \int_\Omega f(t)\,\varphi(t)dt$$

for $\varphi \in C_0{}^\infty(\Omega)$ is a regular distribution and so $W^{k,M}(\Omega) = L^M(\Omega)$ if $k = 0$.

2. Results

We start with general results concerning to Orlicz-Sobolev spaces. Denote:

$$l = \sum_{|\alpha|\le k} 1 \quad \text{and} \quad L_l{}^M(\Omega) = \prod_{i=1}^{l} L^M(\Omega).$$

The space $L_l{}^M(\Omega)$ with the Luxemburg norm generated by a convex modular of the form

$$\rho(f) = \sum_{i=1}^{l} I_0(f_i)$$

is a Banach space. On the space $L_l{}^M(\Omega)$ is defined also the Orlicz norm by

$${}^0\|f\|_{L_l{}^M} = \sup\left\{ \left| \sum_{i=1}^{l} \int_\Omega f_i(t) g_i(t)dt \right| : \|g\|_{L_l{}^N} \le 1 \right\},$$

where N is the complementary function to M in the sense of Young. The Orlicz norm and the Luxemburg norm are equivalent.

Let us suppose that l multiindices α satisfying $|\alpha| \le k$ are linearly ordered in some convenient fashion so that each $f \in W^{k,M}(\Omega)$ we may associate the well-defined vector Pf in $L_l{}^M(\Omega)$ of the norm

$$Pf = (D^\alpha f)_{|\alpha|\le k} \tag{1}$$

defining a mapping $W^{k,M}(\Omega)$ onto a subspace of $L_l{}^M(\Omega)$. Since $\|f\|_{W^{k,M}} = \|Pf\|_{L_l{}^M}$, so P is an isometric isomorphism of $W^{k,M}(\Omega)$ onto $PW^{k,M}(\Omega) \subset L_l{}^M(\Omega)$. If $k > 0$, then $PW^{k,M}(\Omega)$ is a closed proper subspace of $L_l{}^M(\Omega)$.

If M is an N-function satisfying conditions (5), (6) and Δ_2, then every functional $f^* \in ({W_0}^{k,M}(\Omega))'$ is an extension to $W^{k,M}(\Omega)$ of some distribution $T \in D'(\Omega)$ of the form

$$T = \sum_{|\alpha| \le k} (-1)^{|\alpha|} D^\alpha T_{f_\alpha}, \tag{2}$$

where $f = \left(f_\alpha\right)_{|\alpha| \le k} \in {L_l}^N(\Omega)$ is an element determining the functional f^* of the form

$$f^*(g) = \sum_{|\alpha| \le k} \int_\Omega D^\alpha g(t) f_\alpha(t) dt.$$

On the other hand if T is any element of $D'(\Omega)$ having the form (2) for some $f \in {L_l}^N(\Omega)$ then T possesses a unique such extension to ${W_0}^{k,M}(\Omega)$, (see [7]). Thus there holds the following theorem.

Theorem 1. ([7]) *If M is an N-function satisfying conditions* (5), (6) *and Δ_2, then the dual $({W_0}^{k,M}(\Omega))'$ is the space consisting of those distributions $T \in D'(\Omega)$ satisfying* (2) *for some $f = (f_\alpha)_{|\alpha| \le k} \in {L_l}^N(\Omega)$, normed by*

$$\|T\| = \inf \left\{ {}^0\|f\|_{{L_l}^N} : f \text{ satisfies } (2) \right\}. \tag{3}$$

The space of distribution, which are discussed above theorem is denoted by $W^{-k,N}(\Omega)$. Thus $W^{-k,N}(\Omega)$ is isometrically isomorphic to $({W_0}^{k,M}(\Omega))'$ and the latter is isometrically isomorphic to ${L_l}^N/R$, where ${L_l}^N/R$ is the space of equivalence classes identifying those elements of ${L_l}^N(\Omega)$, which determine the same linear bounded functional over ${W_0}^{k,M}(\Omega)$.

The above remarks show that each element $g \in L^N(\Omega)$ determines a functional $T_g \in ({W_0}^{k,M}(\Omega))'$ by means of

$$T_g(f) = \int_\Omega g(t) f(t) dt.$$

The space $L^N(\Omega)$ may be normed in the following manner. We take the norm $\| \; \|_{-k,N}$ of $g \in L^N(\Omega)$ as the norm of corresponding to the functional T_g, that is

$$\begin{aligned} |g\|_{-k,N} &= \|T_g\|_{({W_0}^{k,M}(\Omega))'} \\ &= \sup \left\{ |T_g(f)| \colon f \in {W_0}^{k,M}(\Omega), \, \|f\|_{{W_0}^{k,M}} \le 1 \right\}. \end{aligned} \tag{4}$$

Then the set $V = \{T_g \colon g \in L^N(\Omega)\}$ is dense in $({W_0}^{k,M}(\Omega))'$, (see [7]). Hence and from (4) follows that $({W_0}^{k,M}(\Omega))'$ is a completion of V with respect to the norm $\| \; \|_{-k,N}$.

Let $H^{-k,N}(\Omega)$ denote the completion of $L^N(\Omega)$ with respect to $\| \ \|_{-k,N}$. Since V and $L^N(\Omega)$ are isometrically isomorphic, then we obtain $H^{-k,N}(\Omega)$ is isometrically isomorphic to the space $(W_0{}^{k,M}(\Omega))'$.

Thus we have

Theorem 2. *If M and N are complementary functions satisfying conditions Δ_2,* (6) *and M satisfies additionally* (5)*, then the space $H^{-k,N}(\Omega)$ is isometrically isomorphic to $W^{-k,N}(\Omega)$.*

Throughout the following discussion we will indicate some class of domains $\Omega \subset R^n$ for which is true that $W^{k,M}(\Omega) = W_0{}^{k,M}(\Omega)$.

Let F be a closed subset of R^n. The closed set F is (k,N)-polar if the only distribution T in $W^{-k,N}(R^n)$ having support in F is the zero distribution, that is $Tf = 0$ for every $f \in W_0{}^{k,M}(R^n)$, ([1]).

For arbitrary nonnegative and integer number k there holds the embedding

$$W_0^{k+1,M}(R^n) \subset W_0^{k,M}(R^n).$$

If M and N are complementary φ-functions, by the inequality $\|u\|_{k,M} \leq \|u\|_{k+1,M}$ we obtain that any bounded linear functional on the $W_0{}^{k,M}(R^n)$ is bounded on the $W_0{}^{k+1,M}(R^n)$ as well. Thus there holds the inclusion

$$W^{-k,N}(R^n) \subset W^{-k-1,N}(R^n). \tag{5}$$

From above remarks it follows immediatelly

Lemma 1. *For each N-function N any $(k+1,N)$-polar set is also (k,N)-polar set.*

Proof. Let $F \subset R^n$ be $(k+1,N)$-polar and let T be any distribution in $W^{-k,N}(R^n)$ such that *supp* $T \subset F$. Hence and by (5) we get $T \in W^{-k-1,N}(R^n)$. Since F is $(k+1,N)$-polar, then $T = 0$. Thus F is (k,N)-polar.

For any function f defined on the open set $\Omega \subset R^n$ we denote by $f^\bullet$ zero extension of f outside Ω

$$f^\bullet(t) = \begin{cases} f(t) & \text{if} \quad t \in \Omega \\ 0 & \text{if} \quad t \in \Omega' = R^n - \Omega \end{cases} \tag{6}$$

■

Lemma 2. *Let both the function M and N complementary to M, satisfy conditions* (5), (6) *and Δ_2. Let $f \in W_0{}^{k,M}(\Omega)$. Then for any $|\alpha| \leq k$ there exists distributional derivative of $f^\bullet$ on R^n and*

$$D^{\alpha}f^{\bullet}(t) = \begin{cases} D^{\alpha}f(t) & \text{if} \quad t \in \Omega \\ 0 & \text{if} \quad t \in \Omega' \end{cases}$$

Moreover, $f^{\bullet} \in W^{k,M}(R^n)$.

Proof. Let $f \in {W_0}^{k,M}(\Omega)$. Then $I\left(\frac{f}{\|f\|_{k,M}}\right) \leq 1$. For any $|\alpha| \leq k$ we have $D^{\alpha}f \in L^M(\Omega)$ and

$$I_0\left(\frac{1}{\|f\|_{k,M}}D^{\alpha}f\right) = \int_{\Omega} M\left(t, \frac{1}{\|f\|_{k,M}}\left|D^{\alpha}f(t)\right|\right)dt \leq I\left(\frac{1}{\|f\|_{k,M}}f\right) \leq 1.$$

Hence we obtain

$$\|D^{\alpha}f\|_{L^M} \leq \|f\|_{k,M}.$$

Let (u_n) be a sequence in ${C_0}^{\infty}(\Omega)$ converning to f in ${W_0}^{k,M}(\Omega)$. Thus, for any $\varphi \in D(R^n)$ and $|\alpha| \leq k$ we have

$$\left|\int_{\Omega} D^{\alpha}f(t)\varphi(t)dt - \int_{\Omega} D^{\alpha}u_n(t)\varphi(t)dt\right| \leq c\, \|f - u_n\|_{k,M}.$$

Hence

$$\begin{aligned} \int_{R^n} f^{\bullet}(t)D^{\alpha}\varphi(t)dt &= \int_{\Omega} f(t)D^{\alpha}\varphi(t)dt = \lim_{n\to\infty}\int_{\Omega} u_n(t)D^{\alpha}\varphi(t)dt \\ &= (-1)^{|\alpha|}\lim_{n\to\infty}\int_{\Omega} D^{\alpha}u_n(t)\varphi(t)dt = (-1)^{|\alpha|}\int_{\Omega} D^{\alpha}f(t)\varphi(t)dt \\ &= (-1)^{|\alpha|}\int_{R^n} \left(D^{\alpha}f\right)^{\bullet}(t)\varphi(t)dt. \end{aligned}$$

Thus $D^{\alpha}f^{\bullet} = (D^{\alpha}f)^{\bullet}$ in the distributional sense on R^n. Hence we obtain

$$\begin{aligned} \int_{\Omega} M\left(t, |D^{\alpha}f(t)|\right)dt &= \int_{R^n} M\left(t, |(D^{\alpha}f)^{\bullet}(t)|\right)dt \\ &= \int_{R^n} M\left(t, |D^{\alpha}f^{\bullet}(t)|\right)dt \end{aligned}$$

for every $|\alpha| \leq k$. Thus $\|f\|_{W^{k,M}(\Omega)} = \|f^{\bullet}\|_{W^{k,M}(R^n)}$. By last equality we conclude $f^{\bullet} \in W^{k,M}(R^n)$. ■

The following theorem delivers a necessary and sufficient condition on Ω that mapping (6) carry ${W_0}^{k,M}(\Omega)$ isometrically onto $W^{k,M}(R^n)$.

Theorem 3. *Let M be an N-function and let N be complementary to M, both functions satisfies* (5) , (6) *and* Δ_2. ${C_0}^{\infty}(\Omega)$ *is dense in* $W^{k,M}(R^n)$ *if and only if the complement* $\Omega' = R^n - \Omega$ *is* (k, N)*-polar.*

Proof. The proof is similar to the proof of the respective theorem for the space $W^{k,p}(\Omega)$, $p \geq 1$ (Theorem 3.23 in [1]).

Let us assume $C_0{}^\infty(\Omega)$ is dense in $W^{k,M}(R^n)$. Let T be any distribution in $W^{-k,N}(R^n)$ such that supp $T \subset \Omega'$. If $f \in W^{k,M}(R^n)$, then there exists a sequence $(u_n) \subset C_0{}^\infty(\Omega)$ converning to f with respect to the norm of $W^{k,M}(R^n)$. By continuity of T we obtain $Tu_n \to Tf$, $n \to \infty$. Since T has the support in $R^n - \Omega$, so $Tu_n = 0$, $n = 1, 2, ...$ and hence $Tf = 0$. Thus Ω' is (k, N)-polar.

Now let us suppose $C_0{}^\infty(\Omega)$ is not dense in $W^{k,M}(R^n)$. Thus there exist an element $f \in W^{k,M}(R^n)$ and a constant $c > 0$ such that $\|f - \varphi\|_{W^{k,M}(R^n)} \geq c$ for every $\varphi \in C_0{}^\infty(\Omega)$ and the constant being independent of φ. By the Hahn-Banach theorem there exists a functional $T \in W^{-k,N}(R^n)$ such that $T\varphi = 0$ for all $\varphi \in C_0{}^\infty(\Omega)$ and $Tf = 1$. Thus we have *supp* $T \subset \Omega'$ but $T \neq 0$. Hence Ω' cannot be (k, N)-polar. ■

For differentiable functions is true that identical vanishing of first derivatives over rectangle $B \subset R^n$ implies constancy of this function on that rectangle. This result has extension to functions possessing distributional derivatives. There holds the following lemma.

Lemma 3. ([1]) *Let*

$$B = (a_1, b_1) \times (a_2, b_2) \times \ ... \ \times (a_n, b_n)$$

be an open rectangular box in R^n and let f possesses distributional derivatives $D^\alpha f = 0$ for all $|\alpha| = 1$. Then there exists a constant c such that $f(t) = c$ almost everywhere in B.

Theorem 4. *Let N-functions M and N satisfy the condition Δ_2 and let M satisfies* (5) *and* (6)*.*

1. *If $C_0{}^\infty(\Omega)$ is dense in $W^{k,M}(\Omega)$, then $\Omega' = R^n - \Omega$ is (k, N)-polar.*
2. *If Ω' is both $(1, M)$-polar and (k, N)-polar, then $C_0{}^\infty(\Omega)$ is dense in $W^{k,M}(\Omega)$.*

Proof. The idea of this proof is derived from the proof of the respective theorem for the space $W^{k,p}(\Omega)$, $p \geq 1$ (Theorem 3.28 in [1]).

1. Let $W^{k,M}(\Omega) = W_0{}^{k,M}(\Omega)$. We shall show that Ω' has measure zero. Let us suppose that Ω' has a positive measure. Then there exists an open rectangle $P \subset R^n$ such that $P \cap \Omega$ and $P \cap \Omega'$ are set of positive measure.

Denote by f the function which is the restriction to Ω of some function $g \in C_0{}^\infty(R^n)$ which is identically one on $P \cap \Omega$. Then $f \in W^{k,M}(\Omega)$ and so $f \in W_0{}^{k,M}(\Omega)$. By Lemma 2 we have $f^\bullet \in W^{k,M}(R^n)$ and $D^\alpha f^\bullet = (D^\alpha f)^\bullet$ in the distributional sense on R^n for $|\alpha| = 1$. Since $D^\alpha f = 0$ on $P \cap \Omega$ then for the zero extension there holds $(D^\alpha f)^\bullet = 0$ on P. Thus also $D^\alpha f^\bullet = 0$

on P for $|\alpha| = 1$ as a distribution on P. By Lemma 3 $f^\bullet$ must have a constant value almost everywhere in P, a contradiction because $f^\bullet(t) = 1$ for $t \in P \cap \Omega$ and $f^\bullet(t) = 0$ for $t \in P \cap \Omega'$. Thus Ω' has measure zero.

Now, we will apply the above fact in order to prove density of ${C_0}^\infty(\Omega)$ in $W^{k,M}(\Omega)$. Let $g \in W^{k,M}(R^n)$ and let f be the restriction of g to Ω. Then $f \in W^{k,M}(\Omega)$ and hence $f \in {W_0}^{k,M}(\Omega)$ Thus f can be approximated by functions of ${C_0}^\infty(\Omega)$. By Lemma 2 $f^\bullet \in W^{k,M}(R^n)$ and $f^\bullet$ can be also approximated by elements of ${C_0}^\infty(\Omega)$. Since $g(t) = f^\bullet(t)$ a.e. in R^n, then g and $f^\bullet$ have the same distributional derivatives and so coincide in $W^{k,M}(R^n)$. Thus g can be approximated by elements ${C_0}^\infty(\Omega)$. Therefore ${C_0}^\infty(\Omega)$ is dense in $W^{k,M}(R^n)$. By Theorem 3 the set Ω' is (k, N)-polar.

2. Let $f \in W^{k,M}(\Omega)$. Since $f \in L^M(\Omega)$, then $f^\bullet \in L^M(R^n)$. Thus $f^\bullet$ generate a distribution $T_{f^\bullet}$ as a locally integrable function in R^n and there exists $D^\alpha T_{f^\bullet}$, $|\alpha| = 1$. Consequently, there exists

$$\int_{R^n} D^\alpha f^\bullet(t)\varphi(t)dt$$

for any $|\alpha| = 1$ and for all $\varphi \in D(R^n)$. This integral ia a regular distribution generated by $D^\alpha f^\bullet$. Thus $D^\alpha T_{f^\bullet} = T_{D^\alpha f^\bullet}$ for $|\alpha| = 1$ and $T_{D^\alpha f^\bullet} \in W^{-1,M}(R^n)$.

We have $D^\alpha f \in L^M(\Omega)$ for any $|\alpha| = 1$. Hence the zero extension $(D^\alpha f)^\bullet$ is an element of $L^M(R^n)$. Since $L^M(R^n) \subset H^{-1,M}(R^n)$, then $T_{(D^\alpha f)^\bullet} \in W^{-1,M}(R^n)$. Hence, we obtain

$$T_{D^\alpha f^\bullet - (D^\alpha f)^\bullet} = T_{D^\alpha f^\bullet} - T_{(D^\alpha f)^\bullet} \in W^{-1,M}(R^n)$$

for all $|\alpha| = 1$. Moreover

$$D^\alpha f^\bullet(t) - (D^\alpha f)^\bullet(t) \; = \; 0$$

for every $t \in \Omega$ So *supp* $T_{D^\alpha f^\bullet - (D^\alpha f)^\bullet} \subset \Omega'$ for all $|\alpha| = 1$. Since Ω' is $(1, M)$-polar we obtain

$$T_{D^\alpha f^\bullet - (D^\alpha f)^\bullet}(\varphi) \; = \; 0$$

for all $\varphi \in D(R^n)$. This implies $D^\alpha f^\bullet = (D^\alpha f)^\bullet$ almost everywhere in R^n for $|\alpha| = 1$. Thus $D^\alpha f^\bullet \in L^M(R^n)$ for $|\alpha| = 1$ and we have $f^\bullet \in W^{1,M}(R^n)$. Using the induction principle with respect $|\alpha|$, we obtain that $D^\alpha f^\bullet = (D^\alpha f)^\bullet$ in the distributional sense, for $|\alpha| \le k$.

Finally $f^\bullet \in W^{k,M}(R^n)$. By (k, M)-polarity of Ω' and Theorem 3 we obtain that ${C_0}^\infty(\Omega)$ is dense in $W^{k,M}(R^n)$. Thus we have the closure of ${C_0}^\infty(\Omega)$ in norm $\| \, \|_{k,M}$ is the space $W^{k,M}(R^n)$. Simultaneously this same closure is, by definition, the space ${W_0}^{k,M}(\Omega)$. Since $f^\bullet \in W^{k,M}(R^n)$ and $f^\bullet(t) = f(t)$ for $t \in \Omega$, then $f \in {W_0}^{k,M}(\Omega)$. ■

Let M_1 and M_2 be φ-functions. *A* function M_2 is nonweaker than M_1 if the following condition holds

$$M_1(t,u) \le K_1 M_2(t, K_2 u) + h(t) \tag{7}$$

for all $u \ge 0$ and a.e. $t \in \Omega$, where h is nonnegative, integrable function in Ω and K_1, K_2 are positive constans.

The condition (7) we write $M_1 \prec M_2$, (see [12]). If the inequality (7) is satisfied for every $u \ge u_0$, where $u_0 > 0$ is fixed, then we say that M_2 is nonweaker than M_1 for large u.

If φ-functions M_1 and M_2 satisfy (6) and $M_1 \prec M_2$, then the embedding

$$W^{k,M_2}(\Omega) \subset W^{k,M_1}(\Omega) \tag{8}$$

holds for every nonnegative integer k. If Ω has finite measure, then embedding (8) holds when $M_1 \prec M_2$ for large u, (see [5]).

Lemma 4. *Let an N-function M satisfies conditions* (5), (6), Δ_2 *and let N be a complementary function to M. The set $F \subset R^n$ is (k,N)-polar if and only if $F \cap K$ is (k,N)-polar for every compact set $K \subset R^n$.*

The proof is analogous as in the case, when $M(t,u) = u^p$, $p > 1$.

Lemma 5. *Let M_1 and M_2 be N-functions satisfying* (5), (6) *and* Δ_2. *Let N_1, N_2 be complementary functions respectively. If $M_1 \prec M_2$ for large u and the set $F \subset R^n$ is (k,N_2)-polar, then F is also (k,N_1)-polar.*

Proof. Let K be a compact set in R^n and let $F \subset R^n$ be (k,N_2)-polar. We will show that $F \cap K$ is (k,N_1)-polar. Let us denote by G an arbitrary open and bounded set in R^n such that $K \subset G$. There holds the embedding ${W_0}^{k,M_2}(G) \subset {W_0}^{k,M_1}(G)$. Hence we have $W^{-k,N_1}(G) \subset W^{-k,N_2}(G)$.

Let $T \in W^{-k,N_1}(R^n)$ be such that *supp* $T \subset F \cap K$. Then $T \in W^{-k,N_1}(G)$ and hence $T \in W^{-k,N_2}(G)$. Since $F \cap K$ is (k,N_2)-polar, so $T = 0$. Thus $F \cap K$ is (k,N_1)-polar and, by Lemma 4, F is (k,N_1)-polar. ■

Theorem 5. *Let M and N be complementary N-functions satisfying the condition Δ_2 and let M satisfies* (5) *and* (6). *Let furthermore M be such that the complementary function N satisfies $N \prec M$ for large u. Then $W^{k,M}(\Omega) \subset {W_0}^{k,M}(\Omega)$ if and only if the set Ω' is (k,N)-polar.*

Proof. Let us suppose Ω' is (k,N)-polar. Since $N \prec M$ then, by Lemma 5, Ω' is (k,M)-polar. Thus Ω' is $(1,M)$-polar. The result now follows by Theorem 4(1). ■

Examples.

1. Let $M(t,u) = u^{p(t)}$, where $1 \leq p(t) < \infty$. If $p(t) \geq 2$, then the complementary function N is $N(t,u) = u^{q(t)}$, where $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$ and $q(t) \leq p(t)$ for every $t \in \Omega$. So $N \prec M$ for $u \geq 1$ and $h = 0$.

2. Let $M(u) = e^u - u - 1$, $u \geq 0$. Then the complementary function is $N(u) = (1+u)\ln(1+u) - u$ satisfies $N \prec M$ for all $u \geq 0$.

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MARIAN LISKOWSKI
INSTITUTE OF MATHEMATICS, POZNAŃ UNIVERSITY OF TECHNOLOGY
PIOTROWO 3A, 60-965 POZNAŃ, POLAND
e-mail: mliskows@math.put.poznan.pl

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