F A S C I C U L I M A T H E M A T I C I

Nr 36 2005

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REMARKS ON FIRST ORDER IMPULSIVE ORDINARY DIFFERENTIAL EQUATIONS WITH ANTI-PERIODIC BOUNDARY CONDITIONS

Abstract: The paper deals with the anti-periodic boundary value problem for impulsive ordinary differential equations. The impulsive differential inequalities generated by this problem are considered and a uniqueness criterion is obtained.

Key words: impulsive ordinary differential equations, coupled lower and upper solutions, uniqueness criterion.

1. Introduction

The theory of impulsive differential equations has become an important area of investigation. In recent years many authors have considered different problems involving impulsive differential equations. We mention here papers [1]-[9].

The anti-periodic solutions are not as widely discussed but there are some papers on this subject: [5], [6], [8]. In this paper we give simple remarks on lower and upper solutions for first order impulsive ordinary differential equations with anti-periodic boundary conditions. As an application some uniqueness result is obtained. We assume, different than usually, that the solution is right continuous. The method of upper and lower solution coupled with the monotone iterative technique has been widely used in the treatment of differential equations in [5], [8], [9].

2. Preliminaries

We consider the following anti-periodic boundary value problem

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \in J, \ \ t \neq t_k, \\ \triangle x \,|_{t=t_k} &= I_k(x(t_k)), \quad k = 1, ..., p, \\ x(0) &= -x(T), \end{aligned} \tag{1}$$

where $J = [0, T]$, $0 < t_1 < t_2 < ... < t_p < t_{p+1} = T$, are fixed points $f : J \times R \to R$ and $I_k : R \to R$ for each $k = 1, 2, ..., p$.

Denote by $PC[J, R]$ the set of all functions $x : J \to R$ which are continuous at $t \neq t_k$, $x(t^-)$ and $x(t^+)$ exist, and $x(t^+) = x(t)$ for $t = t_k$, $k = 1, 2, ..., p$. Let $J' = J \setminus \cup_{i=1}^{p} \{t_i\}$. We denote by $PC^1[J, R]$ the set of all functions $x \in PC[J, R]$ that are continuously differentiable for $t \in J'$; $x'(0^+)$, $x'(T^-)$, $x'(t_k^+)$, $x'(t_k^-)$ exist, $k = 1, ..., p$.

Definition 1. *We say that a function x is a solution for Eq.* (1) *if $x \in PC^1[J, R]$ and it satisfies Eq.* (1)*.*

It is possible to define the concept of lower and upper solution for Eq. (1) as follows (see [5]).

Definition 2. *We say that $u \in PC^1[J, R]$ and $v \in PC^1[J, R]$ are pair of lower and upper related solutions for Eq.* (1) *if they satisfy*

$$\begin{aligned} &u'(t) \le f(t, u(t)), \quad t \in J', \\ &\triangle u \,|_{t=t_k} \le I_k(u(t_k)), \quad k = 1, ..., p, \\ &u(0) + v(T) \le 0 \end{aligned}$$

and

$$\begin{aligned} &v'(t) \ge f(t, v(t)), \quad t \in J', \\ &\triangle v \,|_{t=t_k} \ge I_k(v(t_k)), \quad k = 1, ..., p, \\ &v(0) + u(T) \ge 0. \end{aligned}$$

The following theorem is the most important result of this section:

Theorem 1. *Let u and v be a pair of lower and upper related solution of Eq.* (1)*. Suppose that:*

(i) *For each $t \in J$ the function $f(t, \cdot) : R \to R$ is strictly decreasing on R.*

(ii) *For each $k = 1, ..., p$ the function $I_k : R \to R$ is strictly decreasing on R.*

Then we have

$$u(t) \le v(t), \qquad t \in J. \tag{2}$$

Proof. Let $\epsilon = \sup\{u(t) - v(t),\ t \in J\}$. Suppose (2) is not true, then $\epsilon > 0$ and there exists $\bar{t} \in J$ such that

$$\begin{aligned} &u(\bar{t}) - v(\bar{t}) = \epsilon, \\ &u(t) - v(t) \le \epsilon, \quad t \in J. \end{aligned} \tag{3}$$

We consider the following four cases.

Case 1: Suppose that $\bar{t} \in (0,T)$, $\bar{t} \neq t_k$, $k = 1,...,p$. Since $u - v$ attains its maximum at $t = \bar{t}$ we have $(u-v)'(\bar{t}) = 0$. On the other hand from the definition of the functions u and v it follows that

$$0 \;=\; (u-v)'(\bar{t}) \;\leq\; f(\bar{t}, u(\bar{t})) - f(\bar{t}, v(\bar{t})).$$

Then

$$f(\bar{t}, u(\bar{t})) \;\geq\; f(\bar{t}, v(\bar{t}))$$

which is a contradiction because for each $t \in J$, the function $f(t,\cdot) : R \to R$ is strictly decreasing on R.

Case 2: Suppose that $\bar{t} = t_k$ for some $k, 1 \leq k \leq p$. Then we have

$$\begin{aligned} u(t_k) - v(t_k) \;&=\; \epsilon, \\ u(t_k^-) - v(t_k^-) \;&\leq\; \epsilon. \end{aligned} \tag{4}$$

From the definition of the function u, v and by the monotone character of the function I_k, we obtain

$$u(t_k) - v(t_k) \;\leq\; u(t_k^-) + I_k(u(t_k)) - [v(t_k^-) + I_k(v(t_k))] \;<\; \epsilon,$$

which is a contradiction with (4).

Case 3: If $u(t) - v(t) < \epsilon$ for all $t \in J$, then

$$u(t_k^-) - v(t_k^-) \;=\; \epsilon$$

for some $1 \leq k \leq p+1$.

Let $m : [t_{k-1}, t_k] \to R$ be defined by

$$m(t) = \begin{cases} u(t) - v(t), & t \in [t_{k-1}, t_k), \\ \epsilon, & t = t_k. \end{cases} \tag{5}$$

Thus m is continuous function. Since $m(t) < \epsilon$ for $t \in [t_{k-1}, t_k)$, then there exists a sequence $\{\tau_\nu\}$ such that $\tau_\nu \in [t_{k-1}, t_k)$, $\tau_\nu < \tau_{\nu+1}$ and $\lim_{\nu\to\infty} \tau_\nu = t_k$ and $D_- m(\tau_\nu) \geq 0$ for $\nu = 1, 2, ...,$ where $D_- m(\tau_\nu) = \liminf_{h\to 0^-} \dfrac{m(\tau_\nu + h) - m(\tau_\nu)}{h}$.

Let N_0 be such a positive integer that for $\nu \geq N_0$ we have $u(\tau_\nu) - v(\tau_\nu) > 0$. Then it follows that

$$0 \;\leq\; D_- m(\tau_\nu) \;\leq\; u'(\tau_\nu) - v'(\tau_\nu) \;\leq\; f(\tau_\nu, u(\tau_\nu)) - f(\tau_\nu, v(\tau_\nu)).$$

Thus,

$$f(\tau_\nu, u(\tau_\nu)) - f(\tau_\nu, v(\tau_\nu)) \;\geq\; 0$$

which contradicts condition (i) of the theorem.

Case 4: Suppose that $\bar{t} = 0$. Then we have

$$u(0) - v(0) = \epsilon,$$
$$u(t) - v(t) \le \epsilon, \quad t \in J.$$

In particular $u(T) - v(T) \le \epsilon$. From the definition of the functions u, v we have

$$\epsilon = u(0) - v(0) \le -v(T) + u(T) \le \epsilon.$$

Hence

$$u(T) - v(T) = \epsilon\,.$$

However, this imply again a contradiction as in Case 3. The proof of Theorem 1 is completed. ■

Example 1. Let $J = [0, 1]$, $p = 1$, $t_1 = \frac{1}{2}$ and suppose that $f(t, x) = e^{-x}$, $I_1(x) = -12x$, $x \in R$. Conditions (i) and (ii) of Theorem 1 are fulfilled. For instance,

$$u(t) = \begin{cases} -\dfrac{t}{2} - 3, & t \in [0, \dfrac{1}{2}) \\ -\dfrac{t}{2}, & t \in [\dfrac{1}{2}, 1] \end{cases} \qquad \text{and} \qquad v(t) = \begin{cases} t + 1, & t \in [0, \dfrac{1}{2}) \\ t - \dfrac{1}{4}, & t \in [\dfrac{1}{2}, 1] \end{cases}$$

are pair of lower and upper related solutions for Eq. (1). Emploing Theorem 1 we obtain $u(t) \le v(t)$, $t \in [0, 1]$.

We apply Theorem 1 to obtain a uniqueness result for the problem (1).

Theorem 2. *Let the following conditions hold:*

(i) *For each $t \in J$ the function $f(t, \cdot) : R \to R$ is strictly decreasing on R.*

(ii) *For each $k = 1, ..., p$ the function $I_k : R \to R$ is strictly decreasing on R.*

Then problem (1) *has at most one solution in the class $PC^1[J, R]$.*

Proof. Let $u_1, u_2 \in PC^1[J, R]$ be two distinct solutions of (1). From boundary conditions we have

$$u_1(0) + u_1(T) = 0,$$
$$u_2(0) + u_2(T) = 0.$$

Hence

$$u_1(0) + u_1(T) + u_2(0) + u_2(T) = 0,$$
$$u_1(0) + u_2(T) = -(u_1(T) + u_2(0)).$$

We have either $u_1(0) + u_2(T) \geq 0$ or $u_1(0) + u_2(T) \leq 0$.
We assume without of generality that $u_1(0)+u_2(T) \geq 0$ and $u_1(T)+u_2(0) \leq 0$. We have

$$\begin{aligned} u_1'(t) &= f(t, u_1(t)), \quad t \in J', \\ \triangle u_1 \mid_{t=t_k} &= I_k(u_1(t_k)), \quad k = 1, ..., p, \\ u_1(0) + u_2(T) &\geq 0 \end{aligned}$$

and

$$\begin{aligned} u_2'(t) &= f(t, u_2(t)), \quad t \in J', \\ \triangle u_2 \mid_{t=t_k} &= I_k(u_2(t_k)), \quad k = 1, ..., p, \\ u_2(0) + u_1(T) &\leq 0. \end{aligned}$$

Emploing Theorem 1 we obtain

$$u_2(t) \leq u_1(t), \qquad t \in J.$$

In particular $u_2(0) \leq u_1(0)$, $u_2(T) \leq u_1(T)$, This together with boundary conditions yields

$$0 \leq u_1(0) - u_2(0) = -u_1(T) + u_2(T) \leq 0.$$

Hence $u_1(0) = u_2(0)$, $u_1(T) = u_2(T)$. The functions u_1, u_2 satisfy the following systems

$$\begin{aligned} u_1'(t) &= f(t, u_1(t)), \quad t \in J', \\ \triangle u_1 \mid_{t=t_k} &= I_k(u_1(t_k)), \quad k = 1, ..., p, \\ u_1(0) + u_2(T) &= 0 \end{aligned}$$

and

$$\begin{aligned} u_2'(t) &= f(t, u_2(t)), \quad t \in J', \\ \triangle u_2 \mid_{t=t_k} &= I_k(u_2(t_k)), \quad k = 1, ..., p, \\ u_2(0) + u_1(T) &= 0. \end{aligned}$$

Using Theorem 1 we obtain that
(i) $u_1(t) \leq u_2(t), \quad t \in J,$
(ii) $u_1(t) \geq u_2(t), \quad t \in J,$
which proves the statement of the theorem. ■

Example 2. Suppose that $J = [0, T]$, $0 < t_1 < t_2 < ... < t_p < t_{p+1} = T$ are fixed points, $f(t, u) = -u^5$, $I_k(u) = -u^3$, $t \in J$, $u \in R$, $k = 1, ..., p$. Then the unique anti-periodic solution of the problem

$$\begin{aligned} u'(t) &= -u^5, \quad t \in J', \\ \triangle u \mid_{t=t_k} &= -u^3(t_k), \quad k = 1, ..., p, \\ u(0) + u(T) &= 0 \end{aligned}$$

is the zero solutions.

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Received on 22.01.2004 and, in revised from, on 09.06.2004.