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ON THE BOUNDS FOR MEAN-VALUES OF SOLUTIONS TO CERTAIN THIRD-ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

Abstract: In this paper, the issue of bounds for the mean-values of solutions of some third-order differential equations with non-linear terms is considered. It is shown that these bounds are independent of the solutions for the considered equations.

Key words: Mean-values, bounded solutions, and third-order equations.

1. Introduction and Basic Notions

The stimulation of this work comes from the paper [7] of Barbălat and Halanay, where bounds for the mean values of solutions as well as their first two derivatives for Liénard and Rayleigh differential equations were considered when such solutions are bounded and globally exponentially stable.

The problem of interest here is to determine the bounds for the mean values of solutions of third-order differential equations of the form:

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = q(t) \tag{1}$$

and

$$\dddot{x} + a\ddot{x} + g_1(x)\dot{x} + h(x) = q(t) \tag{2}$$

where $a > 0$ and functions g, g_1, h and q are continuous in their respective arguments. A natural question arises: how these bounds affect the solutions and the forcing term $q(t)$ of the equations (1) and (2)? We shall later provide answers to these questions under the condition that the solutions considered are bounded and globally exponentially stable.

Already, it has been shown in [1], that subject to the functions g, g_1 and h satisfying

$$b \leq \frac{g(z) - g(\bar{z})}{z - \bar{z}} \leq b + \mu_1, \quad z \neq \bar{z}, \tag{3}$$

$$b \leq \frac{1}{z}\int_0^z g_1(s)ds \leq b+\mu_1, \quad z \neq 0, \tag{4}$$

$$c \leq \frac{h(z)-h(\bar{z})}{z-\bar{z}} \leq c+\mu_2, \quad z \neq \bar{z}, \tag{5}$$

for some positive constants b, c and positive parameters μ_1 and μ_2, the equations (1) and (2) have unique bounded solutions which are globally exponentially stable if $ab - c > \mu_2$. In the generalized topological degree methods developed by Mahwin [8], one of the key steps is to find bounds for the mean values of solutions of differential equations. These bounds make it possible to give interesting conclusions about the existence of periodic solutions and even some other qualitative properties of such solutions (see e.g [2-6] and references therein).

2. Notations

Let $x(t)$ be a solution of the equation (1) and denote the mean-values of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, and $\dddot{x}\ (t)$ for $T > 0$, by

$$(*) \qquad \begin{array}{lcl} K^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt; \\ L^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T \dot{x}^2(t)dt; \\ M^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T \ddot{x}^2(t)dt; \\ N^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T \dddot{x}^2\ (t)dt. \end{array}$$

Moreover, if $x(t)$ is a solution of equation (2), let us denote its mean-values by

$$(**) \qquad \begin{array}{lcl} {K_1}^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt; \\ {L_1}^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T \dot{x}^2(t)dt; \\ {M_1}^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T \ddot{x}^2(t)dt; \\ {N_1}^2 & = & \limsup\limits_{T\to\infty} \frac{1}{T}\int_0^T \dddot{x}^2\ (t)dt. \end{array}$$

3. Main Results

Before we state the main results of this paper, let us remark that if equations (1) and (2) admit a bounded and globally exponentially stable solution, then the mean values stated in Section Two do not depend on the solution considered, and hence represent general characteristics of the equations.

Throughout this paper it is assumed that for every $t \geq 0$ the solutions of the equations (1) and (2) exist. Moreover, the functions g, g_1, h and q are real valued and continuous in their respective arguments. We now state our main results.

Theorem 1. *If the solutions of the equation* (1) *are bounded and globally exponentially stable, then the mean-values K, L, M and N of any solution $x(t)$, are bounded independent of $x(t)$, with bounds given by*

$$(***)\qquad \begin{aligned} K &\leq \tfrac{A}{c}\left(1 + \left[\mu_1 + (ac)^{\frac{1}{2}}\right]\omega\right); \\ L &\leq A\omega; \\ M &\leq \tfrac{A}{2a}\omega_1; \quad \textit{and} \\ N &\leq \tfrac{A}{2a}\left(a + \left[a^2 + 2(b+\mu_1)(\omega_1 + 2a(c+\mu_2)\omega^2)\right]^{\frac{1}{2}}\right); \end{aligned}$$

where

$$\begin{aligned} A^2 &= \limsup_{T\to\infty} \tfrac{1}{T}\int_0^T q^2(t)dt; \\ \omega &= \frac{\left(a + [\frac{c+\mu_2}{a}]^{\frac{1}{2}} + \left[a^2 + b + 2(a(c+\mu_2))^{\frac{1}{2}} + \frac{3(ab-c-\mu_2)}{a}\right]^{\frac{1}{2}}\right)}{2(ab-c-\mu_2)}; \quad \textit{and} \\ \omega_1 &= \left(1 + \left[1 + 4a(c+\mu_2)\omega^2\right]\right). \end{aligned}$$

Theorem 2. *If the solutions of the equation* (2) *are bounded and globally exponentially stable such that $g_1(x(t))$ satisfies*

$$b \leq \frac{1}{x(t)}\int_0^{x(t)} g_1(s)ds \leq b + \mu_1, \quad x(t) \neq 0,$$

then the mean-values K_1, L_1, M_1 and N_1 of any solution $x(t)$, are bounded independent of $x(t)$, with bounds given by

$$\begin{aligned} K_1 &\leq \tfrac{A}{2c}\left(1 + \left[1 + 4ac\omega^2\right]^{\frac{1}{2}}\right); \\ L_1 &\leq A\,\omega; \\ M_1 &\leq \tfrac{A}{2a}\,\omega_1; \quad \textit{and} \\ N_1 &\leq \tfrac{A}{2a}\left(a + \left[a^2 + 2(b+\mu_1)(\omega_1 + 2a(c+\mu_2)\omega^2)\right]^{\frac{1}{2}}\right); \end{aligned}$$

where ω and ω_1 are as given in Theorem 1.

4. Preliminary Results

Let us denote by $\alpha_i(T)$, $\beta_j(T)$ and $\gamma_\ell(T)$, $(i = 1,\ldots,6; j = 1,\ldots,6; \ell = 1,2,3)$, functions of T, $T > 0$, such that

$$\lim_{T\to\infty} \alpha_i(T) = \lim_{T\to\infty} \beta_j(T) = \lim_{T\to\infty} \gamma_\ell(T) = 0.$$

Suppose that the solution $x(t)$ of the equation (1) is bounded and globally exponentially stable, then we have the following results.

Lemma 1. *For $T > 0$, the following identities are valid:*

$$\begin{array}{lll} (i) & \frac{1}{T}\int_0^T \ddot{x}(t)\dddot{x}(t)dt & = \alpha_1(T); \\ (ii) & \frac{1}{T}\int_0^T \ddot{x}(t)\dot{x}(t)dt & = \alpha_2(T); \\ (iii) & \frac{1}{T}\int_0^T \dot{x}(t)x(t)dt & = \alpha_3(T); \\ (iv) & \frac{1}{T}\int_0^T x(t)\dddot{x}(t)dt & = \alpha_4(T), \\ (v) & \frac{1}{T}\int_0^T \dot{x}(t)\dddot{x}(t)dt & = \alpha_5(T) - \frac{1}{T}\int_0^T \ddot{x}^2(t)dt; \\ (vi) & \frac{1}{T}\int_0^T x(t)\ddot{x}(t)dt & = \alpha_6(T) - \frac{1}{T}\int_0^T \dot{x}^2(t)dt. \end{array}$$

Proof. Identities (i), (ii) and (iii) follow easily, since

$$\begin{array}{lll} \frac{2}{T}\int_0^T \dddot{x}(t)\ddot{x}(t)dt & = & \frac{2}{T}\left[\ddot{x}^2(T) - \ddot{x}^2(0)\right]; \\ \frac{2}{T}\int_0^T \ddot{x}(t)\dot{x}(t)dt & = & \frac{1}{T}\left[\dot{x}^2(T) - \dot{x}^2(0)\right]; \\ \frac{2}{T}\int_0^T \dot{x}(t)x(t)dt & = & \frac{1}{T}\left[x^2(T) - x^2(0)\right]. \end{array}$$

Integrating by parts the product of $\frac{1}{T}$ and each of $\int_0^T x(t)\dddot{x}(t)dt$, $\int_0^T \dot{x}(t)\dddot{x}(t)dt$ and $\int_0^T x(t)\ddot{x}(t)dt$, we obtain identities (iv), (v) and (vi) respectively. ■

Lemma 2. *Let $g(\dot{x}(t)) = b\dot{x}(t) + \hat{g}(\dot{x}(t))$, where $b > 0$ and $\hat{g}(\dot{x}(t))$ is the non-linear part of $g(\dot{x}(t))$. Suppose that g satisfies inequalities (3), then, for $T > 0$,*
(i) $\frac{1}{T}\int_0^T \ddot{x}(t)g(\dot{x}(t))dt = \beta_1(T)$;
(ii) $\frac{1}{T}\int_0^T \dot{x}(t)g(\dot{x}(t))dt \geq \frac{b}{T}\int_0^T \dot{x}^2(t)dt$;
(iii) $\frac{1}{T}\int_0^T \dddot{x}(t)g(\dot{x}(t))dt = \beta_2(T) - \frac{1}{T}\int_0^T \ddot{x}(t)d\hat{g}(\dot{x}(t))dt - \frac{b}{T}\int_0^T \ddot{x}^2(t)dt$; *and*
(iv) $|\frac{1}{T}\int_0^T x(t)g(\dot{x}(t))dt| \leq \beta_3(T) + \mu_1\left(\frac{1}{T}\int_0^T \dot{x}^2(t)dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T x^2(t)dt\right)^{\frac{1}{2}}$.

Proof. As in Lemma 1, identity (i) follows from

$$\frac{1}{T}\int_0^T \ddot{x}(t)g(\dot{x}(t))dt = \frac{1}{T}\left[G(\dot{x}(T)) - G(\dot{x}(0))\right] \quad \text{where} \quad G(z) = \int_0^z g(s)ds.$$

Also if $g(z) = bz + \hat{g}(z)$, $b > 0$, then $\hat{g}$ satisfies

$$\hat{g}(0) = 0 \quad 0 \leq \frac{\hat{g}(z_1) - \hat{g}(z_2)}{z_1 - z_2} \leq \mu_1, \quad (z_1 \neq z_2). \tag{6}$$

Thus $\hat{g}$ satisfies

$$z(t)\hat{g}(z(t)) \geq 0, \quad \text{for all} \quad z(t).$$

We therefore have

$$\frac{1}{T}\int_0^T \dot{x}(t)g(\dot{x}(t))dt \geq \frac{b}{T}\int_0^T \dot{x}^2(t)dt,$$

and it is (ii).

In a similar way, we obtain identity (iii), if we integrate by parts and we take into account the identity (v) of Lemma 1 and we put

$$\beta_2(T) = b\alpha_5(T) + \frac{\ddot{x}(T)\hat{g}(\dot{x}(T)) - \ddot{x}(0)\hat{g}(\dot{x}(0))}{T}.$$

Finally, by the definition of $g(z)$ and Schwarz's inequality, we obtain (iv) with $\beta_3(T) = b\alpha_3(T)$. ■

Lemma 3. *Let $h(x(t)) = cx(t) + \hat{h}(x(t))$, where $c > 0$ and $\hat{h}(x(t))$ is the non-linear part of $h(x(t))$. Suppose that h satisfies inequalities (5). Then,*
(i) $\frac{1}{T}\int_0^T \dot{x}(t)h(x(t))dt = \beta_4(T)$;
(ii) $\frac{1}{T}\int_0^T x(t)h(x(t))dt \geq \frac{c}{T}\int_0^T x^2(t)dt$;
(iii) $\frac{1}{T}\int_0^T \ddot{x}(t)h(x(t))dt = \beta_5(T) - \frac{1}{T}\int_0^T \dot{x}(t)d\hat{h}(\dot{x}(t))dt - \frac{c}{T}\int_0^T \dot{x}^2(t)dt$; *and*
(iv) $\frac{1}{T}\int_0^T \dddot{x}(t)h(x(t))dt = \beta_6(T)$.

Proof. This is similar to the proof of Lemma 2. We use the fact that $h(0) = 0$ and for $x_1(t) \neq x_2(t)$;

$$0 \leq \frac{h(x_1(t)) - h(x_2(t))}{x_1(t) - x_2(t)} \leq \mu_2.$$

■

5. Proof of the Main Results

Proof of Theorem 1. We multiply the equation (1) by $\frac{1}{T}\ddot{x}(t)$ and integrate it from 0 to T, and we obtain

$$\frac{1}{T}\int_0^T \dddot{x}(t)\ddot{x}(t)dt + \frac{a}{T}\int_0^T \ddot{x}(t)dt + \frac{1}{T}\int_0^T \ddot{x}(t)g(\dot{x}(t))dt + \frac{1}{T}\int_0^T \ddot{x}(t)h(x(t))dt = \frac{1}{T}\int_0^T \ddot{x}(t)q(t)dt.$$

By Lemmas 1, 2, 3 and Schwarz's inequality, we have

$$\alpha_1(T) + \frac{a}{T}\int_0^T \ddot{x}^2(t)dt + \beta_1(T) + \beta_5(T) \leq \frac{c+\mu_2}{T}\int_0^T \ddot{x}^2(t)dt + \left(\frac{1}{T}\int_0^T \ddot{x}^2(t)dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T q^2(t)dt\right)^{\frac{1}{2}}.$$

Taking limits as $T \to \infty$, we have

$$a\ M^2 \ \le\ A\ M + (c+\mu_2)L^2 \tag{7}$$

where

$$A^2 \ =\ \limsup_{T\to\infty}\,(\frac{1}{T})\int_0^T q^2(t)dt$$

is the mean-value of $q(t)$ and M, L are as given in Section Two. Next, we multiply the equation (1) by $\frac{1}{T}\dot{x}(t)$ and integrate from 0 to T. Then, by Lemmas 1, 2, 3 and Schwarz's inequality, we obtain

$$\begin{aligned}\alpha_5(T)\ +\ a\alpha_3(T)\ +\ &\frac{b}{T}\int_0^T \dot{x}^2(t)dt + \beta_4(T)\\ &\le\ \frac{1}{T}\int_0^T \ddot{x}^2(t)dt + \left(\frac{1}{T}\int_0^T \dot{x}^2(t)dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T q^2(t)dt\right)^{\frac{1}{2}}.\end{aligned}$$

Taking limits as $T \to \infty$, we have

$$bL^2\ \le\ AL + M^2. \tag{8}$$

From inequality (7), we obtain

$$M\ \le\ \frac{1}{2a}\left(A + \left[A^2 + 4a(c+\mu_2)L^2\right]^{\frac{1}{2}}\right).$$

Substituting this in (8) and solving for L, we have

$$L\ \le\ A\ \omega \tag{9}$$

where

$$\omega\ =\ \frac{\left(a + [\frac{c+\mu_1}{a}]^{\frac{1}{2}} + \left[a^2 + b + 2(a(c+\mu_2))^{\frac{1}{2}} + \frac{3(ab-c-\mu_2)}{a}\right]^{\frac{1}{2}}\right)}{2(ab-c-\mu_2)}.$$

At Last, when this is put in M, we have

$$M\ \le\ \frac{A}{2a}\left(1 + \left[1 + 4a(c+\mu_2)\omega^2\right]^{\frac{1}{2}}\right). \tag{10}$$

For the remaining part of the proof, we multiply the equation (1) by $\frac{1}{T}x(t)$ and we integrate from 0 to T; and making use of Lemmas 1, 2, 3 and Schwarz's inequality, and this way we obtain

$$\begin{aligned}\alpha_4(T) + a\alpha_6(T) + \frac{c}{T}\int_0^T x^2(t)dt - \frac{a}{T}\int_0^T \dot{x}^2(t)dt \le \mu_1\left(\frac{1}{T}\int_0^T x^2(t)dt\right)^{\frac{1}{2}}\\ \times\left(\frac{1}{T}\int_0^T \dot{x}^2(t)dt\right)^{\frac{1}{2}} + \left(\frac{1}{T}\int_0^T q^2(t)dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T x^2(t)dt\right)^{\frac{1}{2}} + \beta_3(T).\end{aligned}$$

Taking limits as $T \to \infty$, we have

$$c\,K^2 \;\leq\; (A+\mu_1 L)K + a\;L^2 \tag{11}$$

where K is as given in Section Two. Substituting for L from inequality (9), we obtain

$$\begin{aligned} K \;&\leq\; \frac{A}{2a}\left(1+\mu_1\omega+\left[(1+\mu_1\omega)^2+4ac\omega^2\right]^{\frac{1}{2}}\right) \\ &\leq\; \frac{A}{c}\left[1+\left(\mu_1+(ac)^{\frac{1}{2}}\right)\omega\right]. \end{aligned} \tag{12}$$

Finally, to complete the proof of the theorem, we multiply the equation (1) by $\frac{1}{T}\dddot{x}\,(t)$ and integrate from 0 to T. By Lemmas 1, 2, 3 and Schwarz's inequality, we have

$$\begin{aligned} \frac{1}{T}\int_0^T \dddot{x}\,(t)dt + a\alpha_1(T)+\beta_2(T)+\beta_6(T) \leq \frac{b+\mu_1}{T}\int_0^T \ddot{x}^2(t)dt \\ + \left(\frac{1}{T}\int_0^T \dddot{x}^2\,(t)dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T q^2(t)dt\right)^{\frac{1}{2}}. \end{aligned}$$

Taking limits as $T \to \infty$, we obtain

$$N^2 \;\leq\; AN+(b+\mu_1)M^2 \tag{13}$$

where N is as given in Section 2. Substituting for M from inequality (10), and solving for N, we have

$$N \;\leq\; \frac{A}{2a}\left(a+\left[a^2+2(b+\mu_1)\left(\omega_1+2a(c+\mu_2)\omega^2\right)\right]^{\frac{1}{2}}\right),$$

and the theorem is proved. ■

Proof of Theorem 2. Let $x(t)$ be a bounded and globally exponentially stable solution of equation (2) and

$$\int_0^{x(t)} g_1(s)ds \;=\; bx(t)+\hat{g}_1(x(t)), \quad b>0,$$

then from the inequality

$$b \;\leq\; \frac{1}{x(t)}\int_0^{x(t)} g_1(s)ds \;\leq\; b+\mu_1, \;\; x(t)\neq 0,$$

we have

$$0 \leq \frac{\hat{g}_1(x(t))}{x(t)} \leq \mu_1, \quad (x(t)\neq 0).$$

Now

$$\begin{aligned}\frac{1}{T}\int_0^T x(t)\dot{x}(t)g_1(x(t))dt &= \frac{b}{2T}\int_0^T d(x^2(t)) + \frac{1}{T}\int_0^T x(t)d\hat{g}_1(x(t))\\ &= \frac{b}{2T}\int_0^T d(x^2(t)) + \frac{[x(T)\hat{g}_1(x(t)) - x(0)\hat{g}_1(x(0))]}{T}\\ &\quad - \frac{\left[\hat{G}_1(x(T)) - \hat{G}_1(x(0))\right]}{T}\\ &:= \gamma_1(T) \quad \text{where} \quad \hat{G}_1(z) = \int_0^z g_1(s)ds.\end{aligned}$$

From the inequality

$$b \le \frac{1}{x(t)}\int_0^{x(t)} g_1(s)ds \le b+\mu_1, \quad x(t)\neq 0,$$

follows $\frac{1}{T}\int_0^T \dot{x}^2(t)g_1(x(t))dt \ge \frac{b}{T}\int_0^T \dot{x}^2(t)dt$. Also since

$$\begin{aligned}\frac{1}{T}\int_0^T \ddot{x}(t)\dot{x}(t)g_1(x(t))dt &= \frac{b}{T}\int_0^T \ddot{x}(t)\dot{x}(t)dt + \frac{1}{T}\int_0^T \ddot{x}(t)d\hat{g}_1(x(t))\\ &\le (b+\mu_1)\alpha_2(T),\end{aligned}$$

then, inequality $\frac{1}{T}\int_0^T \ddot{x}(t)\dot{x}(t)g_1(x(t))dt \le \gamma_2(T)$ follows with $\gamma_2 = (b+\mu_1)\alpha_2(T)$. Finally, by Lemma 1 and the definition of $g_1(x(t))$, we have

$$\frac{1}{T}\int_0^T \dddot{x}(t)\dot{x}(t)g_1(x(t))dt = b\alpha_5(T) - \frac{b}{T}\int_0^T \ddot{x}^2(t)dt + \frac{1}{T}\int_0^T \dddot{x}(t)d\hat{g}_1(x(t)).$$

But by Lemma 1,

$$\left|\frac{1}{T}\int_0^T \dddot{x}(t)d\hat{g}_1(x(t))\right| \le \mu_1\alpha_5(T) - \frac{\mu_1}{T}\int_0^T \ddot{x}^2(t)dt.$$

Thus

$$\frac{1}{T}\int_0^T \dddot{x}(t)\dot{x}(t)g_1(x(t))dt \le \gamma_3(T) - \frac{b+\mu_1}{T}\int_0^T \ddot{x}^2(t)dt,$$

where $\gamma_3(T) = (b+\mu_1)\alpha_5(T)$.

For the completion of the Proof of the Theorem 2, let us proceed as in the proof of the Theorem 1; we first multiply the equation (2) in turns by $\frac{1}{T}\ddot{x}(t)$ and $\frac{1}{T}\dot{x}(t)$. Integrating the resulting equation from 0 to T and applying Lemmas 1 and 3, and Schwarz's inequality and finally taking limits as $T\to\infty$, we obtain

$$aM_1{}^2 \le AM_1 + (c+\mu_2)L_1{}^2 \tag{14}$$

and

$$bL_1{}^2 \leq AL_1 + M_1{}^2. \tag{15}$$

These are equivalent to inequalities (7) and (8), respectively. Hence when solved in a similar manner, we have

$$L_1 \leq A\omega \tag{16}$$

and

$$M_1 \leq \frac{A}{2a}\,\omega_1. \tag{17}$$

Secondly, we multiply the equation (2) by $\frac{1}{T}x(t)$ and integrate from 0 to T. Using Lemmas 1 and 3, and Schwarz's inequality, we obtain on taking the limits as $T \to \infty$,

$$cK_1{}^2 \leq AK_1 + aL_1{}^2. \tag{18}$$

Substituting for L_1 from inequality (17), and solving for K_1, we have

$$K_1 \leq \frac{A}{2c}\left[1 + (1 + 4ac\omega^2)^{\frac{1}{2}}\right]. \tag{19}$$

At last, we multiply the equation (2) by $\frac{1}{T}\ddot{x}(t)$ and integrate from 0 to T. Again, using Lemmas 1, 3, 6 and Schwarz's inequality, we obtain when taking the limits as $T \to \infty$,

$$N_1{}^2 \leq AN_1 + (b + \mu_1)M_1{}^2. \tag{20}$$

Making use of the estimation (19) we have

$$N_1 \leq \frac{A}{2a}\left(a + \left[a^2 + 2(b+\mu_1)\left(\omega_1 + 2a(c+\mu_2)\omega^2\right)\right]^{\frac{1}{2}}\right) \tag{21}$$

as required. K_1, L_1, M_1, N_1 and ω_1 are as given in Sections Two and Three respectively. This completes the proof of the Theorem 2. ■

6. Remarks

We note that from the Theorems 1 and 2, the bounds for the mean-values of the solutions are independent of the considered solutions. However, they all depend on the mean-value of the forcing term $q(t)$. This dependence makes it possible to give conclusions such as the periodicity of solutions whenever $q(t)$ is periodic and have mean-value zero (see [8]-[9]).

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