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A SYSTEM OF IMPLICIT IMPULSIVE NONLINEAR PARABOLIC FUNCTIONAL-DIFFERENTIAL INEQUALITIES

Abstract: A theorem about a system of weak implicit impulsive nonlinear parabolic functional-differential inequalities in an arbitrary parabolic set is proved. As a consequence of the theorem, maximum principles for the system of implicit impulsive nonlinear parabolic functional-differential inequalities and the uniqueness of a classical solution of a mixed implicit impulsive problem for a system of nonlinear parabolic functional-differential equations are obtained.

Key words: implicit system, impulsive system, parabolic system, functional-differential inequalities, comparison theorem, maximum principles, uniqueness of solution.

1. Introduction

In this paper we consider the following diagonal system of the implicit nonlinear parabolic functional-differential inequalities:

$$\begin{aligned} &F_i(t,x,u(t,x),u_t^i(t,x),u_x^i(t,x),u_{xx}^i(t,x),u) \\ &\geq F_i(t,x,v(t,x),v_t^i(t,x),v_x^i(t,x),v_{xx}^i(t,x),v) \quad (i=1,...,m), \end{aligned} \tag{1.1}$$

where $(t,x) \in D \setminus \bigcup_{j=1}^{s}(\{t_j\} \times \mathbb{R}^n)$, $t_0 < t_1 < ... < t_s < t_0 + T$ and D is a relatively arbitrary set more general than the cylindrical domain $(t_0, t_0 + T] \times \Omega \subset \mathbb{R}^{n+1}$. In

$$F_i(t,x,w(t,x),w_t^i(t,x),w_x^i(t,x),w_{xx}^i(t,x),w) \quad (i=1,...,m)$$

the symbol w denotes a function

$$w : \tilde{D} \ni (t,x) \longrightarrow w(t,x) = (w^1(t,x),...,w^m(t,x)) \in \mathbb{R}^m,$$

where $\tilde{D}$ is an arbitrary set such that $\bar{D} \subset \tilde{D} \subset (-\infty, t_0+T] \times \mathbb{R}^n$, $w_x^i(t,x) = grad_x w^i(t,x)$ $(i=1,...,m)$ and $w_{xx}^i(t,x) = \left[\frac{\partial^2 w^i(t,x)}{\partial x_j \partial x_k}\right]_{n\times n}$ $(i=1,...,m)$.

We assume that w is continuous in $\bar{D}\setminus\bigcup_{j=1}^{s}(\{t_i\}\times\mathbb{R}^n)$, the finite different limits $w(t_j^-,x), w(t_j^+,x)$ $(j=1,...,s)$ exist for all admissible $x\in\mathbb{R}^n$ and $w(t_j,x):=w(t_j^+,x)$ $(j=1,...,m)$ for all admissible $x\in\mathbb{R}^n$.

System (1.1) is studied together with impulsive and boundary inequalities. The impulsive inequalities are of the form

$$\begin{gathered} u^i(t_j,x)-u^i(t_j^-,x)-h_i(t_j,x,u(t_j^-,x),u) \\ \leq v^i(t_j,x)-v^i(t_j^-,x)-h_i(t_j,x,v(t_j^-,x),v) \\ (i=1,...,m;j=1,...,s), \end{gathered} \tag{1.2}$$

where h_i $(i=1,...,m)$ are real functions.

A theorem about weak inequalities is proved for system (1.1) together with impulsive inequalities (1.2) and boundary inequalities. As a consequence of the theorem, weak and strong maximum principles for the system of weak implicit impulsive nonlinear parabolic functional-differential inequalities and the uniqueness of a classical solution of a mixed implicit impulsive problem for a system of nonlinear parabolic functional-differential equations are obtained.

Impulsive problems were investigated by V. Lakshmikantham, D. Bainov and P. Simeonov in [6], and by D. Bainov, Z. Kamont and E. Minchev in [1]. The results obtained in the paper are generalizations of those given in [2], [7] - [11] and [3] - [5].

2. Preliminares

We use the notation: $\mathbb{N}_0=\{0,1,2,...\}$. For vectors $z=(z_1,...,z_m)\in\mathbb{R}^m, \tilde{z}=(\tilde{z}_1,...,\tilde{z}_m)\in\mathbb{R}^m$ we write $z\leq\tilde{z}$ in the sense $z_i\leq\tilde{z}_i$ $(i=1,...,m)$. Let t_0 be a real number, $0<T<\infty$ and $x=(x_1,...,x_n)\in\mathbb{R}^n$. By D we denote a bounded or unbounded set contained in $(t_0,t_0+T]\times\mathbb{R}^n$ and satisfying the following conditions:

(a) The projection of the interior of D on the t-axis is the interval (t_0,t_0+T).

(b) For every $(\tilde{t},\tilde{x})\in D$ there exists a number $\rho>0$ such that

$$\{(t,x):(t-\tilde{t})^2+\Sigma_{i=1}^n(x_i-\tilde{x}_i)^2<\rho,\ t<\tilde{t}\}\subset D.$$

(c) All the boundary points $(\tilde{t},\tilde{x})$ of D for which there is a positive number ρ such that

$$\{(t,x):(t-\tilde{t})^2+\Sigma_{i=1}^n(x_i-\tilde{x}_i)^2<\rho,\ t\leq\tilde{t}\}\subset D$$

belong to D.

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t := \{x \in \mathbb{R}^n : (t, x) \in \bar{D}\}$$

and

$$\sigma_t := \bar{D} \cap (\{t\} \times \mathbb{R}^n).$$

Let $s \in \mathbb{N}$ and $t_1, ..., t_s$ be arbitrary fixed real numbers such that

$$t_0 < t_1 < ... < t_s < t_0 + T.$$

We introduce the following sets:

$$D_j := D \cap [(t_j, t_{j+1}) \times \mathbb{R}^n] \ \ (j = 0, 1, ..., s-1),$$
$$D_s := D \cap [(t_s, t_0 + T] \times \mathbb{R}^n],$$
$$D_* := \bigcup_{j=0}^{s} D_j \ \text{ and } \ \sigma_* := \bigcup_{j=1}^{s} \sigma_{t_j}.$$

Let $\tilde{D}$ be an arbitrary set such that

$$\bar{D} \subset \tilde{D} \subset (-\infty, t_0 + T] \times \mathbb{R}^n$$

and let

$$\partial_p D := \tilde{D} \backslash D.$$

For each $j \in \{0, 1, ..., s\}$ and for each $(\tilde{t}_j, \tilde{x}^j) \in D_j$ we denote by $S_j^-(\tilde{t}_j, \tilde{x}^j)$ the set of points $(t, x) \in D_j$ that can be joined with $(\tilde{t}_j, \tilde{x}^j)$ by a polygonal line contained in D_j along which the $t-$coordinate is weakly increasing from (t, x) to $(\tilde{t}_j, \tilde{x}^j)$.

By $PC_m(\tilde{D})$ we denote the linear space of functions

$$w : \tilde{D} \ni (t, x) \to w(t, x) = (w^1(t, x), ..., w^m(t, x)) \in \mathbb{R}^m$$

such that w is continuous in $\bar{D} \backslash \sigma_*$, the finite different limits $w(t_j^-, x), w(t_j^+, x)$ $(j = 1, ..., s)$ exist for all admissible $x \in \mathbb{R}^n$ and $w(t_j, x) := w(t_j^+, x)$ $(j = 1, ..., s)$ for all admissible $x \in \mathbb{R}^n$.

In the set of functions w belonging to $PC_m(\tilde{D})$ and bounded from above in $\tilde{D}$ we define the functional

$$[w]_t = \max_{i=1,...,m} sup\{0, w^i(\tilde{t}, x) : (\tilde{t}, x) \in \tilde{D}, \ \ \tilde{t} \leq t\},$$

where $t \in [t_0, t_0 + T]$.

Assumption (A). For each $i \in \{1,...,m\}$ we assume that Σ_i is a subset (possibly empty) of $[(\tilde{D}\backslash D)\backslash\sigma_*] \cap [(t_0, t_0+T) \times \mathbb{R}^n]$ and $l_i = l_i(t,x)$ is a direction such that for every $(t,x) \in \Sigma_i$ the direction l_i is orthogonal to the t-axis and the interior of some segment starting at (t,x) of the straight half line from (t,x) in the direction l_i is contained in D.

For the sets Σ_i $(i=1,...,m)$ and the directions l_i $(i=1,...,m)$ satisfying Assumption (A), a function $w \in PC_m(\tilde{D})$ is said to belong to $PC_m^{1,2}(\tilde{D})$ if w_t^i, w_x^i, w_{xx}^i $(i=1,...,m)$ are continuous in D_* and the derivatives $\frac{dw^i}{dl_i}$ $(i=1,...,m)$ are finite on Σ_i $(i=1,...,m)$, respectively.

By $M_{n\times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n\times n}$. For each $i \in \{1,...,m\}$ by F_i we denote the mapping

$$F_i : D_* \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n}(\mathbb{R}) \times PC_m^{1,2}(\tilde{D})$$

$$\ni (t,x,z,p,q,r,w) \longrightarrow F_i(t,x,z,p,q,r,w) \in \mathbb{R},$$

where $q = (q_1,...,q_n)$ and $r = [r_{jk}]_{n\times n}$.

We use the notation

$$F_i[t,x,w] := F_i(t,x,w(t,x),\ w_t^i(t,x),\ w_x^i(t,x), w_{xx}^i(t,x), w)\ (i=1,...,m)$$

for all $(t,x) \in D_*$ and $w \in PC_m^{1,2}(\tilde{D})$.

By Z we denote a fixed subset of $PC_m^{1,2}(\tilde{D})$. Functions u and v belonging to Z are called *solutions* of the system

$$F_i[t,x,u] \geq F_i[t,x,v] \ \ (i=1,...,m) \tag{2.1}$$

in D_*, if they satisfy (2.1) for all $(t,x) \in D_*$.

For each $i \in \{1,...,m\}$ the function F_i is said to be *uniformly parabolic* in a subset $S \subset D_*$ with respect to a function $w \in PC_m^{1,2}(\tilde{D})$ if there exists a constant $C > 0$ (depending on S) such that for every $r = [r_{jk}], \tilde{r} = [\tilde{r}_{jk}] \in M_{n\times n}(\mathbb{R})$ and $(t,x) \in S$ the following implication holds:

$$\begin{aligned} r \leq \tilde{r} \Longrightarrow\ & F_i(t,x,w(t,x), w_t^i(t,x), w_x^i(t,x), \tilde{r}, w) \\ & - F_i(t,x,w(t,x), w_t^i(t,x), w_x^i(t,x), r, w) \geq C\Sigma_{j=1}^n(\tilde{r}_{jj} - r_{jj}), \end{aligned} \tag{2.2}$$

where $r \leq \tilde{r}$ means that the inequality $\Sigma_{j,k=1}^n (r_{jk} - \tilde{r}_{jk})\lambda_j\lambda_k \leq 0$ is satisfied for each $(\lambda_1,...,\lambda_n) \in \mathbb{R}^n$.

If (2.2) is satisfied for $r = w_{xx}^i(t,x), \tilde{r} = w_{xx}^i(t,x) + \hat{r}$, where $\hat{r} \geq 0$, with $C = 0$ then F_i is called *parabolic* with respect to w in S.

For every $w, \tilde{w} \in PC_m(\tilde{D})$ and for every $t \in \{t_1, ..., t_s\}$, we write $w \overset{t}{\leq} \tilde{w}$ if $w(t^-, x) \leq \tilde{w}(t^-, x)$ for $x \in S_t$, and $w(\tau, x) \leq \tilde{w}(\tau, x)$ for $(\tau, x) \in \tilde{D}, \tau < t$.

For each $i \in \{1, ..., m\}$, by h_i we denote the function

$$h_i : \sigma_* \times \mathbb{R}^m \times PC_m(\tilde{D}) \longrightarrow \mathbb{R}.$$

Assumption (H). We assume that functions h_i $(i = 1, ..., m)$ satisfy the following condition:

$$(w, \tilde{w} \in PC_m(\tilde{D}), w \overset{t}{\leq} \tilde{w}) \Longrightarrow$$

$$\Longrightarrow (h_i(t, x, w(t^-, x), w) \leq h_i(t, x, \tilde{w}(t^-, x), \tilde{w})$$

$$\text{for } \ (t, x) \in \sigma_* \quad (i = 1, ..., m).$$

3. Theorem about weak inequalities

Theorem 3.1. *Assume that:*

1_1. *The functions F_i $(i = 1, ..., m)$ are weakly increasing with respect to $z_1, ..., z_{i-1}, z_{i+1}, ..., z_m$ $(i = 1, ..., m)$, respectively, and there exists $L > 0$ such that*

$$F_i(t, x, z, p, q, r, w) - F_i(t, x, \tilde{z}, p, \tilde{q}, \tilde{r}, \tilde{w})$$

$$\leq L\Big(\max_{k=1,...,m}(z_k - \tilde{z}_k) + \mid x \mid \Sigma_{j=1}^n \mid q_j - \tilde{q}_j \mid + \mid x \mid^2 \Sigma_{j,k=1}^n \mid r_{jk} - \tilde{r}_{jk} \mid + [w - \tilde{w}]_t\Big)$$

$$(i = 1, ..., m)$$

for

(i) $(t, x) \in D_*, \mid x \mid > L, z \geq \tilde{z}, p \in \mathbb{R}, q, \tilde{q} \in \mathbb{R}^n, r, \tilde{r} \in M_{n\times n}(\mathbb{R})$ *and* $w, \tilde{w} \in Z$ *satisfying the condition* $\sup_{(t,x)\in\tilde{D}} (w(t, x) - \tilde{w}(t, x)) < \infty$, *and*

(ii) $(t, x) \in D_*, \mid x \mid \leq L, z \geq \tilde{z}, p \in \mathbb{R}, q = \tilde{q}, r = \tilde{r}$ *and* $w, \tilde{w} \in Z$ *satisfying the condition* $\sup_{(t,x)\in\tilde{D}} (w(t, x) - \tilde{w}(t, x)) < \infty$.

1_2. *There exists $C > 0$ such that*

$$F_i(t, x, z, p, q, r, w) - F_i(t, x, z, \tilde{p}, q, r, w) < C(\tilde{p} - p) \ \ (i = 1, ..., m)$$

for all $(t, x) \in D_*, z \in \mathbb{R}^m, p > \tilde{p}, q \in \mathbb{R}^n, r \in M_{n\times n}(\mathbb{R}), w \in Z$.

2. For the given sets Σ_i $(i = 1, ..., m)$ and the directions l_i $(i = 1, ..., m)$ satisfying Assumption (A), for given functions $a_i(t, x)$ $(i = 1, ..., m)$ defined and nonnegative for $(t, x) \in \Sigma_i$ $(i = 1, ..., m)$, for the given functions

$g_i(t,x,\xi)$ $(i=1,...,m)$ defined for $(t,x)\in\Sigma_i$ $(i=1,...,m), \xi\in\mathbb{R}$ and strictly increasing with respect to ξ and for the given functions h_i $(i=1,..,m)$ satisfying Assumption (H), functions u and v belonging to Z and such that $\sup\limits_{(t,x)\in D}(u(t,x)-v(t,x))<\infty$, satisfy the inequalities

$$\begin{aligned}
&u^i(t,x)\le v^i(t,x)\\
&\textit{for}\,(t,x)\in\partial_p D\backslash[\sigma_*\cup(\Sigma_i\cap\{(t,x)\in\mathbb{R}^{n+1}:|\,x\,|\le L\})]\,(i=1,...,m),\\
&g_i(t,x,u^i(t,x))-g_i(t,x,v^i(t,x))\le a_i(t,x)\frac{d[u^i(t,x)-v^i(t,x)]}{dl_i}\\
&\textit{for}\,(t,x)\in\Sigma_i\cap\{(t,x)\in\mathbb{R}^{n+1}:|\,x\,|\le L\}\,(i=1,...,m)
\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}
&u^i(t,x)-u^i(t^-,x)-h_i(t,x,u(t^-,x),u)\\
&\le v^i(t,x)-v^i(t^-,x)-h_i(t,x,v(t^-,x),v)\\
&\textit{for}\;\;(t,x)\in\sigma_*\;(i=1,...,m).
\end{aligned}\tag{3.2}$$

3. $F_i(i=1,..,m)$ are parabolic with respect to u in D_*, and u,v are solutions of system (2.1) in D_*.

Then

$$u(t,x)\le v(t,x)\;\;\text{for}\;\;(t,x)\in\tilde{D}.\tag{3.3}$$

Proof. To prove Theorem 3.1 consider the following problem:

$$\left.\begin{aligned}
&F_i[t,x,u]\ge F_i[t,x,v]\;\;\text{for}\;\;(t,x)\in D\cap[(t_0,T_1]\times\mathbb{R}^n]\\
&(i=1,...,m),\\
&u^i(t,x)\le v^i(t,x)\;\;\text{for}\;\;(t,x)\in[\partial_p D\cap((-\infty,T_1]\times\mathbb{R}^n)]\\
&\backslash\Big[\Sigma_i\cap\{(t,x):t\in(t_0,T_1],\;|\,x\,|\le L\}\Big]\;(i=1,...,m),\\
&g_i(t,x,u^i(t,x))-g_i(t,x,v^i(t,x))\\
&\le a_i(t,x)\frac{d[u^i(t,x)-v^i(t,x)]}{dl_i}\;\;\text{for}\;\;(t,x)\in\Sigma_i\\
&\cap\{(t,x):t\in(t_0,T_1],\;|\,x\,|\le L\}\;(i=1,...,m),\\
&\text{where } T_1 \text{ is an arbitrary number such that } t_0<T_1<t_1.
\end{aligned}\right\}\tag{3.4}$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.4) we obtain, by Theorem 2.1 from [3] applied to set $D\cap[(t_0,T_1]\times\mathbb{R}^n]$, the inequality

$$u(t,x)\le v(t,x)\;\;\text{for}\;\;(t,x)\in D\cap[(t_0,T_1]\times\mathbb{R}^n].$$

By the above inequality, by (3.1) and by the continuity of u, v in $\bar{D}\cap([t_0, T_1]\times\mathbb{R}^n)$, we have

$$u(t,x) \le v(t,x) \ \text{ for } \ (t,x) \in \tilde{D} \cap [(-\infty, T_1] \times \mathbb{R}^n].$$

Consequently,

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \tilde{D} \cap [(-\infty, t_1) \times \mathbb{R}^n]. \tag{3.5}$$

Therefore

$$u(t^-,x) \le v(t^-,x) \text{ for } (t,x) \in \sigma_{t_1}. \tag{3.6}$$

Inequalities (3.5), (3.6), and Assumption (H) imply that

$$\begin{aligned} &h_i(t,x,u(t^-,x),u) \le h_i(t,x,v(t^-,x),v) \\ &\text{for } \ (t,x) \in \sigma_{t_1} \ (i = 1, ..., m). \end{aligned} \tag{3.7}$$

From (3.2), (3.6) and (3.7), we obtain

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \sigma_{t_1}. \tag{3.8}$$

By (3.5) and (3.8)

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \tilde{D} \cap [(-\infty, t_1] \times \mathbb{R}^n]. \tag{3.9}$$

Now, write the following problem:

$$\left.\begin{aligned} &F_i[t,x,u] \ge F_i[t,x,v] \ \text{ for } \ (t,x) \in D \cap [(t_1, T_2] \times \mathbb{R}^n] \\ &(i = 1, ..., m), \\ &u^i(t,x) \le v^i(t,x) \text{ for } (t,x) \in [\tilde{D} \cap ((-\infty, T_2] \times \mathbb{R}^n)] \\ &\setminus \Big[(D \cap [(t_1, T_2] \times \mathbb{R}^n]) \cup \Big(\Sigma_i \cap \{(t,x) : t \in (t_1, T_2], \mid x \mid \le L\}\Big)\Big] \\ &(i = 1, ..., m), \\ &g_i(t,x,u^i(t,x)) - g_i(t,x,v^i(t,x)) \\ &\le a_i(t,x)\frac{d[u^i(t,x) - v^i(t,x)]}{dl_i} \ \text{ for } \ (t,x) \in \Sigma_i \\ &\cap \{(t,x) : t \in (t_1, T_2], \ \mid x \mid \le L\} \ \ (i = 1, ..., m), \\ &\text{where } T_2 \text{ is an arbitrary number such that } t_1 < T_2 < t_2. \end{aligned}\right\} \tag{3.10}$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.10) we obtain, by Theorem 2.1 from [3] applied to set $D \cap [(t_1, T_2] \times \mathbb{R}^n]$, the inequality

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in D \cap [(t_1, T_2] \times \mathbb{R}^n].$$

By the above inequality, by (3.1), by the continuity of u, v in $\bar{D} \cap ([t_1, T_2] \times \mathbb{R}^n)$ and by (3.9), we have

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \tilde{D} \cap [(-\infty, T_2] \times \mathbb{R}^n].$$

Consequently

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \tilde{D} \cap [(-\infty, t_2) \times \mathbb{R}^n]. \tag{3.11}$$

Therefore

$$u(t^-,x) \le v(t^-,x) \text{ for } (t,x) \in \sigma_{t_2}. \tag{3.12}$$

Inequalities (3.11), (3.12), and Assumption (H) imply that

$$\begin{aligned} &h_i(t,x,u(t^-,x),u) \le h_i(t,x,v(t^-,x),v) \\ &\text{for } (t,x) \in \sigma_{t_2} \ (i = 1,...,m). \end{aligned} \tag{3.13}$$

From (3.2), (3.12) and (3.13), we obtain

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \sigma_{t_2}. \tag{3.14}$$

By (3.11) and (3.14),

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \tilde{D} \cap [(-\infty, t_2] \times \mathbb{R}^n]. \tag{3.15}$$

Repeating the above procedure $(s-2)$-times, we have

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \tilde{D} \cap [(-\infty, t_s] \times \mathbb{R}^n]. \tag{3.16}$$

Finally, consider the problem

$$\left.\begin{aligned} &F_i[t,x,u] \ge F_i[t,x,v] \text{ for } (t,x) \in D \cap [(t_s, t_0+T] \times \mathbb{R}^n] \\ &(i = 1,...,m), \\ &u^i(t,x) \le v^i(t,x) \text{ for } (t,x) \in \tilde{D} \setminus \Big[(D \cap [(t_s, t_0+T] \times \mathbb{R}^n]) \\ &\cup \Big(\Sigma_i \cap \{(t,x) : t \in (t_s, t_0+T], \mid x \mid \le L\}\Big)\Big] \\ &(i = 1,...,m), \\ &g_i(t,x,u^i(t,x)) - g_i(t,x,v^i(t,x)) \\ &\le a_i(t,x)\frac{d[u^i(t,x) - v^i(t,x)]}{dl_i} \text{ for } (t,x) \in \Sigma_i \\ &\cap \{(t,x) : t \in (t_s, t_0+T], \ \mid x \mid \le L\} \ (i = 1,...,m). \end{aligned}\right\} \tag{3.17}$$

According to the assumptions of Theorem 3.1 corresponding to problem (3.17) we obtain, by Theorem 2.1 from [3] applied to set $D\cap[(t_s,t_0+T]\times\mathbb{R}^n]$, the inequality

$$u(t,x)\le v(t,x) \text{ for } (t,x)\in D\cap[(t_s,t_0+T]\times\mathbb{R}^n].$$

By the above inequality, by (3.1), by the continuity of u,v in $\bar{D}\cap([t_s,t_0+T]\times\mathbb{R}^n)$ and by (3.16), we get (3.3).

The proof of Theorem 3.1 is complete. ■

Remark 3.1. Theorem 3.1 can be formulated for

$v(t,x):=$

$$\begin{cases} M_1=(M_1^1,...,M_1^m) \text{ for } (t,x)\in\tilde{D}\cap((-\infty,t_1)\times\mathbb{R}^n),\\ M_2=(M_2^1,...,M_2^m) \text{ for } (t,x)\in\tilde{D}\cap([t_1,t_2)\times\mathbb{R}^n),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ M_{s+1}=(M_{s+1}^1,..,M_{s+1}^m) \text{ for } (t,x)\in\tilde{D}\cap([t_s,t_0+T]\times\mathbb{R}^n), \end{cases}$$

where $M_j=(M_j^1,..,M_j^m)\in Z$ $(j=1,...,s+1)$ are constant functions.

This form of Theorem 3.1 said to be a *weak maximum principle* for a system of implicit impulsive nonlinear parabolic functional-differential inequalities.

4. Strong maximum principle and uniqueness criterion

As a consequence of Theorem 3.1 we obtain the following theorem about a strong maximum principle for a system of implicit impulsive nonlinear parabolic functional-differential inequalities:

Theorem 4.1. *Assume that:*

1_1. *The functions F_i $(i=1,...,m)$ are weakly increasing with respect to $z_1,...,z_{i-1},z_{i+1},...,z_m$ $(i=1,...,m)$, respectively, and there exists $L>0$ such that*

$$F_i(t,z,p,q,r,w)-F_i(t,x,\tilde{z},p,\tilde{q},\tilde{r},\tilde{w})$$

$$\le L\Big(\max_{k=1,...,m}|z_k-\tilde{z}_k|+|x|\Sigma_{j=1}^n|q_j-\tilde{q}_j|+|x|^2\Sigma_{j,k=1}^n|r_{jk}-\tilde{r}_{jk}|+[w-\tilde{w}]_t)$$

$$(i=1,..,m)$$

for all $(t,x)\in D_*$, $z,\tilde{z}\in\mathbb{R}^m$, $p\in\mathbb{R}$, $r,\tilde{r}\in M_{n\times n}(\mathbb{R})$ and $w,\tilde{w}\in Z$ satisfying the condition $\sup_{(t,x)\in\tilde{D}}(w(t,x)-\tilde{w}(t,x))<\infty$.

1_2. *There are constants* $C_i > 0$ $(i = 1, 2)$ *such that*

$$F_i(t, x, z, p, q, r, w) - F_i(t, x, z, \tilde{p}, q, r, w) < C_1(\tilde{p} - p) \ (i = 1, ..., m)$$

for all $(t, x) \in D_*, z \in \mathbb{R}^m, p > \tilde{p}, q \in \mathbb{R}^n, r \in M_{n\times n}(\mathbb{R}), w \in Z$ *and*

$$F_i(t, x, z, p, q, r, w) - F_i(t, x, z, \tilde{p}, q, r, w) < C_2(\tilde{p} - p) \ (i = 1, ..., m)$$

for all $(t, x) \in D_*,\ z \in \mathbb{R}^m,\ p < \tilde{p},\ q \in \mathbb{R}^n,\ r \in M_{n\times n}(\mathbb{R}),\ w \in Z.$

2. Functions u, v, F_i *and* h_i $(i = 1, ..., m)$ *satisfy assumptions 2 and 3 of Theorem 3.1. Moreover,* F_i $(i = 1, ...m)$ *are uniformly parabolic with respect to* v *in any compact subset of* D_*.

Then inequality (3.3) holds. Moreover,

(i) if for some $(\tilde{t}_j, \tilde{x}^j) \in D_j$ $(j = 0, 1, ..., s)$ *and for some* $k_j \in \{1, ..., m\}$

$$u^{k_j}(\tilde{t}_j, \tilde{x}^j) = v^{k_j}(\tilde{t}_j, \tilde{x}^j) \ (j = 0, 1, ..., s)$$

then

$$u^{k_j}(t, x) = v^{k_j}(t, x) \text{ for } (t, x) \in S_j^-(\tilde{t}_j, \tilde{x}^j) \ (j = 0, 1, ..., s). \tag{4.1}$$

Additionally,

(ii) if D *is a cylindrical domain in* $\mathbb{R}^{n+1}$ *and if for each* $j \in \{1, ..., s\}$ *there exist a point* $P_j = (t_j, \tilde{x}^j)$ *and a segment* $\overline{P_jQ_j}$, *where* $Q_j = (t_j - \delta, \tilde{x}^j), \delta > 0$, *satisfying the conditions*

$$P_j \in \text{ int } (\sigma_{t_j}) \text{ and int } \overline{P_jQ_j} \subset D_j \tag{4.2}$$

and such that

$$u(t, x) = v(t, x) \text{ for } (t, x) \in \bigcup_{j=1}^{s} \text{ int } \overline{P_jQ_j} \tag{4.3}$$

then

$$u(t, x) = v(t, x) \text{ for } (t, x) \in \bar{D} \cap ([t_0, t_s] \times \mathbb{R}^n). \tag{4.4}$$

Proof. Since the assumptions of Theorem 3.1 are satisfied then inequality (3.3) holds.

To prove the second part of the thesis write the following problems:

$$
\left.
\begin{array}{l}
F_i[t,x,u] \geq F_i[t,x,v] \ \text{ for } \ (t,x) \in D \cap [(t_0,t_1] \times \mathbb{R}^n] \\
(i = 1, ..., m), \\
u^i(t,x) \leq v^i(t,x) \text{ for } (t,x) \in [\partial_p D \cap ((-\infty, t_1) \times \mathbb{R}^n)] \\
\setminus \Big[\Sigma_i \cap \{(t,x) : t \in (t_0,t_1), \mid x \mid \leq L\} \Big] (i = 1, ..., m), \\
g_i(t,x,u^i(t,x)) - g_i(t,x,v^i(t,x)) \\
\leq a_i(t,x) \dfrac{d[u^i(t,x) - v^i(t,x)]}{dl_i} \text{ for } (t,x) \in \Sigma_i \\
\cap \{(t,x) : t \in (t_0,t_1), \ \mid x \mid \leq L\} \ (i = 1, ..., m), \\
u^{k_0}(\tilde{t}_0, \tilde{x}^0) = v^{k_0}(\tilde{t}_0, \tilde{x}^0) \text{ for some } (\tilde{t}_0, \tilde{x}^0) \in D_0 \\
\text{and some } k_0 \in \{1, ..., m\},
\end{array}
\right\} \tag{4.5}
$$

$$
\left.
\begin{array}{l}
F_i[t,x,u] \geq F_i[t,x,v] \ \text{ for } \ (t,x) \in D \cap [(t_1,t_2] \times \mathbb{R}^n] \\
(i = 1, ..., m), \\
u^i(t,x) \leq v^i(t,x) \text{ for } (t,x) \in [\tilde{D} \cap [(-\infty, t_2) \times \mathbb{R}^n]] \\
\setminus \Big[(D \cap [(t_1,t_2) \times \mathbb{R}^n]) \cup \\
\Big(\Sigma_i \cap \{(t,x) : t \in (t_1,t_2), \mid x \mid \leq L\} \Big) \Big] (i = 1, ..., m), \\
g_i(t,x,u^i(t,x)) - g_i(t,x,v^i(t,x)) \\
\leq a_i(t,x) \dfrac{d[u^i(t,x) - v^i(t,x)]}{dl_i} \text{ for } (t,x) \in \Sigma_i \\
\cap \{(t,x) : t \in (t_1,t_2), \ \mid x \mid \leq L\} \ (i = 1, ..., m), \\
u^{k_1}(\tilde{t}_1, \tilde{x}^1) = v^{k_1}(\tilde{t}_1, \tilde{x}^1) \text{ for some } (\tilde{t}_1, \tilde{x}^1) \in D_1 \\
\text{and some } k_1 \in \{1, ..., m\},
\end{array}
\right\} \tag{4.6}
$$

. . . .

$$
(4.7)\quad \left.\begin{array}{l}
F_i[t,x,u] \geq F_i[t,x,v] \ \text{ for } \ (t,x) \in D \cap [(t_s, t_0+T] \times \mathbb{R}^n] \\
(i = 1, ..., m), \\
u^i(t,x) \leq v^i(t,x) \text{ for } (t,x) \in \tilde{D} \setminus \Big[(D \cap [(t_s, t_0+T] \times \mathbb{R}^n]) \cup \\
\Big(\Sigma_i \cap \{(t,x) : t \in (t_s, t_0+T], \mid x \mid \leq L\} \Big) \Big] (i = 1, ..., m), \\
g_i(t,x,u^i(t,x)) - g_i(t,x,v^i(t,x)) \\
\leq a_i(t,x) \dfrac{d[u^i(t,x) - v^i(t,x)]}{dl_i} \text{ for } (t,x) \in \Sigma_i \\
\cap \{(t,x) : t \in (t_s, t_0+T), \ \mid x \mid \leq L\} \ (i = 1, ..., m), \\
u^{k_s}(\tilde{t}_s, \tilde{x}^s) = v^{k_s}(\tilde{t}_s, \tilde{x}^s) \text{ for some } (\tilde{t}_s, \tilde{x}^s) \in D_s \\
\text{and some } k_s \in \{1, ..., m\}.
\end{array}\right\}
$$

Applying to (4.5)-(4.7) the strong maximum principle from [3] we have

$$
\begin{cases}
u^{k_0}(t,x) = v^{k_0}(t,x) \text{ for } (t,x) \in S_0^-(\tilde{t}_0, \tilde{x}^0), \\
u^{k_1}(t,x) = v^{k_1}(t,x) \text{ for } (t,x) \in S_1^-(\tilde{t}_1, \tilde{x}^1), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
u^{k_s}(t,x) = v^{k_s}(t,x) \text{ for } (t,x) \in S_s^-(\tilde{t}_s, \tilde{x}^s).
\end{cases}
$$

Therefore, (4.1) holds.

Formula (4.4) is a consequence of (4.3) and (4.2), of thesis (i) of Theorem 4.1, and of the fact that D is a cylindrical domain in $\mathbb{R}^{n+1}$ and $u, v \in PC_m(\tilde{D})$.

The proof of Theorem 4.1 is complete. ■

To make easier the notation of this part of Section 4 assume that the sets $\Sigma_i (i = 1, ..., m)$ and the constant L from Theorem 3.1 satisfy the conditions

$$\Sigma_i \cap \{(t,x) : t \in (t_0, t_0+T), \mid x \mid \leq L\} = \Sigma_i \ (i = 1, .., m).$$

Definition 4.1. *Being given:*

(i) the functions $F_i(t,x,z,p,q,r,w)$ $(i = 1, ..., m)$ defined for $(t,x) \in D_*, z \in \mathbb{R}^m, p \in \mathbb{R}, q \in \mathbb{R}^n, r \in M_{n\times n}(\mathbb{R}), w \in Z$,

(ii) the functions $h_i(t,x,z,w)$ $(i = 1, ..., m)$ defined for $(t,x) \in \sigma_*, z \in \mathbb{R}^m, w \in Z$,

(iii) the functions $a_i(t,x)$ $(i = 1, ..., m)$ defined and nonnegative for $(t,x) \in \Sigma_i$ $(i = 1, ..., m)$, and the functions $g_i(t,x,\xi)$ $(i = 1, ..., m)$ defined for $(t,x) \in \Sigma_i$ $(i = 1, ..., m)$ and $\xi \in \mathbb{R}$,

(iv) the functions $\tilde{F}_i(t,x)$ $(i=1,...,m)$ *defined for* $(t,x)\in\partial_p D\backslash(\sigma_*\cup\Sigma_i)$ $(i=1,...,m)$*, the functions* $\tilde{G}_i(t,x)$ $(i=1,...,m)$ *defined for* $(t,x)\in\Sigma_i$ $(i=1,...m)$ *and the functions* $\tilde{H}_i(t,x)$ $(i=1,...,m)$ *defined for* $(t,x)\in\sigma_*$*,*

the MIXED IMPLICIT IMPULSIVE PARABOLIC FUNCTIONAL-DIFFERENTIAL PROBLEM *consists on finding a function* $u\in PC_m^{1,2}(\tilde{D})$ *satisfying the system of equations*

$$F_i[t,x,u]=0 \ \text{for} \ (t,x)\in D_* \ (i=1,...,m) \tag{4.9}$$

and the system of the mentioned below initial-boundary and impulsive conditions

$$u_i(t,x)=\tilde{F}_i(t,x) \ \text{for} \ (t,x)\in\partial_p D\backslash(\sigma_*\cup\Sigma_i) \ (i=1,...,m), \tag{4.10}$$

$$\begin{aligned} &g_i(t,x,u^i(t,x))-a_i(t,x)\frac{du^i(t,x)}{dl_i}=\tilde{G}_i(t,x)\\ &\text{for} \ (t,x)\in\Sigma_i \ \ (i=1,...,,m)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &u^i(t,x)-u^i(t^-,x)-h_i(t,x,u(t^-,x),u)=\tilde{H}_i(t,x)\\ &\text{for} \ (t,x)\in\sigma_* \ \ (i=1,...,m)\end{aligned} \tag{4.12}$$

As a consequence of Theorem 3.1 we obtain the following theorem about the uniqueness of a classical solution of the mixed implicit impulsive parabolic functional-differential problem:

Theorem 4.2. *Suppose that assumptions* 1_1 *and* 1_2 *of Theorem 3.1 are satisfied, the sets* $\sum_i$ $(i=1,...,m)$ *and the constant* L *from assumptions* $1_1,2$ *of Theorem 3.1 satisfy conditions (4.8), the functions* $g_i(t,x,\xi)$ $(i=1,...,m)$ *are strictly increasing with respect to* $\xi\in\mathbb{R}$ *for all* $(t,x)\in\sum_i$ $(i=1,...,m)$*, and the functions* h_i $(i=1,...,m)$ *satisfy Assumption* (H)*. Then in the class of all functions* w *belonging to* $PC_m^{1,2}(\tilde{D})$*, bounded in* $\tilde{D}$ *and such that the functions* F_i $(i=1,...,m)$ *are parabolic with respect to* w *in* D_**, there exists at most one function* u *satisfying the mixed implicit impulsive parabolic functional-differential problem (4.9)-(4.12).*

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