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ON THE STRONG APPROXIMATION OF FUNCTIONS BY THE BERNSTEIN POLYNOMIALS

Abstract: In this note we introduce the strong differences of function and its Bernstein polynomials and give approximation theorems for them.
This note is motivated by results on the strong summability of trigonometric Fourier series given in [2] and by the paper [5].

Key words: Bernstein polynomial, strong difference, degree of approximation.

1. The Bernstein polynomials of function of one variable

1.1. Let $C(I)$ be the space of real-valued functions f continuous on the interval $I = [0, 1]$ with the norm

(1) $$\|f\| = \max_{x\in I} |f(x)|$$

It is know ([1],[3],[4]) that the Bernstein polynomials of $f \in C(I)$ are defined by the formula

(2) $$B_n(x; f) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in I, \quad n \in N = \{1, 2, \ldots\},$$

where

(3) $$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \le k \le n.$$

From (2) and (3) it follows that

(4) $$\sum_{k=0}^{n} p_{n,k}(x) = 1, \quad x \in I, \ \ n \in N,$$

and

(5) $$B_n(x; f) - f(x) = \sum_{k=0}^{n} p_{n,k}(x) \left(f\left(\frac{k}{n}\right) - f(x) \right),$$

for every $f \in C(I), x \in I$ and $n \in N$. Moreover, it is known ([1]) that

$$\|B_n(f;\cdot) - f(\cdot)\| \le \frac{3}{2}\omega\left(f;n^{-1/2}\right), \qquad n \in N, \tag{6}$$

for every $f \in C(I)$, where $\omega(f;\cdot)$ is the modulus of continuity of f defined by

$$\omega(f;t) = \sup\{|f(x) - f(y)| : x, y \in I, |x - y| \le t\}, \qquad t \in I. \tag{7}$$

1.2. Similarly as in [2] and [5] we introduce the following strong difference of $f \in C(I)$ and $B_n(f)$

$$H_n^q(f;x) := \left(\sum_{k=0}^{n} p_{n,k}(x)\left|f\left(\frac{k}{n}\right) - f(x)\right|^q\right)^{1/q}, \quad x \in I, \ \ n \in N. \tag{8}$$

It is obvious that H_n^q is well defined for every $f \in C(I), x \in I, n \in N$ and $q > 0$ and moreover by (1),(4),(5) and (8) we have

$$\|H_n^q(f)\| \le 2\|f\|, \quad n \in N,$$

and

$$\|B_n(f;x) - f(x)\| \le \|H_n^1(x)\|, \quad n \in N. \tag{9}$$

Using the Hölder inequality and (4) we easily obtain the following.

Lemma 1. *For every $f \in C(I)$ and $0 < q < p < \infty$ we have*

$$H_n^q(f;x) \le H_n^p(f;x), \quad x \in I, n \in N,$$

which implies

$$\|H_n^q(f)\| \le \|H_n^p(f)\| \quad \textit{for} \quad n \in N.$$

In this paper we shall apply the following auxiliary inequality.

Lemma 2. *For every $s \in N$ there exists a positive constants $M_1(s)$ depending only on s such that*

$$\max_{x\in I}\sum_{k=0}^{n} p_{n,k}(x)\left|\frac{k}{n} - x\right|^s \le M_1(s)\cdot n^{-s/2} \quad \textit{for} \quad n \in N. \tag{10}$$

Proof. In [4], p.248, was proved the following inequality

$$\left|\sum_{k=0}^{n} p_{n,k}(x)(k - nx)^{2s}\right| \le M_2(s)n^s, \quad x \in I, \ \ n \in N, \tag{11}$$

for every $s \in N$, where $M_2(s)$ is a positive constant depending only on s. Applying the Hölder inequality, we get

$$\sum_{k=0}^{n} p_{n,k}(x)\left|\frac{k}{n}-x\right|^s \le n^{-s}\left(\sum_{k=0}^{n}(x)(k-nx)^{2s}\right)^{1/2}\left(\sum_{k=0}^{n}p_{n,k}(x)\right)^{1/2},$$

which by (4) and (11) yields the inequality (10). ■

1.3. Now we shall prove the main theorem.

Theorem 1. *Suppose that $q > 0$ is a fixed number. Then there exists a positive constant $M_3(q)$, depending only on q, such that for every $f \in C(I)$ and $n \in N$ there holds*

$$\|H_n^q(f;\cdot)\| \le M_3(q)\omega\left(f;n^{-1/2}\right), \tag{12}$$

where $\omega(f;\cdot)$ is defined by (7).

Proof. a) First let $q \in N$. Then by (7) and the inequality $\omega(f;\lambda t) \le (\lambda+1)\omega(f;t)$ for $\lambda, t \in I$ (see [4,5]) we have

$$\left|f\left(\frac{k}{n}\right)-f(x)\right| \le \omega\left(f;\left|\frac{k}{n}-x\right|\right) \le \omega\left(f;n^{-1/2}\right)\left(\sqrt{n}\left|\frac{k}{n}-x\right|+1\right)$$

and by (8) and the Minikowski inequality we can write

$$\begin{aligned} H_n^q(f;x) &\le \omega(f;n^{-1/2})\left[\sum_{k=0}^{n}p_{n,k}(x)\left(\sqrt{n}\left|\frac{k}{n}-x\right|+1\right)^q\right]^{1/q} \\ &\le \omega(f;n^{-1/2})\left[\sqrt{n}\left(\sum_{k=0}^{n}p_{n,k}(x)\left|\frac{k}{n}-x\right|^q\right)^{1/q}+\left(\sum_{k=0}^{n}p_{n,k}(x)\right)^{1/q}\right]. \end{aligned}$$

Applying (4) and Lemma 2, we immediately obtain the desired assertion (12) for $q \in N$.

b) Let $0 < q \notin N$. Then $0 < q < [q]+1$, where $[q]$ is the integral part of q. Since $([q]+1) \in N$, we have by Lemma 1 and (12) with the power $[q]+1$:

$$\|H_n^q(f)\| \le \|H_n^{[q]+1}(f)\| \le M_3(q)\omega(f;n^{-1/2}) \quad \text{for} \quad n \in N.$$

Thus the proof is completed. ■

From Theorem 1 we derive the following two corollaries

Corollary 1. *For every $f \in C(I)$ and $q > 0$ there holds*

$$\lim_{n\to\infty}\|H_n^q(f)\| = 0.$$

Corollary 2. *For every $f \in C(I)$ having the derivative f' bounded on I and for every $q > 0$ we have*

$$|H_n^q(f)\| = O(n^{-1/2}), \qquad n \in N.$$

Remark. The inequality (9) shows that Theorem 1 generalizes the result (6).

2. The Bernstein polynomials of function of two variables

Considering functions of two variables we shall prove analogues of results given in Section 1.

2.1. Let $C(I^2)$ be the space of all real-valued functions f continuous on $I^2 = I \times I$ with the norm

(13) $$\|f\| = \max_{(x,y)\in I^2} |f(x,y)|.$$

For $f \in C(I^2)$ we define modulus of continuity ([6])

(14) $$\omega(f;t,s) = \sup\{|f(x,y) - f(u,v)| : (x,y),(u,v) \in I^2, \\ |x-y| \le t, |y-v| \le s\}$$

for $t, s \in I$. It is known ([6]) that $\lim_{\substack{t\to 0+\\ s\to 0+}} \omega(f;t,s) = 0$, for every $f \in C(I^2)$. Moreover for every $f \in C(I^2)$ and $0 \le s_1 < s_2 < 1$, $0 \le t_1 < t_2 < 1$ we have

$$\begin{aligned} \omega(f;s_1,t_1) &\le \omega(f;s_2,t_1) \le \omega(f;s_2,t_2), \\ \omega(f;s_1,t_1) &\le \omega(f;s_1,t_2) \le \omega(f;s_2,t_2) \end{aligned}$$

and

$$\begin{aligned} \omega(f;s,t) &\le \omega(f;s,0) + \omega(f;0,t), \\ \omega(f;\lambda_1 s,\lambda_2 t) &\le (\lambda_1+1)\omega(f;0,t) + (\lambda_2+1)\omega(f;0,t) \\ &\le (\lambda_1+\lambda_2+2)\omega(f;s,t), \end{aligned}$$

for $\lambda_1 s, \lambda_2 t \in I$.

2.2. We shall consider the following Bernstein polynomials of $f \in C(I^2)$:

(15) $$B_{m,n}(f;x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} p_{m,j}(x)p_{n,k}(y)\, f\Big(\frac{j}{m},\frac{k}{n}\Big),$$

$(x, y) \in I^2$, $m, n \in N$, where $p_{m,j}(x)$ and $p_{n,k}(y)$ are defined by (3) (see [3]). By (4) and (15) we have

$$\text{(16)} \qquad B_{m,n}(f; x, y) - f(x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n} p_{m,j}(x)p_{n,k}(y)\left[f\left(\frac{j}{m}, \frac{k}{n}\right) - f(x, y)\right],$$

for $(x, y) \in I^2$ and $m, n \in N$. In [3] was proved that if $f \in C(I^2)$, then

$$\text{(17)} \qquad \|B_{n,n}(f; \cdot, \cdot) - f(\cdot; \cdot)\| \le 3\omega(f; n^{-1/2}, n^{-1/2}), \qquad n \in N.$$

From the proof of this inequality ([3]) we deduce that for $B_{m,n}(f)$ and $f \in C(I^2$ there holds

$$\|B_{m,n}(f; \cdot, \cdot) - f(\cdot, \cdot)\| \le 3\omega(f; m^{-1/2}, n^{-1/2}, \qquad m, n \in N.$$

2.3. Similarly to Section 1 we introduce strong differences of $f \in C(I^2)$ and $B_{m,n}(f)$:

$$\text{(18)} \quad H^q_{m,n}(f; x, y) := \left(\sum_{j=0}^{m}\sum_{k=0}^{n} p_{m,j}(x)p_{n,k}(y)\left|f\left(\frac{j}{m}, \frac{k}{n}\right) - f(x, y)\right|^q\right)^{1/q},$$

$(x, y) \in I^2, m, n \in N$ and $q > 0$. Applying (13),(16) and (18) and arguing as in Section 1, we get

$$\|H^q_{m,n}(f)\| \le 2\|f\|, \quad q > 0$$

$$\text{(19)} \qquad \|B_{m,n}(f) - f\| \le \|H^1_{m,n}(f)\|$$

$$\text{(20)} \qquad \|H^q_{m,n}(f)\| \le \|H^p_{m,n}(f)\|, \quad 0 < q < p < \infty$$

for every $f \in C(I^2)$ and $m, n \in N$.

2.4. Now we shall prove an analogue of (13).

Theorem 2. *Suppose that $q > 0$ is a fixed number. Then there exists a positive constant $M_4(q)$, depending only on q, such that for every $f \in C(I^2)$ and $n \in N$ there holds*

$$\text{(21)} \qquad \|H^q_{m,n}(f)\| \le M_4(q)\,\omega(f; m^{-1/2}, n^{-1/2}),$$

where $\omega(f)$ is the modulus of continuity defined by (14).

Proof. a) Let $q \in N$. By (14) and properties of the modulus of continuity given in Section 2.1, we have

$$\begin{aligned}\left|f\left(\frac{j}{m},\frac{k}{n}\right)-f(x,y)\right| &\le \omega\left(f;\left|\frac{j}{m}-x\right|,\left|\frac{k}{n}-y\right|\right)\\ &\le \omega\left(f;\left|\frac{j}{m}-x\right|,0\right)+\omega\left(f;0,\left|\frac{k}{n}-y\right|\right)\\ &\le \omega(f;m^{-1/2},0)\left(\sqrt{m}\left|\frac{j}{m}-x\right|+1\right)+\\ &\quad+\omega(f;0,n^{-1/2})\left(\sqrt{n}\left|\frac{k}{n}-y\right|+1\right)\\ &\le \omega(f;m^{-1/2},n^{-1/2})\left(\sqrt{m}\left|\frac{j}{m}-x\right|+\sqrt{n}\left|\frac{k}{n}-y\right|+2\right)\end{aligned}$$

Using the above inequality to (18) and by the Minikowski inequality, we get

$$\begin{aligned}H^q_{n,n}(f;x,y) \le \omega(f;m^{-1/2},n^{-1/2})\Bigg\{&\sqrt{m}\left(\sum_{j=0}^{m}\sum_{k=0}^{n}p_{m,j}(x)p_{n,k}(y)\left|\frac{j}{m}-x\right|^q\right)^{1/q}\\ &+\sqrt{n}\left(\sum_{j=0}^{m}\sum_{k=0}^{n}p_{m,j}(x)p_{n,k}(y)\left|\frac{k}{n}-y\right|^q\right)^{1/q}\\ &+\left(\sum_{j=0}^{m}\sum_{k=0}^{n}p_{m,j}(x)p_{n,k}(y)\right)^{1/q}\Bigg\}\\ :=\omega(f;m^{-1/2},n^{-1/2})\{&A_{m,n,1}(x,y)+A_{m,n,2}(x,y)+A_{m,n,3}(x,y)\}\end{aligned}$$

for $(x,y) \in I^2$ and $n \in N$. But by (4) and Lemma 2 we can write

$$\begin{aligned}A_{m,n,1}(x,y) &= \sqrt{m}\left(\sum_{j=0}^{m}p_{m,j}(x)\left|\frac{j}{m}-x\right|^q\right)^{1/q}\left(\sum_{k=0}^{n}p_{n,k}(y)\right)^{1/q} \le \sqrt{M_1(q)},\\ A_{m,n,2}(x,y) &= \sqrt{n}\left(\sum_{j=0}^{m}p_{m,j}(x)\right)^{1/q}\left(\sum_{k=0}^{n}p_{n,k}(y)\left|\frac{k}{n}-y\right|^q\right)^{1/q} \le \sqrt{M_1(q)},\\ A_{m,n,3}(x,y) &= 1,\end{aligned}$$

for all $(x,y) \in I^2$ and $m,n \in N$. Combining these we obtain (21) for $q \in N$.

b) If $0 < q \notin N$, then by (20) and (21) for $[q]+1$ (analogously as in the proof of Theorem 1) we have

$$\|H^q_{m,n}(f)\| \le \|H^{[q]+1}_{m,n}(f)\| \le M_4(q)\omega\left(f;m^{-1/2},n^{-1/2}\right), \quad m,n \in N,$$

which completes the proof. ■

Theorem 2 implies the following

Corollary 3. *For every $f \in C(I^2)$ and $q > 0$ there holds*

$$\lim_{m,n\to\infty} \|H^q_{m,n}(f)\| = 0.$$

If $f \in C(I^2)$ is function having partial derivatives f'_x and f'_y bounded on I^2, then

$$\|H^q_{m,n}(f)\| = O\left(m^{-1/2} + n^{-1/2}\right) \quad as \quad m, n \in N.$$

Finally we remark that inequality (19) and Theorem 2 with $q = 1$ yield

$$\|B_{m,n}(f) - f\| \leq M_1(1)\omega\left(f; m^{-1/2}, n^{-1/2}\right) \quad \text{for} \quad f \in C(I^2)\ ,\ m, n \in N,$$

Hence we see that Theorem 2 generalizes the result (17).

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