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ASYMPTOTIC BEHAVIOUR OF OSCILLATORY SOLUTIONS OF n-TH ORDER DIFFERENTIAL EQUATIONS

Abstract: In this paper, sufficient conditions have been obtained so that all oscillatory solutions of the n-th order differential equations with quasi derivatives tend to zero as t tends to infinity.

Key words: Oscillatory, asymptotic behaviour, delay-differential equations.

1.

Recently Singh [3] obtained sufficient conditions to ensure that all oscillatory solutions of the general n-th order equation

$$(r(t)y'(t))^{(n-1)} + a(t)y(t-\tau(t)) = f(t) \tag{1}$$

approach to zero as $t \to \infty$, where $a(t)$, $r(t)$, $\tau(t)$ and $f(t)$ are real valued continuous functions on the whole real line with $r(t) > 0$, $\tau(t) > 0$ and $\tau(t)$ is bounded above by a real constant $k > 0$. He obtained the following result:

Theorem 1.1. *Suppose that*

$$\int^{\infty} t^{n-2}|f(t)|\, dt < \infty, \tag{2}$$

$$\int^{\infty} t^{n-2}|a(t)|\, dt < \infty \tag{3}$$

and

$$\int^{\infty} \frac{1}{r(t)}\, dt < \infty. \tag{4}$$

Then all oscillatory solutions of (1) tend to zero as $t \to \infty$.

Theorem 1.1 does not cover an important class of differential equations of the form (1) with $\int^{\infty} \frac{1}{r(t)}\, dt = \infty$. It may be noted that the example cited in Singh [4] shows that the condition (4) on $r(t)$ cannot be changed keeping the conditions (2) and (3) intact in Theorem 1.1. Most probably, this is the reason for which Singh [4] obtained a result with the same conclusion of this theorem for the equations of the type (1) with $\int^{\infty} \frac{1}{r(t)}\,dt = \infty$ by relaxing the conditions (2) and (3). In [2], Chen and Yeh improved Singh's result [4] to a more general n-th order equation

$$(\frac{1}{r_{n-1}(t)}(\frac{1}{r_{n-2}(t)}(...(\frac{x(t)}{r_0(t)})'...)')')' + f(t, x[g(t)]) = h(t), \quad t \geq 0, \quad n \geq 2,$$

where $\int^{\infty} r_i(t)\,dt = \infty, i = 1, 2, ..., n-1$. More recently, in the monograph [1], Bainov and Mishev gave results concerning the asymptotic decay of the oscillatory solutions of the operator differential equations of the form

$$(r_{n-1}(t)(r_{n-2}(t)(...(r_1(t)x'(t))'...)')')' + F(t, x(t), (A(x)))(t) = b(t),$$

where A is an operator with certain properties (see Theorem 3.3.3, [1]).

The motivation for the present work has come from Theorem 1.1 and the paper by Chen and Yeh [2] and Theorem 3.3.3 in [1]. Our purpose is to improve the conditions of Theorem 1.1 and extend the result to a more general equation

$$(r_{n-1}(t)(r_{n-2}(t)(...(r_1(t)y'(t))'...)')')' + p(t)h(y(g(t))) = f(t), \tag{5}$$

$t \geq 0$, $n \geq 2$, where $r_i(t) > 0, i = 1, 2, ..., n-1; p, f$ and $h \in (R, R), g(t) \leq t$ and $g(t) \to \infty$ as $t \to \infty$. Our result is stronger and more easily verifiable than the results in [1, 2, 3, 4]. In [1], sufficient conditions are given for the oscillation of (5).

We always assume that h satisfies the condition:

(6) *$uh(u) > 0$ for $u \neq 0$ and there exists a positive real m and $\gamma \in (0, 1]$ such that $|h(u)| \leq m|u|^{\gamma}$.*

2.

Define

$$L_0 y(t) = y(t), L_i y(t) = r_i(t)\frac{dL_{i-1}y(t)}{dt}, i = 1, 2, ..., n$$

and $r_n(t) = 1$. Then (5) can be written in the form

$$L_n y(t) + p(t)h(y(g(t))) = f(t).$$

Theorem 2.1. *Let*

$$\int_0^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\dots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |p(s_n)|\,ds_n\dots ds_2ds_1<\infty \tag{7}$$

and

$$\int_0^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\dots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |f(s_n)|\,ds_n\dots ds_2ds_1<\infty \tag{8}$$

hold. Then all oscillatory solutions of (5) satisfy the property

$$\lim_{t\to\infty}(L_iy)(t)=0,\qquad i=0,1,2,\dots,n-1. \tag{9}$$

Proof. From (7) and (8), it follows that there exist a $T>0$ and a real $\beta>0$ such that

$$\int_T^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\dots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |p(s_n)|\,ds_n\dots ds_2ds_1<\frac{\beta^{1-\gamma}}{m} \tag{10}$$

and

$$\int_T^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\dots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |f(s_n)|\,ds_n\dots ds_2ds_1<\frac{\beta}{4}. \tag{11}$$

Suppose that $y(t)$ is an oscillatory solution of (5). Since $y(t)$ is oscillatory, then $L_iy(t), i=1,2,\dots,n$ is also oscillatory. Let $T\le t_0<t_1<t_2<\dots<t_{n-1}$ be the sequence of zeros of $L_0y(t), L_1y(t), L_2y(t),\dots,L_{n-1}y(t)$ respectively. We claim that $y(t)$ is bounded. If not, then $y(t)$ is unbounded. Let $M=\max\{|y(t)|; T\le t\le t_{n-1}\}>\beta$. Integrating (5) from $t(\ge T)$ to t_{n-1}, we have

$$-L_{n-1}y(t)=-\int_t^{t_{n-1}} p(s_n)h(y(g(s_n)))\,ds_n+\int_t^{t_{n-1}} f(s_n)\,ds_n,$$

that is

$$-(L_{n-2}y(t))'=-\frac{1}{r_{n-1}(t)}\int_t^{t_{n-1}} p(s_n)h(y(g(s_n)))\,ds_n+\frac{1}{r_{n-1}(t)}\int_t^{t_{n-1}} f(s_n)\,ds_n.$$

repeating the integration of the above from $t(\geq T)$ to $t_{n-2}, t_{n-3}, ..., t_2$ and t_1 respectively, we have

$$(-1)^n y(t) = -\int_T^{t_1} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} p(s_n) h(y(g(s_n)))\, ds_n ... ds_2 ds_1$$
$$+ \int_T^{t_1} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} f(s_n)\, ds_n ds_{n-1} ... ds_2 ds_1.$$

Hence

$$|y(t)| = \int_T^{t_1} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |p(s_n)||h(y(g(s_n)))|\, ds_n ... ds_2 ds_1$$
$$+ \int_T^{t_1} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |f(s_n)|\, ds_n ds_{n-1} ... ds_2 ds_1.$$

Taking maximum $y(t)$ in $[T, t_{n-1}]$ and using (6), we have

$$M \leq m M^\gamma \int_T^{t_1} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |p(s_n)|\, ds_n ... ds_2 ds_1$$
$$+ \int_T^{t_1} \frac{1}{r_1(s_1)} \int_{s_1}^{t_2} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{t_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{t_{n-1}} |f(s_n)|\, ds_n ds_{n-1} ... ds_2 ds_1$$
$$\leq m M^\gamma \int_T^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |p(s_n)|\, ds_n ... ds_2 ds_1$$
$$+ \int_T^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |f(s_n)|\, ds_n ds_{n-1} ... ds_2 ds_1,$$

that is,

$$1 \leq \frac{m}{M^{1-\gamma}} \int_T^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |p(s_n)|\, ds_n ... ds_2 ds_1$$
$$+ \frac{1}{M} \int_T^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |f(s_n)|\, ds_n ds_{n-1} ... ds_2 ds_1,$$

or,

$$1 \leq \frac{m}{\beta^{1-\gamma}} \int_T^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |p(s_n)|\, ds_n ... ds_2 ds_1$$

$$+\frac{1}{\beta}\int_T^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}...\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |f(s_n)|\,ds_n ds_{n-1}...ds_2 ds_1.$$

Using (10) and (11), the above inequality yields a contradiction. Hence our claim holds, that is, $y(t)$ is bounded. Thus there exist a constant $\lambda > 0$ and a real T_1 such that $|y(t)| < \lambda$ and $|y(g(t))| < \lambda$ for $t \geq T_1$. Then the rest of the proof is same as in the lines of proof of Theorem 3.3.1 in [1]. However, for the sake of completeness, we give the proof.

Since $y(t)$ is oscillatory, then $L_i(t)$ is oscillatory, $i = 1, 2, ..., n$. Let $\{t'_k\}, t'_k \geq T_1$ be a sequence of numbers such that $L_{n-1}y(t'_k) = 0$. Let $\alpha'_k \in (t'_k, t'_{k+1})$ and

$$|L_{n-1}y(\alpha'_k)| = \max\{|L_{n-1}y(t)|; t'_k \leq t \leq t'_{k+1}\}.$$

Integrating (5) from t'_k to α'_k, we see that

$$|L_{n-1}y(\alpha'_k)| \leq m\lambda^\gamma \int_{t'_k}^{\alpha'_k} |p(s)|\,ds + \int_{t'_k}^{\alpha'_k} |f(s)|\,ds.$$

Taking sum with respect to k, we have

$$\sum_{k=1}^{\infty} |L_{n-1}y(\alpha'_k)| \leq m\lambda^\gamma \int_{t'_k}^{\infty} |p(s)|\,ds + \int_{t'_k}^{\infty} |f(s)|\,ds.$$

Since (7) and (8) hold, then $\lim_{k\to\infty} L_{n-1}y(\alpha'_k) = 0$ and hence

$$\lim_{t\to\infty} L_{n-1}y(t) = 0.$$

Then integrating (5) from t to ∞, we get

$$L_{n-1}y(t) = \int_t^\infty p(s)h(y(g(s)))\,ds - \int_t^\infty f(s)\,ds. \tag{12}$$

Now we shall prove that $\lim_{t\to\infty} L_{n-2}y(t) = 0$. let $\{t''_k\}, t''_k \geq T_1$ be a sequence of numbers such that $L_{n-2}y(t''_k) = 0$. Let $\alpha''_k \in (t''_k, t''_{k+1})$ and

$$|L_{n-2}y(\alpha''_k)| = \max\{|L_{n-2}y(t)|; t''_k \leq t \leq t''_{k+1}\}.$$

Integrating (12) from t''_k toα''_k, we have

$$|L_{n-2}y(\alpha''_k)| \leq m\lambda^\gamma \int_{t''_k}^{\alpha''_k} \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |p(s_n)|\,ds_n ds_{n-1}$$

$$+\int_{t''_k}^{\alpha''_k} \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty |f(s_n)|\,ds_n ds_{n-1}.$$

Now, summing the above integral inequality with respect to k, we obtain

$$\sum_{k=1}^{\infty} |L_{n-2}y(\alpha_k'')| \le m\lambda^{\gamma} \int_{t_k''}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |p(s_n)|\, ds_n ds_{n-1}$$
$$+ \int_{t_k''}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} |f(s_n)|\, ds_n ds_{n-1},$$

which in turn implies that

$$\lim_{t\to\infty} L_{n-2}y(t) = 0.$$

Proceeding as above repeatedly, we see that

$$\lim_{t\to\infty} L_0 y(t) = 0,$$

that is,

$$\lim_{t\to\infty} y(t) = 0.$$

Thus the theorem is proved. ■

Remark. Our Theorem 2.1 is more general than Theorem 1.1. In Theorem 2.1, one may consider either $\int^{\infty} \frac{1}{r_i(t)}\, dt = \infty$ or $\int^{\infty} \frac{1}{r_i(t)}\, dt < \infty, i = 1, 2, ..., n-1$. However, in both these cases, $\int^{\infty} |p(t)|\, dt < \infty$ and $\int^{\infty} |f(t)|\, dt < \infty$. Further our Theorem 2.1 cannot be comparable with the results in Section 3.2-3.3 in [1]. The function $H(t)$ cannot be reduced to a constant.

The following example strengthens Theorem 2.1.

Example 2.2. Consider

$$(t(ty'(t))')' + \tfrac{1}{t^2} y^{1/2}(\tfrac{t}{2}) = -\tfrac{4\sin 2t}{t^2} - \tfrac{18\cos 2t}{t^3} + \tfrac{16\sin t/2}{t^4} + \tfrac{21\sin 2t}{t^4} + \tfrac{16 sin2t}{t^5} - \tfrac{80(\sin t)^2}{t^6}, \tag{13}$$

$t \ge 1$. By Theorem 2.1, all oscillatory solution of (13) satisfy the property (9). In particular, $y(t) = \frac{1}{t^4}(\sin t)^2$ is such a solution satisfying (9).

Remark. In Example 2.2, the case $\int^{\infty} \frac{1}{r_i(t)}\, dt = \infty, i = 1, 2$ are satisfied. In the following, we give an example for the case $\int^{\infty} \frac{1}{r_i(t)}\, dt < \infty, i = 1, 2$.

Example 2.2 Consider

$$(e^t(e^t y'(t))')' + 2e^{-t}(\cos\frac{t}{2}) y(\frac{t}{2}) = 46e^{-3t}\cos t - 48e^{-3t}\sin t + e^{\frac{-7t}{2}}\sin t,\ t \ge 1. \tag{14}$$

All the conditions of Theorem 2.1 are satisfied and $y(t) = e^{-5t}\sin t$ is a solution of (14) satisfying (9).

The following example is of interest.

Example 2.3. Clearly $y(t) = e^{-5t}\sin t$ is a solution of the equation

$$(e^{-t}(e^t y'(t))')' + 2e^{-2t}(\cos\frac{t}{2})y(\frac{t}{2}) = 64e^{-5t}\cos t - 86e^{-5t}\sin t + e^{\frac{-9t}{2}}\sin t, \quad t \geq 1.$$

all the conditions of Theorem 2.1 are satisfied.

Remark. In the above example, one may observe that $\int^{\infty} \frac{1}{r_2(t)}\,dt = \infty$ and $\int^{\infty} \frac{1}{r_1(t)}\,dt < \infty$. Thus, the above remarks and examples ensure that our Theorem 2.1 is more general than the results in [1, 2, 3, 4].

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