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OSCILLATION AND NON-OSCILLATION OF NEUTRAL DIFFERENCE EQUATIONS OF FIRST ORDER WITH POSITIVE AND NEGATIVE COEFFICIENTS

Abstract: In this paper necessary and sufficient conditions have been obtained so that every solution of the Neutral Delay Difference Equation (NDDE)

$$\Delta(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) - r_n G(y_{n-\ell}) = f_n$$

where different symbols have there usual meaning, oscillates or tends to zero as $n \to \infty$ for different ranges of $\{p_n\}$. This paper generalizes some recent work. The results of this paper hold for linear, sublinear or super linear equations and also for homogeneous equations, i.e. when $f_n \equiv 0$.

Key words: oscillation, non–oscilation, asymptotic behaviour, neutral difference equations.

1. Introduction

During the last several years many research papers on the oscillatory behaviour of solutions of neutral delay difference equations (NDDEs) have appeared in the literature, as these equations occur as mathematical models of some real world problems (see [2, 4]). In this paper we study oscillatory and asymptotic behaviour of solutions of the first order NDDE

$$\Delta(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) - r_n G(y_{n-\ell}) = f_n, \tag{1}$$

where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}$, $\{q_n\}$, $\{r_n\}$, $\{f_n\}$ are infinite sequences of real numbers with $q_n \geq 0$, $r_n \geq 0$, m, k, ℓ are non negative integers, $G \in C(R, R)$ such that $xG(x) > 0$ for $x \neq 0$ and G is non-decreasing. If we put $r_n = 0$ for every n in (1) then (1) takes the form

$$\Delta(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) = f_n, \tag{2}$$

which is a NDDE of first order or a delay difference equation of $(k+1)$-th order (take $p_n = 0$). In [6,8,9] the authors have obtained necessary and sufficient conditions for the oscillatory and asymptotic behaviour of solutions of (2). In the present work an attempt is made to generalize the work of [6,8,9] and find the necessary and sufficient conditions for the oscillatory behaviour of solutions of (1). The motivation of the present problem came from the fact that the study of difference equations and differential equations run parallel. But at times we may note that the oscillatory behaviour of ordinary differential equation and their discrete analogues can be quite different (see [6]). Although many authors (see [1, 5, 10-12]) have studied the oscillatory behaviour of solutions of the corresponding neutral delay differential equations with positive and negative coefficients

$$(y(t)-p(t)y(t-\tau))'+Q(t)G(y(t-\sigma))-R(t)G(y(t-\delta))=f(t), \tag{3}$$

but most of these results have $f(t)\equiv 0$, $G(u)=u$, $p(t)\equiv p$ and $Q(t)\equiv q$, p, q are constants. Moreover, these results give information for the oscillatory behaviour of only bounded solutions of (3). It seems that (1) which is the discrete analogue of (3) is not studied yet. Also our work sufficiently indicates that the work of [1, 5, 11, 12] can be improved substantially. Results of this paper hold when G is linear, sublinear or superlinear and also for $f_n\equiv 0$.

In the present work we assume the following conditions for its use in the sequel

(H_1) $\displaystyle\sum_{n=0}^{\infty} q_n=\infty$,

(H_2) $\displaystyle\sum_{n=0}^{\infty} r_n<\infty$,

(H_3) There exists a sequence $\{F_n\}$ such that $\Delta F_n=f_n$ and $\lim_{n\to\infty} F_n=0$,

(H_4) G is Lipschitzian in every interval of the form $[a,b]$, where $0<a<b$.

In this paper the following conditions are assumed for $\{p_n\}$.

(A_1) $0\le p_n\le b<1$, (A_2) $-1<-b\le p_n\le 0$,
(A_3) $-b_2\le p_n\le -b_1<-1$, (A_4) $1<b_1\le p_n\le b_2$,
(A_5) $-b_2\le p_n\le 0$, (A_6) $0\le p_n\le b_2$,

where b, b_1, b_2 are positive real numbers.

Let us choose a positive integer $s>\max\{m,k,\ell\}$. By a solution of (1) on $[0,\infty)$ we mean a sequence $\{y_n\}$ of real numbers which is defined for $n\ge -s$ and which satisfies (1) (for $n=0,1,2,\ldots$). A solution $\{y_n\}$ of (1) on $(0,\infty)$ is said to be oscillatory if for every positive integer $N_0>0$, there exists $n\ge N_0$ such that $y_ny_{n+1}\le 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

2. Main Results

First we quote a lemma from [8]

Lemma 2.1. *Let $\{f_n\}$, $\{q_n\}$ and $\{p_n\}$ be sequences of real numbers defined for $n \geq N_0 \geq 0$ such that*

$$f_n = q_n - p_n q_{n-m}, \qquad n \geq N_0 + m$$

where $m \geq 0$ is an integer. Suppose that there exists real numbers b, b_1, b_2 such that p_n satisfies one of the conditions (A_2), (A_3) or (A_6). If $q_n > 0$ for $n \geq N_0$, $\liminf_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} f_n = L$ exists then $L = 0$.

Remark 1. The above lemma holds when p_n satisfies one of the conditions (A_1) or (A_4).

Theorem 2.2. *Suppose that p_n satisfies one of the conditions (A_1), (A_2) or (A_3). Let (H_1)-(H_3) hold. Then every solution of (1) oscillates or tends to zero as $n \to \infty$.*

Proof. Let $\{y_n\}$ be any non-oscillatory positive solution of (1) for $n \geq N_0 > 0$. Setting for $n \geq N > N_0$

$$z_n = y_n - p_n y_{n-m} \tag{4}$$

and

$$w_n = z_n + \sum_{i=n}^{\infty} r_i G(y_{i-\ell}) - F_n \tag{5}$$

we obtain

$$\Delta w_n = -q_n G(y_{n-k}) \leq 0. \tag{6}$$

Hence $w_n < 0$ or $w_n > 0$ for large n and $\lim_{n\to\infty} w_n = L$, where $-\infty \leq L < \infty$. We claim that $\{y_n\}$ is bounded. Otherwise, there exists a sequence $\{y_{n_k}\}$ such that $y_{n_k} \to \infty$ as $k \to \infty$ and $y_{n_k} = \max\{y_n : N \leq n \leq n_k\}$. We may choose n_k large enough such that $n_k - s > N$. Suppose that $\{p_n\}$ satisfies (A_1). Then using (H_3) we obtain

$$w_{n_k} = y_{n_k} - p_{n_k} y_{n_k - m} + \sum_{i=n_k}^{\infty} r_i G(y_{i-\ell}) - F_{n_k} \geq (1-b) y_{n_k} - F_{n_k} \to \infty \tag{7}$$

a contradiction. If $\{p_n\}$ satisfies (A_2) or (A_3) then we use (H_3) to obtain the following inequality in place of (7)

$$w_{n_k} > y_{n_k} - F_{n_k} \to +\infty \quad \text{as} \quad k \to \infty.$$

Thus we see that $L = \infty$, a contradiction. Hence our claim that $\{y_n\}$ is bounded, holds and $-\infty < L < \infty$. Consequently $\lim_{n\to\infty} z_n = L$ by (H_2) and (H_3). Next we claim $\liminf_{n\to\infty} y_n = 0$. Otherwise for $n \geq N_1 > N$ we have $y_n > \alpha > 0$ and since G is nondecreasing then (H_1) yields

$$\sum_{n=N_1+s}^{\infty} q_n G(y_{n-k}) > G(\alpha) \sum_{n=N_1+s}^{\infty} q_n = \infty. \tag{8}$$

However, taking summation in (6) we obtain

$$\sum_{n=N_1+s}^{i-1} q_n G(y_{n-k}) = -\sum_{n=N_1+s}^{i-1} \Delta w_n = -(w_i - w_{N_1+s}).$$

Taking limit $i \to \infty$ in the above expression we obtain

$$\sum_{n=N_1+s}^{\infty} q_n G(y_{n-k}) < \infty,$$

which contradicts (8). Hence $\liminf_{n\to\infty} y_n = 0$. Application of Lemma 2.1 yields $L = 0$. If $\{p_n\}$ satisfies (A_1), then

$$\begin{aligned} 0 &= \lim_{n\to\infty} z_n = \limsup_{n\to\infty}(y_n - p_n y_{n-m}) \\ &\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}(-p_n y_{n-m}) \geq (1-b) \limsup_{n\to\infty} y_n, \end{aligned}$$

which implies $\lim_{n\to\infty} y_n = 0$. If $\{p_n\}$ satisfies (A_2) or (A_3) then the fact $y_n \leq z_n$ implies $\lim_{n\to\infty} y_n = 0$. The proof of the case $y_n < 0$ for $n \geq N_0 > 0$ is similar. Thus the theorem is proved. ∎

From the proof of the above Theorem we note the following result as:

Corollary 2.3. *Suppose all the conditions of Theorem 2.2 hold then every unbounded solution of (1) oscillates.*

Remark 2. **(i)** Theorem 2.2 holds when G is linear, sublinear or superlinear. **(ii)** (H_3) implies and implied by

$$\left| \sum_{n=0}^{\infty} f_n \right| < \infty.$$

(iii) In the paper [5] where the oscillatory behaviour of solutions of (3) is studied the assumption $\sigma > \delta$ and $Q(t) > R(t)$ seems redundant. The discrete analogue of the above two conditions are $k > \ell$ and $q_n > r_n$. Since these conditions are not required for Theorem 2.2, therefore it seems the above paper [5] can be improved with the technique and method of this paper. Further, most of the work in [5] are for bounded solutions only whereas Theorem 2.2 holds for both bounded and unbounded solutions of (1).

Theorem 2.4. *Suppose that $\{p_n\}$ satisfies (A_1) and (H_2), (H_3), (H_4) hold. If every solution of (1) oscillates or tends to zero as $n \to \infty$ than (H_1) holds.*

Proof. If possible let

$$\sum_{n=0}^{\infty} q_n < \infty. \tag{9}$$

From (9), (H_2) and (H_3), we can find $N_1 > 0$ such that for $n \geq N_1$ implies

$$k\sum_{i=n}^{\infty} q_i < \frac{1-b}{20}, \quad k\sum_{i=n}^{\infty} r_n < \frac{1-b}{20} \quad \text{and} \quad |F_n| < \frac{1-b}{20},$$

where $k = \max\{k_1, G(1)\}$, k_1 is the Lipschitz constant of G in $[(1-b)/10, 1]$. Let $X = \ell_\infty^N$, Banach space of real bounded sequences $x = \{x_n\}$ with supremum norm

$$\|x\| = \sup\{\,|x_n| : n \geq N_1\}.$$

Define

$$S = \left\{x \in X : \frac{1-b}{10} \leq x_n \leq 1, \ \ n \geq N_1\right\}.$$

Since S is a closed subset of X, Then S is a complete metric space, where the metric is induced by the norm on X. For $y \in S$, define

$$T(y)_n = \begin{cases} (Ty)_{N_1+s}\,, & N_1 \leq n \leq N_1 + a = N_2, \\ p_n y_{n-m} + \sum\limits_{i=n}^{\infty} q_i G(y_{i-k}) - \sum\limits_{i=n}^{\infty} r_i G(y_{i-\ell}) & \\ \qquad\qquad +F_n + (1-b)/5, & n \geq N_2, \end{cases}$$

where a is any positive integer $> \max\{m, k, \ell\}$. Clearly, T maps S into S and $\|Tu - Tv\| < \mu\|u - v\|$ where $0 < \mu = (1+9b)/10 < 1$. Hence T is a contraction admitting a unique fixed point $y = \{y_n\}$ in S which is the required positive solution of (1). Thus the theorem is proved. ∎

Corollary 2.5. *Suppose that p_n satisfies (A_1). If (H_2), (H_3) and (H_4) hold then (H_1) is both necessary and sufficient for every solution of (1) to be oscillatory or tending to zero.*

This follows from Theorem 2.2 and 2.4.

Remark 3. If p_n satisfies (A_2) or (A_3) then we can have similar results like Theorem 2.4 and find positive solutions of (1) under condition (9).

Example. The NDDE

$$\Delta(y_n - (1+e)y_{n-1}) + (e-1)y_{n-1} = \big(2e^2 + e^{-1} - 2 - e\big)\, e^{-n}, \tag{10}$$

$n \geq 0$ has a positive unbounded solution $y = e^n + e^{-n}$ tending to ∞ as $n \to \infty$. Here p_n satisfies (A_4). It may be noted that (10) satisfies all the conditions of Theorem 2.2 (axcept the bounds of p_n).

The above example is a source of motivation for the following theorem.

Theorem 2.6. *Suppose p_n satisfies (A_4). Let (H_1), (H_2), (H_3) hold. Then every bounded solution of (1) oscillates or tends to zero as $n \to \infty$.*

Proof. Proceeding as in the proof of Theorem 2.2 and noting that $0 \leq \lim\limits_{n\to\infty} z_n \leq (1-b_1) \limsup\limits_{n\to\infty} y_n$, we prove the theorem. ■

Theorem 2.7. *Suppose that p_n satisfies (A_4) and (H_2), (H_3), (H_4) hold. If every bounded solution of (1) oscillates or tends to zero as $n \to \infty$ then (H_1) holds.*

Proof. The proof here is similar to that of Theorem 2.4 with the following changes:

$$k\sum_{i=n}^{\infty} q_i < \frac{b_1}{\alpha + b_1}, \quad k\sum_{i=n}^{\infty} r_i < \frac{b_1}{\alpha + b_1}, \quad |F_n| < \frac{b_1}{\alpha + b_1},$$

where $\alpha > (3b_1 + b_1b_2 + 2b_2 - b_1^2)/(b_1 - 1)$. Suppose $L = (\alpha + 3b_1 + b_1b_2 + 2b_2)/b_1(\alpha + b_1)$. It is clear that $L < 1$. Let $S = \Big\{x \in \ell_N^\infty : \frac{b_1}{\alpha+b_1} \leq x \leq L\Big\}$. Let $\lambda = (b_1 + 2)b_2/(\alpha + b_1)$. Define for $y \in S$

$$(Ty)_n = \begin{cases} (Ty)_{N_1+a} & \text{for} \quad N_1 \leq n \leq N_1 + a, \\ \frac{y_{n+m}}{p_{n+m}} - \frac{1}{p_{n+m}} \sum\limits_{i=n+m}^{\infty} q_i G(y_{i-k}) + \frac{1}{p_{n+m}} \sum\limits_{i=n+m}^{\infty} r_i G(y_{i-\ell}) \\ \qquad - \frac{F_{n+m}}{p_{n+m}} + \frac{\lambda}{p_{n+m}}, \quad \text{for } n \geq N_1 + a. \end{cases}$$

$\|Ty_1 - Ty_2\| \leq \mu \|y_1 - y_2\|$ where $\mu = \frac{\alpha+3b_1}{b_1(\alpha+b_1)}$. Hence the theorem is proved. ■

Corollary 2.8. *Suppose that $\{p_n\}$ satisfies (A_4). Let (H_2), (H_3) and (H_4) hold. Then every bounded solution of (1) oscillates or tends to zero as $n \to \infty$ if and only if (H_1) holds.*

The proof follows from Theorem 2.6 and 2.7.

Theorem 2.9. *Suppose that $\{p_n\}$ satisfies (A_5). Let (H_2) and (H_3) hold. Suppose that*

$$(\mathrm{H}_5)\qquad \sum_{n=s}^{\infty} q_n^* = \infty \quad \text{where} \quad q_n^* = \min\{q_n, q_{n-m}\},$$

$$(\mathrm{H}_6)\qquad G(-u) = -G(u),$$

$$(\mathrm{H}_7)\ \ G(u)G(v) \geq G(uv) \ \ and\ \ G(u)+G(v) \geq \delta G(u+v)\ \ for\ \ u>0,\ v>0$$

and some constant $\delta > 0$. Then every solution of (1) oscillates or tends to zero as $n \to \infty$.

Proof. Let $y = \{y_n\}$ be an eventually positive solution of (1) for $n \geq N_0 > 0$. Then we proceed as in the proof of Theorem 2.2 and arrive at (4), (5) and (6). Consequently $\lim\limits_{n\to\infty} w_n = L$, where $-\infty \leq L < \infty$. But $z_n > 0$ and $z_n - F_n = w_n - \sum\limits_{i=n}^{\infty} r_i G(y_{i-\ell}) \leq w_n$. We claim $\{y_n\}$ is bounded. Otherwise $\{z_n\}$ is unbounded and we can find a sequence $\{z_{n_k}\}$ such that $n_k \to \infty$, $z_{n_k} \to \infty$ as $k \to \infty$ and $z_{n_k} = \max\{z_n : N_1 \leq n \leq n_k\}$. Then

$$w_{n_k} = z_{n_k} + \sum_{i=n_k}^{\infty} r_i G(y_{i-\ell}) - F_{n_k} \geq z_{n-k} - F_{n_k}.$$

Taking limit $k \to \infty$, we get $L = \infty$, a contradiction. Hence our claim holds. Thus $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} w_n = L$. Since $z_n > 0$ therefore $L \geq 0$. If $L = 0$ then $\lim\limits_{n\to\infty} y_n = 0$. If $L > 0$ then $z_n > \lambda > 0$ for $n \geq N_3 > N_2$. Using the definition of q_n^*, (H_7) we obtain

$$\begin{aligned} 0 &= \Delta w_n + q_n G(y_{n-k}) + G(-p_{n-k})\{\Delta w_{n-m} + q_{n-m} G(y_{n-k-m})\} \\ &\geq \Delta w_n + G(b_2)\Delta w_{n-m} + \delta q_n^* G(z_{n-k}) \\ &\geq \Delta w_n + G(b_2)\Delta w_{n-m} + \delta q_n^* G(\lambda). \end{aligned}$$

If we take sum form $n = N_2$ to $n = i-1$ and take limit $i \to \infty$ and use (H_5) then we obtain the contradiction $w_i + G(b_2) w_{i-m} \to -\infty$ as $i \to \infty$. The proof for the case $y_n < 0$ for large n is similar. Thus the theorem is proved. ■

Remark 4. The prototype of G satisfying (H_6) and (H_7) is $G(u) = (\beta + |u|^{\mu})|u|^{\lambda}\mathrm{sgn} u$ where $\lambda > 0$, $\mu > 0$, $\beta \geq 1$.

Remark 5. **(i)** This paper generalizes all the results of [8]. **(ii)** If we compare the above theorem with Theorems 4 and 6 of [5] then we see that we do not require the monotonicity of $\{q_n\}$ and $\{r_n\}$. Also the authors of [5] have assumed some other conditions which seem to be unnecessary to prove that every bounded solution of (3) oscillates or tends to zero as $t \to \infty$. Thus the above theorem gives definite indication that the results of [5] can be improved.

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