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ON SINGULAR INITIAL VALUE PROBLEM FOR NONLINEAR FREDHOLM INTEGRODIFFERENTIAL EQUATIONS

Abstract: There are given conditions for the solvability of the singular initial value problem for Fredholm integrodifferential equations. The continuous dependence of solutions on a parameter is investigated as well.

Key words: Fredholm integrodifferential equations, Banach contraction principle.

1. Introduction

The most results concerning of behaviour of singular ordinary and integrodifferential equations were studied by means of modifications of the Wazewki's topological method and by fixed point theorems (see [1-7]). In [4,5] there are investigated singular initial problems for Volterra integrodifferential equations. However, in cases of Fredholm integrodifferential equations there is necessary to modify known results from the theory of Voltera integrodifferential equations which is shown in this paper.

We give sufficient conditions of existence and uniqueness of solutions of nonlinear singular Fredholm integrodifferential equations. Moreover, we shall also investigate a problem of continuous dependence of solutions on a parameter.

Consider the following initial value problem

$$(1)\qquad y'(t) = \mathcal{F}\left(t, y(t), \int_{0^+}^{1} K(t,s,y(t),y(s))ds, \mu\right), \quad y(0^+,\mu) = 0,$$

and suppose

(I) $\mathcal{F} : \Omega \to R^n$, $\mathcal{F} \in C^0(\Omega)$,
$\Omega = \{(t,u_1,u_2,\mu) \in J \times R^n \times R^n \times R : |u_1| \le \phi(t),\ |u_2| \le \psi(t)\}$,
$J = (0,1]$, , $0 < \phi(t) \in C^0(J)$, $\phi(0^+) = 0$, $0 < \psi(t) \in C^0(J)$, $|\cdot|$ denotes the usual norm in R^n. There exist constants $M_i \ge 0$, $i = 1,2$

such that $|\mathcal{F}(t,\overline{u}_1,\overline{u}_2,\mu)-\mathcal{F}(t,\overline{\overline{u}}_1,\overline{\overline{u}}_2,\mu)| \le M_1|\overline{u}_1-\overline{\overline{u}}_1|+M_2|\overline{u}_2-\overline{\overline{u}}_2|$ for all $(t,\overline{u}_1,\overline{u}_2,\mu)$, $(t,\overline{\overline{u}}_1,\overline{\overline{u}}_2,\mu)\in\Omega$.

(II) $K:\Omega^1\to R^n$, $K\in C^0(\Omega_1)$,
$\Omega_1=\{(t,s,v_1,v_2)\in J\times J\times R^n\times R^n : |v_1|\le\phi(t)\ |v_2|\le\phi(t)\}$.
There exist constants $N_i\ge 0,\ i=1,2$ such that
$|K(t,s,\overline{v}_1,\overline{v}_2)-K(t,s,\overline{\overline{v}}_1,\overline{\overline{v}}_1|\le N_1|\overline{v}_1-\overline{\overline{v}}_1|+N_2e^{\lambda(t-s)}|\overline{v}_2-\overline{\overline{v}}_2|$
for all $(t,s,\overline{v}_1,\overline{v}_2)$, $(t,s,\overline{\overline{v}}_1,\overline{\overline{v}}_2)\in\Omega_1$, $\lambda>0$ is a sufficiently large constant such that $\left(\frac{M_1+M_2N_1+M_2N_2}{\lambda}\right)<1$.

2. Main results

Theorem 2.1. *Let the functions $\mathcal{F}(t,u_1,u_2,\mu)$, $K(t,s,v_1,v_2)$ satisfy conditions (I), (II) and, moreover*

$$|\mathcal{F}|\le g_1(t)|u_1|+g_2(t)|u_2|,;\quad 0<g_i(t)\in C^0(J),\ i=1,2,\quad \int_{0^+}^t g_1(s)\phi(s)ds\le\alpha\phi(t),$$

$$\int_{0^+}^t g_2(s)\psi(s)ds\le\beta\phi(t),\qquad \alpha+\beta\le 1.$$

Then the problem (1) *has a unique solution $y(t,\mu)$ for each $\mu\in R$, $t\in J$.*

Proof. Denote H the Banach space of continuous vector-valued functions

$$h:J_0\to R^n,\ J_0=[0,1],\ |h(t)|\le\phi(t)$$

on J with the norm

$$||h||_\lambda=\max_{t\in J_0}\{e^{-\lambda t}|h(t)|\}.$$

The initial value problem (1)is equivalent to the system of integral equations

$$y(t)=\int_{0^+}^t\mathcal{F}\left(s,y(s),\int_{0^+}^1 K(s,w,y(s),y(w))dw,\mu\right)ds \tag{2}$$

Define the operator T by the right-hand side of (2)

$$T(h)=\int_{0^+}^t\mathcal{F}\left(s,h(s),\int_{0^+}^1 K(s,w,h(s),h(w))dw,\mu\right)ds,$$

where $h\in H$. Let $\mu\in R$ be fixed. The transformation T maps H continuously into itself because

$$\begin{aligned}|T(h)|&\le\int_{0^+}^t\left|\mathcal{F}\left(s,h(s),\int_{0^+}^1 K(s,w,h(s),h(w))dw,\mu\right)\right|ds\\ &\le\int_{0^+}^t\left[g_1(s)|h(s)|+g_2(s)\left|\int_{0^+}^1 K(s,w,h(s),h(w))dw\right|\right]ds\\ &\le\int_{0^+}^t\left(g_1(s)\phi(s)+g_2(s)\psi(s)\right)ds\le(\alpha+\beta)\phi(t)\le\phi(t)\end{aligned}$$

for every $h \in H$. Using (I), (II) and the definition $||.||_\lambda$ we have

$$|T(h_2) - T(h_1)| \le \int_{0^+}^{t} \left| \mathcal{F}\left(s, h_2(s), \int_{0^+}^{1} K(s,w,h_2(s),h_2(w))dw, \mu\right)\right.$$

$$\left. - \mathcal{F}\left(s, h_1(s), \int_{0^+}^{1} K(s,w,h_1(s),h_1(w))dw, \mu\right)\right| ds$$

$$\le \int_{0^+}^{t} \left(M_1|h_2(s) - h_1(s)|\right.$$

$$\left. + M_2 \int_{0^+}^{1} |K(s,w,h_2(s),h_2(w)) - K(s,w,h_1(s),h_1(w))|dw\right) ds$$

$$\le \int_{0^+}^{t} \left(M_1|h_2(s) - h_1(s)|\right.$$

$$\left. + M_2 \int_{0^+}^{1} (N_1|h_2(s) - h_1(s)| + N_2 e^{\lambda(s-w)}|h_2(w) - h_1(w)|)dw\right) ds$$

$$\le M_1 ||h_2 - h_1||_\lambda \int_{0^+}^{t} e^{\lambda s} ds$$

$$+ M_2 N_1 ||h_2 - h_1||_\lambda \int_{0^+}^{t} e^{\lambda s} ds + M_2 N_2 ||h_2 - h_1||_\lambda \int_{0^+}^{t}\int_{0^+}^{1} e^{\lambda s} dw ds$$

$$= ||h_2 - h_1||_\lambda \left(M_1(\frac{e^{\lambda t}}{\lambda} - \frac{1}{\lambda}) + M_2 N_1(\frac{e^{\lambda t}}{\lambda} - \frac{1}{\lambda}) + M_2 N_2(\frac{e^{\lambda t}}{\lambda} - \frac{1}{\lambda})\right)$$

$$< ||h_2 - h_1||_\lambda \, e^{\lambda t} \left(\frac{M_1 + M_2 N_1 + M_2 N_2}{\lambda} \right).$$

Thus

$$||T(h_2) - T(h_1)||_\lambda = \max_{t \in J_0} \{ e^{-\lambda t} |T(h_2) - T(h_1)| \} \le q \; ||h_2 - h_1||_\lambda,$$

where

$$q := \frac{1}{\lambda}(M_1 + M_2 N_1 + M_2 N_2) < 1.$$

By Banach theorem the operator T has a unique stationary point h^* in the space H , i.e. $h^*(t) \equiv T(h^*(t))$, $t \in J_0$. Then $y := h^*$ is the desidered solution of (1). ∎

Theorem 2.2. *Let all assumptions of Theorem 2.1 be satisfied and let there exist a constant $L > 0$ and an integrable function $\gamma : J_0 \to J_0$ such that*

$$|\mathcal{F}(t, u_1, u_2, \mu_2) - \mathcal{F}(t, u_1, u_2, \mu_1)| \le \gamma(t)|\mu_2 - \mu_1|,$$

where (t, u_1, u_2, μ_1), $(t, u_1, u_2, \mu_2) \in \Omega$ *and*

$$\max_{t \in J_0} \left\{ e^{-\lambda t} \int_{0^+}^{t} \gamma(s) ds \right\} \leq L.$$

Then the solution $y(t, \mu)$ *of* (1) *is continuous with respect to the variables* $(t, \mu) \in J \times R$.

Proof. Define as above, for $h \in H$ the transformation $T_\mu(h)$ by means of the right-hand side (2) then we obtain

$$||T_\mu(h) - T_\mu(y)||_\lambda \leq \left(\frac{M_1 + M_2 N_1 + M_2 N_2}{\lambda} \right) ||h - y||_\lambda.$$

By the hypothesis of Theorem 2.2 we get

$$\begin{aligned} &e^{-\lambda t} |T_{\mu_2}(h) - T_{\mu_1}(h)| \\ &\quad \leq e^{-\lambda t} \int_{0^+}^{t} \bigg| \mathcal{F}(s, h(s), \int_{0^+}^{1} K(s, w, h(s), h(w)) dw, \mu_2) \\ &\quad\quad - \mathcal{F}(s, h(s), \int_{0^+}^{1} K(s, w, h(s), h(w)) dw, \mu_1) \bigg| ds \\ &\quad \leq e^{-\lambda t} \int_{0^+}^{t} \gamma(s) |\mu_2 - \mu_1| ds \leq L |\mu_2 - \mu_1|. \end{aligned}$$

Hence

$$||T_{\mu_2}(h) - T_{\mu_1}(h)||_\lambda \leq L |\mu_2 - \mu_1|.$$

From this and by Theorem 2.1 we obtain

$$\begin{aligned} &||h(t, \mu_2) - h(t, \mu_1)||_\lambda = ||T_{\mu_2}[h(t, \mu_2)] \\ &\quad - T_{\mu_2}[h(t, \mu_1)] + T_{\mu_2}[h(t, \mu_1)] - T_{\mu_1}[h(t, \mu_1)]||_\lambda \leq ||T_{\mu_2}[h(t, \mu_2)] \\ &\quad - T_{\mu_2}[h(t, \mu_1)]||_\lambda + ||T_{\mu_2}[h(t, \mu_1)] - T_{\mu_1}[h(t, \mu_1)]||_\lambda \\ &\leq \left(\frac{M_1 + M_2 N_1 + M_2 N_2}{\lambda} \right) ||h(t, \mu_2) - h(t, \mu_1)||_\lambda + L |\mu_2 - \mu_1|. \end{aligned}$$

Thus

$$||h(t, \mu_2) - h(t, \mu_1)||_\lambda \leq \left[1 - \left(\frac{M_1 + M_2 N_1 + M_2 N_2}{\lambda} \right) \right]^{-1} L |\mu_2 - \mu_1|.$$

Consequently the function $h(t, \mu)$ is uniformly continuous with respect to the variable $\mu \in R$; so $y(t, \mu)$ is also continuous with respect to two variables $(t, \mu) \in J \times R$. The proof is complete. ∎

Now we shall consider a special form of (1)

$$(3)\quad p_i(t)y_i' - y_i = \mathcal{F}_i\left(t, y(t), \int_{0^+}^1 K(t,s,y(t),y(s))ds, \mu\right), \quad y_i(0^+,\mu) = 0,$$

$i = 1, \ldots, n$. A solution of (3) will be described by means of a solution of an auxiliary system

$$(4)\qquad p_i(t)y_i' - y_i = 0, \quad i = 1, \ldots, n.$$

Put

$$(5)\qquad \eta_i(t, C_i) = C_i\ exp\left(\int_1^t \frac{ds}{p_i(s)}\right), \quad i = 1, \ldots, n.$$

It is obvious that (5) is the general solution of (4). Denote

$$Y = (Y_1, \ldots, Y_n), \quad Y_i(t,\mu) = \frac{y_i(t,\mu) - \eta_i(t,C_i)}{\eta_i(t,C_i)},$$

$$G = (G_1, \ldots, G_n), \quad G_i\left(t, y(t), \int_{0^+}^1 K(t,s,y(t),y(s))ds, \mu\right)$$
$$= (p_i(t)\eta_i(t,C_i))^{-1}\mathcal{F}_i\left(t, y(t), \int_{0^+}^1 K(t,s,y(t),y(s))ds, \mu\right)$$

$$\eta(t)(1+Y(t)) = (\eta_1(t,C_1)(1+Y_1(t,\mu)), \ldots, \eta_n(t,C_n)(1+Y_n(t,\mu)).$$

Then

$$Y_i' = \frac{(y_i'(t,\mu) - \frac{\eta_i(t,C_i)}{p_i(t)})\eta_i(t,C_i) - (y_i(t,\mu) - \eta_i(t,C_i))\frac{\eta_i(t,C_i)}{p_i(t)}}{\eta_i(t,C_i)^2}$$
$$= \frac{p_i(t)y_i'(t,\mu) - y_i(t,\mu)}{p_i(t)\eta_i(t,C_i)}, \quad i = 1, 2, \ldots, n.$$

Now, we can rewrite the equation (3) in the form

$$Y_i' = G_i\left(t, \eta(t)(1+Y(t)), \int_{0^+}^1 K(t,s,\eta(t)(1+Y(t)), \eta(s)(1+Y(s))ds, \mu\right).$$

Denote

$$\mathcal{G}_i\left(t, Y(t), \int_{0^+}^1 \mathcal{K}(t,s,Y(t),Y(s))ds, \mu\right)$$
$$= G_i\left(t, \eta(t)(1+Y(t)), \int_{0^+}^1 K(t,s,\eta(t)(1+Y(t)), \eta(s)(1+Y(s))ds, \mu\right).$$

Then

$$Y_i' = \mathcal{G}_i\left(t, Y(t), \int_{0^+}^{1} \mathcal{K}(t,s,Y(t),Y(s))ds, \mu\right). \tag{6}$$

We suppose:

(i) $\mathcal{G} \in C^0(\overline{\Omega})$,
$\overline{\Omega} = \{(x, Y, u, \mu) \in J \times R^n \times R^n \times R : |Y| \leq \phi(t),\ |u| \leq \psi(t)\}$,
$|\mathcal{G}(t, \overline{Y}, \overline{u}, \mu) - \mathcal{G}(t, \overline{\overline{Y}}, \overline{\overline{u}}, \mu)| \leq M_1|\overline{Y} - \overline{\overline{Y}}| + M_2|\overline{u} - \overline{\overline{u}}|$
for all $(t, \overline{Y}, \overline{u}, \mu),\ (t, \overline{\overline{Y}}(t), \overline{\overline{u}}) \in \overline{\Omega}$.

(ii) $\mathcal{K} \in C^0(\overline{\Omega}_1),\ \overline{\Omega}_1 = \{(t, s, Y(t), Y(s)) \in J \times J \times R^n \times R^n :$
$|Y(t)| \leq \phi(t),\ |Y(s)| \leq \phi(t)\}$,
$|\mathcal{K}(t, s, \overline{Y}(t), \overline{Y}(s)) - \mathcal{K}(t, s, \overline{\overline{Y}}(t), \overline{\overline{Y}}(s)| \leq N_1|\overline{Y}(t) - \overline{\overline{Y}}(t)| + N_2 e^{\lambda(t-s)}$
$\times |\overline{Y}(s) - \overline{\overline{Y}}(s)|$ for all $(t, s, \overline{Y}(t), \overline{Y}(s)),\ (t, s, \overline{\overline{Y}}(t), \overline{\overline{Y}}(s)) \in \overline{\Omega}_1$.

The functions $\phi(t)$, $\psi(t)$ and the constants $M_i, N_i,\ i = 1, 2,\ \lambda$ have the same properties as in Theorem 2.1.

Theorem 2.3. *Let the functions $\mathcal{G}(t, Y(t), \mu),\ \mathcal{K}(t, s, Y(t), Y(s))$ satisfy the conditions (i),(ii) and*

$$|\mathcal{G}| \leq g_1(t)|Y| + g_2(t)|u|, \quad 0 < g_j(t) \in C^0(J),\ j = 1, 2,$$

$$\int_{0^+}^{t} g_1(s)\phi(s)ds \leq \alpha\phi(t), \quad \int_{0^+}^{t} g_2(s)\psi(s)ds \leq \beta\phi(t), \quad \alpha + \beta \leq 1,$$

$$0 < p_i(t) \in C^0(J), \quad \int_{0^+}^{1} \frac{dt}{p_i(t)} = \infty,\ i = 1, \ldots, n.$$

Then there exists a unique solution $y(t, \mu)$ of (3) satisfying an inequality

$$|y_i(t, \mu) - \eta_i(t, C_i)| \leq \phi(t)|\eta_i(t, C_i)|,$$

on J, $i = 1, \ldots, n$.

Proof. The equation (6) satisfies all assumptions of Theorem 2.1. Thus, there exists a unique solution $Y(t, \mu)$ of (6) such that $|Y(t, \mu| \leq \phi(t)$. From this and by the notations above, we have

$$|Y_i(t, \mu)| = \left|\frac{y_i(t, \mu) - \eta_i(t, C_i)}{\eta_i(t, C_i)}\right| \leq \phi(t) \Rightarrow |y_i(t, \mu) - \eta_i(t, C_i)| \leq \phi(t)|\eta_i(t, C_i)|.$$

The proof is complete. ■

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