F A S C I C U L I   M A T H E M A T I C I

Nr 37 2007

Jiří Šremr

ON THE INITIAL VALUE PROBLEM FOR TWO–DIMENSIONAL SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MONOTONE OPERATORS

Abstract: We establish new efficient conditions sufficient for the unique solvability of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators.

Key words: System of functional differential equations with monotone operators, initial value problem, solvability

1. Introduction and notation

On the interval $[a, b]$, we consider two–dimensional differential system

$$u_i'(t) = \sigma_{i1}\, \ell_{i1}(u_1)(t) + \sigma_{i2}\, \ell_{i2}(u_2)(t) + q_i(t) \qquad (i = 1, 2) \tag{1.1}$$

with the initial conditions

$$u_1(a) = c_1, \qquad u_2(a) = c_2\,, \tag{1.2}$$

where $\ell_{ik} : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ are linear nondecreasing operators, $\sigma_{ik} \in \{-1, 1\}$, $q_i \in L([a, b]; \mathbb{R})$, and $c_i \in \mathbb{R}$ $(i, k = 1, 2)$. Under a solution of the problem (1.1), (1.2) is understood an absolutely continuous vector function $u = (u_1, u_2)^T : [a, b] \to \mathbb{R}^2$ satisfying (1.1) almost everywhere on $[a, b]$ and verifying also the initial conditions (1.2).

The problem on the solvability of the Cauchy problem for linear functional differential equations and their systems has been studied by many authors (see, e.g., [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 15, 16, 18] and references therein). There are a lot of interested results but only a few efficient conditions is known at present. Furthermore, most of them is available for the one-dimmensional case only or for the systems with the so–called Volterra operators (see, e.g., [3, 4, 5, 13, 10, 7]). Let us mention that the efficient conditions guaranteeing the unique solvability of the initial value problem for n-dimensional systems of linear functional differential equations are given, e.g., in [12, 2, 15, 14, 11].

In this paper, we establish new efficient conditions sufficient for the unique solvability of the problem (1.1), (1.2) with $\sigma_{11}\sigma_{22} = -1$. The cases, where $\sigma_{11} = \sigma_{22} = 1$ and $\sigma_{11} = \sigma_{22} = -1$ are studied in [9] and [17], respectively.

The integral conditions given in Theorems 2.1–2.6 are optimal in a certain sense which is shown by counter–examples constructed in the last part of the paper.

The following notation is used throughout the paper:

(1) $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[\,$.
(2) $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max\Big\{|u(t)| : t \in [a,b]\Big\}.$$

(3) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $h : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|h\|_L = \int_a^b |h(s)|ds.$$

(4) $L\big([a,b];\mathbb{R}_+\big) = \Big\{h \in L([a,b];\mathbb{R}) : h(t) \geq 0 \text{ for a.a. } t \in [a,b]\Big\}.$
(5) An operator $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is said to be nondecreasing if the inequality

$$\ell(u_1)(t) \leq \ell(u_2)(t) \quad \text{for a.a.} \quad t \in [a,b]$$

holds for every functions $u_1, u_2 \in C([a,b];\mathbb{R})$ such that

$$u_1(t) \leq u_2(t) \quad \text{for} \quad t \in [a,b].$$

(6) $\mathcal{P}_{ab}$ is the set of linear nondecreasing operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

2. Main results

In this section, we present the main results of the paper. The proofs are given later, in Section 3. Theorems formulated below contain the efficient conditions sufficient for the unique solvability of the problem (1.1), (1.2) with $\sigma_{11}\sigma_{22} = -1$. Recall that the operators ℓ_{ik} are supposed to be linear and nondecreasing, i.e., such that $\ell_{ik} \in \mathcal{P}_{ab}$ for $i, k = 1, 2$.

Put

$$A_{ik} = \int_a^b \ell_{ij}(1)(s)ds \quad \text{for} \quad i, k = 1, 2 \tag{2.1}$$

and

$$\varphi(s) = \begin{cases} 1 & \text{for} \quad s \in [0, 1[\\ 1 - \frac{1}{4}(s-1)^2 & \text{for} \quad s \in [1, 3[\end{cases}. \tag{2.2}$$

2.1. The case $\sigma_{12}\sigma_{21} > 0$. At first, we consider the case, where $\sigma_{11} = 1$ and $\sigma_{22} = -1$.

Theorem 2.1. *Let $\sigma_{11} = 1$, $\sigma_{22} = -1$, and $\sigma_{12}\sigma_{21} > 0$. Let, moreover,*

$$A_{11} < 1, \qquad A_{22} < 3, \tag{2.3}$$

and

$$A_{12}A_{21} < (1 - A_{11})\varphi(A_{22}), \tag{2.4}$$

where the numbers A_{ik} $(i, k = 1, 2)$ are defined by (2.1) *and the function φ is given by* (2.2)*. Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.1. Neither one of the strict inequalities in (2.3) and (2.4) can be replaced by the nonstrict one (see Examples 4, 4, 4, and 4).

Remark 2.2. Let H_1 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 3, \quad z < (1 - x)\varphi(y)$$

(see Fig. 2.1). According to Theorem 2.1, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ik} \in \mathcal{P}_{ab}$ $(i, k = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds \right) \in H_1\,.$$

The next statement concerning the case, where $\sigma_{11} = -1$ and $\sigma_{22} = 1$, follows immediately from Theorem 2.1.

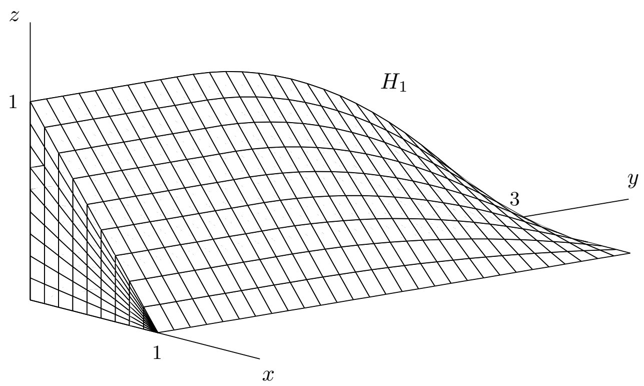

Fig. 2.1.

Theorem 2.2. *Let $\sigma_{11} = -1$, $\sigma_{22} = 1$, and $\sigma_{12}\sigma_{21} > 0$. Let, moreover,*

$$A_{11} < 3, \qquad A_{22} < 1, \tag{2.5}$$

and

$$A_{12}A_{21} < (1 - A_{22})\varphi(A_{11}),$$

where the numbers A_{ik} $(i, k = 1, 2)$ are defined by (2.1) *and the function φ is given by* (2.2)*. Then the problem* (1.1), (1.2) *has a unique solution.*

2.2. The case $\sigma_{12}\sigma_{21} < 0$

At first, we consider the case, where $\sigma_{11} = 1$ and $\sigma_{22} = -1$.

Theorem 2.3. *Let $\sigma_{11} = 1$, $\sigma_{22} = -1$, and $\sigma_{12}\sigma_{21} < 0$. Let, moreover, the condition* (2.3) *be satisfied and*

$$A_{12}A_{21} < (1 - A_{11})(3 - A_{22}), \tag{2.6}$$

where the numbers A_{ik} $(i, k = 1, 2)$ are defined by (2.1)*. Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.3. The strict inequalities (2.3) cannot be replaced by the nonstrict ones (see Examples 4 and 4). Furthermore, the strict inequality (2.6) cannot be replaced by the nonstrict one provided $A_{22} > 1$ (see Example 4).

Remark 2.4. Let H_2 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 3, \quad z < (1 - x)(3 - y)$$

(see Fig. 2.2). According to Theorem 2.3, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds \right) \in H_2\,.$$

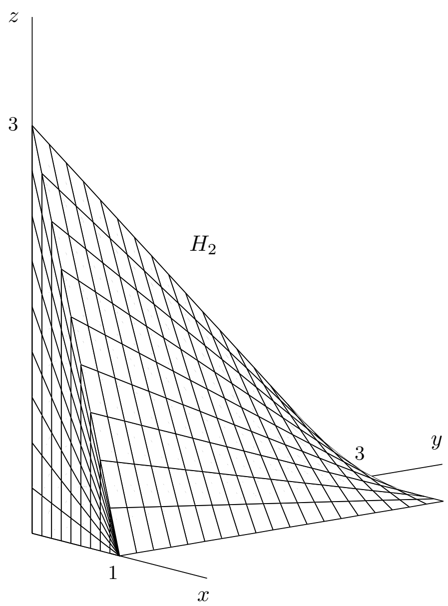

Fig. 2.2.

Example 4 shows that Theorem 2.3 is optimal whenever $1 < A_{22} < 3$. If $A_{22} \leq 1$ then the theorem mentioned can be improved. For example, the next theorem improves Theorem 2.3 if A_{22} is close to zero.

Theorem 2.4. *Let* $\sigma_{11} = 1$, $\sigma_{22} = -1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover,*

$$A_{11} < 1, \qquad A_{22} < 1, \tag{2.7}$$

and

$$A_{12}A_{21} < \frac{\omega(1 - A_{11})\big[1 + A_{22}(1 - A_{22})\big]}{1 - A_{11} + \omega A_{22}}\,, \tag{2.8}$$

where

$$\omega = 4\sqrt{1 - A_{11}} + \left(1 + \sqrt{(1 - A_{11})(1 - A_{22})}\right)^2 \tag{2.9}$$

and the numbers A_{ik} $(i, k = 1, 2)$ *are defined by* (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.5. If $A_{22} = 0$ then the inequality (2.8) can be rewritten as

$$A_{12}A_{21} < 4\sqrt{1 - A_{11}} + \left(1 + \sqrt{1 - A_{11}}\right)^2,$$

which coincides with the assumptions of Theorem 2.2 in [9].

Remark 2.6. Let H_3 be the set of triplets $(x,y,z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < \frac{\omega_0(1-x)\big[1+y(1-y)\big]}{1-x+\omega_0 y}\,,$$

where

$$\omega_0 = 4\sqrt{1-x} + \left(1 + \sqrt{(1-x)(1-y)}\right)^2$$

(see Fig. 2.3). According to Theorem 2.4, the problem (1.1), (1.2) is uniquely

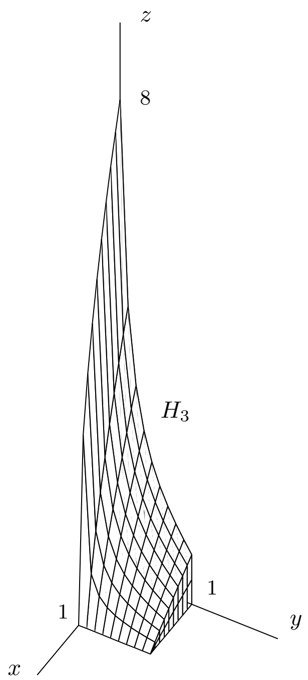

Fig. 2.3.

solvable if $\ell_{ik} \in \mathcal{P}_{ab}$ $(i,k=1,2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds\right) \in H_3\,.$$

The next statements concerning the case, where $\sigma_{11} = -1$ and $\sigma_{22} = 1$, follow immediately from Theorems 2.3 and 2.4.

Theorem 2.5. *Let $\sigma_{11} = -1$, $\sigma_{22} = 1$, and $\sigma_{12}\sigma_{21} < 0$. Let, moreover, the condition* (2.5) *be satisfied and*

$$A_{12}A_{21} < (1-A_{22})(3-A_{11}),$$

where the numbers A_{ik} $(i,k=1,2)$ are defined by (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Theorem 2.6. *Let $\sigma_{11}=-1$, $\sigma_{22}=1$, and $\sigma_{12}\sigma_{21}<0$. Let, moreover, the condition* (2.7) *be satisfied and*

$$A_{12}A_{21} < \frac{\widetilde{\omega}(1-A_{22})\big[1+A_{11}(1-A_{11})\big]}{1-A_{22}+\widetilde{\omega}A_{11}}\,,$$

where

$$\widetilde{\omega} = 4\sqrt{1-A_{22}} + \left(1+\sqrt{(1-A_{11})(1-A_{22})}\right)^2$$

and the numbers A_{ij} $(i,j=1,2)$ are defined by (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

3. Proofs of the Main Results

In this section, we shall prove the statements formulated above. Recall that the numbers A_{ik} $(i,k=1,2)$ are defined by (2.1) and the function φ is given by (2.2).

It is well–known from the general theory of boundary value problems for functional differential equations (see, e.g., [12, 8, 11, 16]) that the following lemma is true.

Lemma 3.1. *The problem* (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u_i'(t) = \sigma_{i1}\,\ell_{i1}(u_1)(t) + \sigma_{i2}\,\ell_{i2}(u_2)(t) \qquad (i=1,2), \tag{3.1}$$

$$u_1(a)=0, \qquad u_2(a)=0 \tag{3.2}$$

has only the trivial solution.

In order to simplify the discussion in the proofs below, we formulate the following obvious lemma.

Lemma 3.2. $(u_1,u_2)^T$ *is a solution of the problem* (3.1), (3.2) *if and only if* $(u_1,-u_2)^T$ *is a solution of the problem*

$$v_i'(t) = (-1)^{i-1}\sigma_{i1}\,\ell_{i1}(v_1)(t) + (-1)^{i}\sigma_{i2}\,\ell_{i2}(v_2)(t) \qquad (i=1,2), \tag{3.3}$$

$$v_1(a)=0, \qquad v_2(a)=0\,. \tag{3.4}$$

Lemma 3.3 ([6, Remark 1.1])**.** *Let $\ell\in\mathcal{P}_{ab}$ be such that*

$$\int_a^b \ell(1)(s)ds < 1.$$

Then every absolutely continuous function $u : [a,b] \to \mathbb{R}$ *such that*

$$u'(t) \geq \ell(u)(t) \quad \textit{for} \quad t \in [a,b], \qquad u(a) \geq 0$$

satisfies $u(t) \geq 0$ *for* $t \in [a,b]$.

Now we are in position to prove Theorems 2.1–2.6.

Proof of Theorem 2.1. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = \ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.5}$$

$$u_2'(t) = \ell_{21}(u_1)(t) - \ell_{22}(u_2)(t) \tag{3.6}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.5), (3.6), (3.2). For $i = 1, 2$, we put

$$M_i = \max\big\{u_i(t) : t \in [a,b]\big\}, \quad m_i = -\min\big\{u_i(t) : t \in [a,b]\big\}. \tag{3.7}$$

Choose $\alpha_i, \beta_i \in [a,b]$ $(i = 1,2)$ such that the equalities

$$u_1(\alpha_1) = M_1\,, \qquad u_1(\beta_1) = -m_1 \tag{3.8}$$

and

$$u_2(\alpha_2) = M_2\,, \qquad u_2(\beta_2) = -m_2\,. \tag{3.9}$$

are satisfied. Obviously, (3.2) guarantees

$$M_i \geq 0, \quad m_i \geq 0 \quad \text{for} \quad i = 1, 2.$$

Furthermore, for $i, k = 1, 2$, we denote

$$B_{ik} = \int\limits_a^{\min\{\alpha_i,\beta_i\}} \ell_{ik}(1)(s)ds, \qquad D_{ik} = \int\limits_{\min\{\alpha_i,\beta_i\}}^{\max\{\alpha_i,\beta_i\}} \ell_{ik}(1)(s)ds. \tag{3.10}$$

It is clear that

$$B_{ik} + D_{ik} \leq A_{ik} \quad \text{for} \quad i, k = 1, 2. \tag{3.11}$$

The integrations of (3.5) from a to α_1 and from a to β_1, in view of (3.7), (3.8), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yield

$$M_1 = \int_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int_a^{\alpha_1} \ell_{12}(u_2)(s)ds \leq \tag{3.12}$$
$$\leq M_1 \int_a^{\alpha_1} \ell_{11}(1)(s)ds + M_2 \int_a^{\alpha_1} \ell_{12}(1)(s)ds \leq M_1 A_{11} + M_2 A_{12}$$

and

$$m_1 = -\int_a^{\beta_1} \ell_{11}(u_1)(s)ds - \int_a^{\beta_1} \ell_{12}(u_2)(s)ds \leq \tag{3.13}$$
$$\leq m_1 \int_a^{\beta_1} \ell_{11}(1)(s)ds + m_2 \int_a^{\beta_1} \ell_{12}(1)(s)ds \leq m_1 A_{11} + m_2 A_{12}\,.$$

Now we shall divide the discussion into the following two cases.

(a) The function u_2 is of a constant sign. Then, without loss of generality we can assume that $u_2(t) \geq 0$ for $t \in [a, b]$.

(b) The function u_2 changes its sign.

Case (a): $u_2(t) \geq 0$ for $t \in [a, b]$. In view of (2.3) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 implies $u_1(t) \geq 0$ for $t \in [a, b]$. Consequently,

$$M_1 \geq 0, \quad M_2 \geq 0, \quad M_1 + M_2 > 0. \tag{3.14}$$

The integration of (3.6) from a to α_2, on account of (3.7), (3.9), and the assumption $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, yields

$$M_2 = \int_a^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_a^{\alpha_2} \ell_{22}(u_2)(s)ds \leq M_1 \int_a^{\alpha_2} \ell_{21}(1)(s)ds \leq M_1 A_{21}\,. \tag{3.15}$$

According to (2.3) and (3.14), it follows from (3.12) and (3.15) that

$$0 \leq M_1(1 - A_{11}) \leq M_2 A_{12}\,, \qquad 0 \leq M_2 \leq M_1 A_{21}. \tag{3.16}$$

Using (2.3) and (3.14) once again, the last relations imply $M_1 > 0$, $M_2 > 0$, and

$$A_{12} A_{21} \geq 1 - A_{11} \geq (1 - A_{11})\varphi(A_{22}),$$

which contradicts (2.4).

Case (b): u_2 changes its sign. It is clear that

$$M_2 > 0, \qquad m_2 > 0. \tag{3.17}$$

We can assume without loss of generality that $\beta_2 < \alpha_2$. The integrations of (3.6) from a to β_2 and from β_2 to α_2, in view of (3.7), (3.9), (3.10), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, result in

$$\text{(3.18)}\quad m_2 = -\int_a^{\beta_2} \ell_{21}(u_1)(s)ds + \int_a^{\beta_2} \ell_{22}(u_2)(s)ds \leq$$
$$\leq m_1 \int_a^{\beta_2} \ell_{21}(1)(s)ds + M_2 \int_a^{\beta_2} \ell_{22}(1)(s)ds = m_1 B_{21} + M_2 B_{22}$$

and

$$\text{(3.19)}\quad M_2 + m_2 = \int_{\beta_2}^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_{\beta_2}^{\alpha_2} \ell_{22}(u_2)(s)ds \leq$$
$$\leq M_1 \int_{\beta_2}^{\alpha_2} \ell_{21}(1)(s)ds + m_2 \int_{\beta_2}^{\alpha_2} \ell_{22}(1)(s)ds = M_1 D_{21} + m_2 D_{22}\,.$$

On the other hand, using (2.3) and (3.17), from (3.12) and (3.13) we get

$$\text{(3.20)}\qquad \frac{M_1}{M_2} \leq \frac{A_{12}}{1 - A_{11}}\,, \qquad \frac{m_1}{m_2} \leq \frac{A_{12}}{1 - A_{11}}\,.$$

If we take the assumption (2.4) into account, (3.20) yields

$$\frac{m_1}{m_2}\, B_{21} \leq \frac{A_{12}A_{21}}{1 - A_{11}} < 1, \qquad \frac{M_1}{M_2}\, D_{21} \leq \frac{A_{12}A_{21}}{1 - A_{11}} < 1.$$

Consequently, it follows from (3.18) and (3.19) that

$$0 < 1 - \frac{m_1}{m_2}\, B_{21} \leq \frac{M_2}{m_2}\, B_{22}\,, \qquad 0 < 1 - \frac{M_1}{M_2}\, D_{21} \leq \frac{m_2}{M_2}\,(D_{22} - 1),$$

whence we get $D_{22} > 1$ and

$$\left(1 - \frac{m_1}{m_2}\, B_{21}\right)\left(1 - \frac{M_1}{M_2}\, D_{21}\right) \leq B_{22}\big(D_{22} - 1\big).$$

Therefore,

$$1 - \frac{m_1}{m_2}\, B_{21} - \frac{M_1}{M_2}\, D_{21} \leq \frac{1}{4}\big(B_{22} + D_{22} - 1\big)^2 \leq \frac{1}{4}\big(A_{22} - 1\big)^2,$$

which, together with (3.20), results in

$$\varphi(A_{22}) = 1 - \frac{1}{4}\big(A_{22} - 1\big)^2 \le \frac{m_1}{m_2}\, B_{21} + \frac{M_1}{M_2}\, D_{21} \le \\ \le \frac{A_{12}}{1 - A_{11}}\,(B_{21} + D_{21}) \le \frac{A_{12}A_{21}}{1 - A_{11}}\,.$$

But this contradicts (2.4).

The contradictions obtained in (a) and (b) prove that the problem (3.5), (3.6), (3.2) has only the trivial solution. □

Proof of Theorem 2.2. The validity of the theorem follows immediately from Theorem 2.1. □

Proof of Theorem 2.3. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = \ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.21}$$
$$u_2'(t) = -\ell_{21}(u_1)(t) - \ell_{22}(u_2)(t) \tag{3.22}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.21), (3.22), (3.2). It is clear that one of the following items is satisfied.

(a) The function u_2 is of a constant sign. Then, without loss of generality, we can assume that $u_2(t) \ge 0$ for $t \in [a, b]$.

(b) The function u_2 changes its sign.

Case (a): $u_2(t) \ge 0$ for $t \in [a, b]$. In view of (2.3) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 implies $u_1(t) \ge 0$ for $t \in [a, b]$. Therefore, by virtue of the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, (3.22) yields $u_2'(t) \le 0$ for $t \in [a, b]$. Consequently, $u_2 \equiv 0$ and Lemma 3.3 once again results in $u_1 \equiv 0$, which is a contradiction.

Case (b): u_2 changes its sign. Define the numbers M_i, m_i $(i = 1, 2)$ by (3.7) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.8) and (3.9) are satisfied. Furthermore, let the numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.10). It is clear that

$$M_1 \ge 0, \quad m_1 \ge 0, \quad M_2 > 0, \quad m_2 > 0.$$

We can assume without loss of generality that $\beta_2 < \alpha_2$. The integrations of (3.22) from a to β_2 and from β_2 to α_2, in view of (3.7), (3.9), (3.10), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, yield

$$m_2 = \int_a^{\beta_2} \ell_{21}(u_1)(s)ds + \int_a^{\beta_2} \ell_{22}(u_2)(s)ds \le \le M_1 \int_a^{\beta_2} \ell_{21}(1)(s)ds + M_2 \int_a^{\beta_2} \ell_{22}(1)(s)ds = M_1 B_{21} + M_2 B_{22} \tag{3.23}$$

and

$$M_2 + m_2 = - \int_{\beta_2}^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_{\beta_2}^{\alpha_2} \ell_{22}(u_2)(s)ds \le \le m_1 \int_{\beta_2}^{\alpha_2} \ell_{21}(1)(s)ds + m_2 \int_{\beta_2}^{\alpha_2} \ell_{22}(1)(s)ds = m_1 D_{21} + m_2 D_{22} . \tag{3.24}$$

By virtue of (3.11) and (3.17), it follows from (3.23) and (3.24) that

$$3 - A_{22} \le 1 + \frac{m_2}{M_2} + \frac{M_2}{m_2} - B_{22} - D_{22} \le \frac{M_1}{M_2} B_{21} + \frac{m_1}{m_2} D_{21} . \tag{3.25}$$

On the other hand, the integrations of (3.21) from a to α_1 and from a to β_1, on account of (3.7), (3.8), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yield (3.12) and (3.13), respectively. Using (2.3) and (3.17), from (3.12) and (3.13) we get (3.20). Consequently, (3.25) implies

$$3 - A_{22} \le \frac{A_{12}}{1 - A_{11}} (B_{21} + D_{21}) \le \frac{A_{12} A_{21}}{1 - A_{11}} ,$$

which contradicts (2.6).

The contradictions obtained in (a) and (b) prove that the problem (3.21), (3.22), (3.2) has only the trivial solution. □

Proof of Theorem 2.4. If $A_{12} A_{21} < (1 - A_{11})(1 - A_{22})$ then the validity of the theorem follows immediately from Theorem 2.3. Therefore, suppose that

$$A_{12} A_{21} \ge (1 - A_{11})(1 - A_{22}). \tag{3.26}$$

According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the problem (3.21), (3.22), (3.2) has only the trivial solution.

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.21), (3.22), (3.2). Define the numbers M_i, m_i $(i = 1, 2)$ by (3.7) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.8) and (3.9) are satisfied. Furthermore, let the numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.10). It is clear that (3.2) guarantees

$$M_i \geq 0, \quad m_i \geq 0 \quad \text{for} \quad i = 1, 2.$$

For the sake of clarity we shall devide the discussion into the following cases.

(a) The function u_2 is of a constant sign. Then, without loss of generality, we can assume that $u_2(t) \geq 0$ for $t \in [a, b]$.

(b) The function u_2 changes its sign. Then, without loss of generality, we can assume that $\beta_2 < \alpha_2$. It is clear that one of the following items is satisfied.

(b1) $u_1(t) \geq 0$ for $t \in [a, b]$.

(b2) $u_1(t) \leq 0$ for $t \in [a, b]$.

(b3) The function u_1 changes its sign.

Case (a): $u_2(t) \geq 0$ for $t \in [a, b]$. In view of (2.7) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 implies $u_1(t) \geq 0$ for $t \in [a, b]$. Therefore, by virtue of the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, (3.22) yields $u_2'(t) \leq 0$ for $t \in [a, b]$. Consequently, $u_2 \equiv 0$ and Lemma 3.3 once again results in $u_1 \equiv 0$, which is a contradiction.

Case (b): u_2 changes its sign and $\beta_2 < \alpha_2$. Obviously, (3.17) is true. The integrations of (3.22) from a to β_2 and from β_2 to α_2, in view of (3.7), (3.9), (3.10), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, yield (3.23) and (3.24), respectively. At first we note that, by virtue of (2.7), the assumption (2.8) implies

$$A_{22}\Big[A_{12}A_{21} - (1 - A_{11})(1 - A_{22})\Big] < 1 - A_{11}\,. \tag{3.27}$$

Now we are in position to discuss the cases (b1)–(b3).

Case (b1): $u_1(t) \geq 0$ for $t \in [a, b]$. This means that $m_1 = 0$. Consequently, (3.24) implies

$$M_2 \leq m_2(D_{22} - 1) \leq m_2(A_{22} - 1),$$

which, together with (2.7), contradicts (3.17).

Case (b2): $u_1(t) \leq 0$ for $t \in [a, b]$. This means that $M_1 = 0$. Consequently, (3.23) and (3.24) yield

$$M_2 \leq m_1 A_{21} - m_2(1 - A_{22}), \qquad m_2 \leq M_2 A_{22}\,. \tag{3.28}$$

On the other hand, the integration of (3.21) from a to β_1, in view of (3.7), (3.8), and the assumption $\ell_{11}, \ell_{21} \in \mathcal{P}_{ab}$, results in (3.13). If we take now (2.7) into account, it follows from (3.13) and (3.28) that

$$m_2(1-A_{11}) \le M_2A_{22}(1-A_{11}) \le \\ \le m_1A_{21}A_{22}(1-A_{11}) - m_2A_{22}(1-A_{11})(1-A_{22}) \le \\ \le m_2A_{12}A_{21}A_{22} - m_2A_{22}(1-A_{11})(1-A_{22}).$$

Since $m_2 > 0$, we get from the last relations that

$$1 - A_{11} \le A_{22}\Big[A_{12}A_{21} - (1-A_{11})(1-A_{22})\Big],$$

which contradicts (3.27).

Case (b3): u_1 changes its sign. Suppose that $\alpha_1 < \beta_1$ (the case, where $\alpha_1 > \beta_1$, can be proved analogously). Obviously,

$$M_i > 0, \quad m_i > 0 \quad \text{for} \quad i = 1,2. \tag{3.29}$$

is true. The integrations of (3.21) from a to α_1 and from α_1 to β_1, on account of (3.7), (3.8), (3.10), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yield

$$M_1 = \int_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int_a^{\alpha_1} \ell_{12}(u_2)(s)ds \le \\ \le M_1 \int_a^{\alpha_1} \ell_{11}(1)(s)ds + M_2 \int_a^{\alpha_1} \ell_{12}(1)(s)ds = M_1B_{11} + M_2B_{12} \tag{3.30}$$

and

$$M_1 + m_1 = -\int_{\alpha_1}^{\beta_1} \ell_{11}(u_1)(s)ds - \int_{\alpha_1}^{\beta_1} \ell_{12}(u_2)(s)ds \le \\ \le m_1 \int_{\alpha_1}^{\beta_1} \ell_{11}(1)(s)ds + m_2 \int_{\alpha_1}^{\beta_1} \ell_{12}(1)(s)ds = m_1D_{11} + m_2D_{12}\,, \tag{3.31}$$

respectively. By virtue of (2.7), (3.29), and (3.11), combining the inequalities (3.23), (3.24) and (3.30), (3.31), we get

$$0 < \frac{m_2}{M_1} + \frac{M_2}{m_1} + \frac{m_2}{m_1}\,(1-D_{22}) \le A_{21} + \frac{M_2}{M_1}\,B_{22} \tag{3.32}$$

and

$$0 < \frac{M_1}{M_2}\,(1-B_{11}) + \frac{m_1}{m_2}\,(1-D_{11}) + \frac{M_1}{m_2} \le A_{12}\,, \tag{3.33}$$

respectively.

On the other hand, in view of (2.7), the relations (3.24) and (3.31) imply

$$M_2(1-A_{11}) \leq m_2\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big].$$

Using (3.23) and (3.26) in the last inequality, we get

$$M_2\Big(1-A_{11}-A_{22}\big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\big]\Big) \leq$$
$$\leq M_1A_{21}\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big].$$

Consequently,

$$A_{21}+\frac{M_2}{M_1}\,B_{22} \leq \frac{(1-A_{11})A_{21}}{1-A_{11}-A_{22}\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big]}\,, \tag{3.34}$$

because the inequality (3.27) is true.

Now, it follows from (3.32)–(3.34) that

$$\frac{(1-A_{11})A_{12}A_{21}}{1-A_{11}-A_{22}\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big]} \geq \frac{m_2}{M_2}\,(1-B_{11})+ \tag{3.35}$$
$$+\frac{m_1}{M_1}\,(1-D_{11})+1+\frac{M_1}{m_1}\,(1-B_{11})+\frac{M_2}{m_2}\,(1-D_{11})+\frac{M_1M_2}{m_1m_2}+$$
$$+\frac{M_1m_2}{M_2m_1}\,(1-B_{11})(1-D_{22})+(1-D_{11})(1-D_{22})+\frac{M_1}{m_1}\,(1-D_{22}).$$

Using the relation

$$x+y \geq 2\sqrt{xy} \quad \text{for} \quad x \geq 0,\ y \geq 0,$$

we get

$$\frac{M_1M_2}{m_1m_2}+\frac{M_1m_2}{M_2m_1}\,(1-B_{11})(1-D_{22}) \geq 2\,\frac{M_1}{m_1}\sqrt{(1-B_{11})(1-D_{22})}\,, \tag{3.36}$$

$$\frac{M_1}{m_1}\,(1-B_{11})+2\,\frac{M_1}{m_1}\sqrt{(1-B_{11})(1-D_{22})}+\frac{M_1}{m_1}\,(1-D_{22}) = \tag{3.37}$$
$$=\frac{M_1}{m_1}\,\Big(\sqrt{1-B_{11}}+\sqrt{1-D_{22}}\Big)^2,$$

$$
\begin{aligned}
(3.38)\quad \frac{M_1}{m_1}\left(\sqrt{1-B_{11}}+\sqrt{1-D_{22}}\right)^2+\frac{m_1}{M_1}(1-D_{11}) &\geq \\
\geq 2\sqrt{1-D_{11}}\left(\sqrt{1-B_{11}}+\sqrt{1-D_{22}}\right) &\geq \\
\geq 2\sqrt{1-B_{11}-D_{11}}+2\sqrt{(1-D_{11})(1-D_{22})} &\geq \\
\geq 2\sqrt{1-A_{11}}+2\sqrt{(1-D_{11})(1-D_{22})}&,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.39)\qquad \frac{m_2}{M_2}(1-B_{11})+\frac{M_2}{m_2}(1-D_{11}) &\geq 2\sqrt{(1-B_{11})(1-D_{11})} \\
&\geq 2\sqrt{1-A_{11}}\,.
\end{aligned}
$$

Finally, in view (3.36)–(3.39), (3.35) implies

$$
\begin{aligned}
\frac{(1-A_{11})A_{12}A_{21}}{1-A_{11}-A_{22}\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big]} &\geq \\
\geq 4\sqrt{1-A_{11}}+1+2\sqrt{(1-D_{11})(1-D_{22})}+(1-D_{11})(1-D_{22}) &\geq \\
\geq 4\sqrt{1-A_{11}}+\left(1+\sqrt{(1-A_{11})(1-A_{22})}\right)^2 &= \omega,
\end{aligned}
$$

which contradicts (2.8).

The contradictions obtained in (a) and (b) prove that the problem (3.21), (3.22), (3.2) has only the trivial solution. □

Proof of Theorem 2.5. The validity of the theorem follows immediately from Theorem 2.3. □

Proof of Theorem 2.6. The validity of the theorem follows immediately from Theorem 2.4. □

4. Counter–examples

In this part, the counter–examples are constructed verifying that the results obtained above are optimal in a certain sense.

Example 4.1. Let $\sigma_{ik} \in \{-1, 1\}$, $h_{ik} \in L\big([a,b];\mathbb{R}_+\big)$ $(i,k=1,2)$ be such that

$$
\sigma_{11} = 1, \qquad \int_a^b h_{11}(s)ds \geq 1.
$$

It is clear that there exists $t_0 \in]a,b]$ such that

$$\int_a^{t_0} h_{11}(s)ds = 1.$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$ $(i,k=1,2)$ be defined by

$$\ell_{ik}(v)(t) \stackrel{\text{def}}{=} h_{ik}(t)v\big(\tau_{ik}(t)\big) \quad \text{for} \quad t \in [a,b],\ v \in C([a,b];\mathbb{R}), \tag{4.1}$$

where $\tau_{11}(t) = t_0$, $\tau_{12}(t) = a$, $\tau_{21}(t) = a$, and $\tau_{22}(t) = a$ for $t \in [a,b]$. Put

$$u(t) = \int_a^t h_{11}(s)ds \quad \text{for} \quad t \in [a,b].$$

It is easy to verify that $(u,0)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i=1,2)$.

This example shows that the constant 1 on the right–hand side of the inequality in (2.3) is optimal and cannot be weakened.

Example 4.2 Let $\sigma_{ik} \in \{-1,1\}$, $h_{ik} \in L\big([a,b];\mathbb{R}_+\big)$ $(i,k=1,2)$ be such that

$$\sigma_{22} = -1, \qquad \int_a^b h_{22}(s)ds \geq 3.$$

It is clear that there exist $t_0 \in]a,b[$ and $t_1 \in]t_0,b]$ such that

$$\int_a^{t_0} h_{22}(s)ds = 1, \qquad \int_{t_0}^{t_1} h_{22}(s)ds = 2.$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$ $(i,k=1,2)$ be defined by (4.1), where $\tau_{11}(t) = a$, $\tau_{12}(t) = a$, $\tau_{21}(t) = a$ for $t \in [a,b]$, and

$$\tau_{22}(t) = \begin{cases} t_1 & \text{for} \quad t \in [a,t_0[\\ t_0 & \text{for} \quad t \in [t_0,b] \end{cases}.$$

Put

$$u(t) = \begin{cases} \int_a^t h_{22}(s)ds & \text{for} \quad t \in [a,t_0[\\ 1 - \int_{t_0}^t h_{22}(s)ds & \text{for} \quad t \in [t_0,b] \end{cases}.$$

It is easy to verify that $(0,u)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

This example shows that the constant 3 on the right–hand side of the inequality in (2.3) is optimal and cannot be weakened.

Example 4.3. Let $\sigma_{11} = 1$, $\sigma_{12} = 1$, $\sigma_{21} = 1$, $\sigma_{22} = -1$ and let the functions $h_{ik} \in L\big([a,b];\mathbb{R}_+\big)$ $(i,k = 1,2)$ be such that

$$\int_a^b h_{11}(s)ds < 1, \qquad \int_a^b h_{22}(s)ds \leq 1,$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq 1 - \int_a^b h_{11}(s)ds.$$

It is clear that there exists $t_0 \in\ ]a,b]$ satisfying

$$\int_a^{t_0} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = 1 - \int_a^{t_0} h_{11}(s)ds.$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$ $(i,k = 1,2)$ be defined by (4.1), where $\tau_{11}(t) = t_0$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_0$, and $\tau_{22}(t) = a$ for $t \in [a,b]$. Put

$$u_1(t) = \int_a^t h_{11}(s)ds + \frac{1 - \int_a^{t_0} h_{11}(s)ds}{\int_a^{t_0} h_{12}(s)ds} \int_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a,b],$$

$$u_2(t) = \int_a^t h_{21}(s)ds \quad \text{for} \quad t \in [a,b].$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

This example shows that the strict inequality (2.4) in Theorem 2.1 cannot be replaced by the nonstrict one provided $A_{22} \leq 1$.

Example 4.4. Let $\sigma_{11} = 1$, $\sigma_{12} = 1$, $\sigma_{21} = 1$, $\sigma_{22} = -1$, and let the functions $h_{11}, h_{22} \in L\big([a,b];\mathbb{R}_+\big)$ be such that

$$\int_a^b h_{11}(s)ds < 1, \qquad 1 < \int_a^b h_{22}(s)ds < 3. \tag{4.2}$$

Obviously, there exists $t_0 \in \,]a,b[$ satisfying

$$\int_a^{t_0} h_{22}(s)ds = \frac{\int_a^b h_{22}(s)ds - 1}{2}. \tag{4.3}$$

Furthermore, we choose $h_{12}, h_{21} \in L\big([a,b];\mathbb{R}_+\big)$ with the properties

$$h_{21}(t) = 0 \quad \text{for} \quad t \in [t_0, b]$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq \left(1 - \int_a^b h_{11}(s)ds\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2\right].$$

It is clear that there exists $t_1 \in \,]a,b]$ such that

$$\int_a^{t_1} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = \left(1 - \int_a^{t_1} h_{11}(s)ds\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2\right].$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$ $(i,k = 1,2)$ be defined by (4.1), where $\tau_{11}(t) = t_1$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a,b]$, and

$$\tau_{22}(t) = \begin{cases} b & \text{for} \quad t \in [a, t_0[\\ t_0 & \text{for} \quad t \in [t_0, b] \end{cases}. \tag{4.4}$$

Put

$$u_1(t) = \frac{\int_a^{t_1} h_{12}(s)ds}{1 - \int_a^{t_1} h_{11}(s)ds} \int_a^t h_{11}(s)ds + \int_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a,b],$$

$$u_2(t) = \begin{cases} \dfrac{\int_a^{t_1} h_{12}(s)ds \int_a^t h_{21}(s)ds}{1 - \int_a^{t_1} h_{11}(s)ds} + \dfrac{\left(\int_a^b h_{22}(s)ds - 1\right)\int_a^t h_{22}(s)ds}{2} & \text{for} \quad t \in [a, t_0[\\ 1 - \int_{t_0}^t h_{22}(s)ds & \text{for} \quad t \in [t_0, b] \end{cases}.$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

This example shows that the strict inequality (2.4) in Theorem 2.1 cannot be replaced by the nonstrict one provided $A_{22} > 1$.

Example 4.5. Let $\sigma_{1i} = 1$, $\sigma_{2i} = -1$ for $i = 1, 2$ and let $h_{11}, h_{22} \in L([a, b]; \mathbb{R}_+)$ be such that (4.2) is true. Obviously, there exists $t_0 \in \,]a, b[$ satisfying

$$\int_a^{t_0} h_{22}(s)ds = 1. \tag{4.5}$$

Furthermore, we choose $h_{12}, h_{21} \in L([a, b]; \mathbb{R}_+)$ with the properties

$$h_{21}(t) = 0 \quad \text{for} \quad t \in [a, t_0]$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq \left(1 - \int_a^b h_{11}(s)ds\right)\left(3 - \int_a^b h_{22}(s)ds\right).$$

It is clear that there exists $t_1 \in \,]a, b]$ such that

$$\int_a^{t_1} h_{12}(s)ds \int_{t_0}^b h_{21}(s)ds = \left(1 - \int_a^{t_1} h_{11}(s)ds\right)\left(2 - \int_{t_0}^b h_{22}(s)ds\right).$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$ $(i, k = 1, 2)$ be defined by (4.1), where $\tau_{11}(t) = t_1$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a, b]$, and τ_{22} is given by (4.4). Put

$$u_1(t) = \frac{\int_a^{t_1} h_{12}(s)ds}{1 - \int_a^{t_1} h_{11}(s)ds} \int_a^t h_{11}(s)ds + \int_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a, b],$$

$$u_2(t) = \begin{cases} 1 - \int_t^{t_0} h_{22}(s)ds & \text{for} \quad t \in [a, t_0[\\ 1 - \frac{\int_a^{t_1} h_{12}(s)ds}{1 - \int_a^{t_1} h_{11}(s)ds} \int_{t_0}^t h_{21}(s)ds - \int_{t_0}^t h_{22}(s)ds & \text{for} \quad t \in [t_0, b] \end{cases}.$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$.

This example shows that the strict inequality (2.6) in Theorem 2.3 cannot be replaced by the nonstrict one provided $A_{22} > 1$.

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The research was supported by the Grant Agency of the Czech Republic, Grant No. 201/04/P183 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

JIŘÍ ŠREMR
MATHEMATICAL INSTITUTE, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC
ŽIŽKOVA 22, 616 62 BRNO, CZECH REPUBLIC
e-mail: sremr@ipm.cz

Received on 15.11.2005 and, in revised from, on 05.06.2006