F A S C I C U L I M A T H E M A T I C I

Nr 37 2007

Ethiraju Thandapani and Bothiah Selvaraj

OSCILLATIONS OF FOURTH ORDER QUASILINEAR DIFFERENCE EQUATIONS

Abstract: Consider the fourth order quasilinear difference equation of the form

$$(*) \qquad \Delta^3 \left(p_{n-1} \left(\Delta x_{n-1}\right)^{\alpha}\right) + q_n x_n^{\beta} = 0, n = 1, 2, \cdots$$

where $\{p_n\}$ is a positive sequence and $\{q_n\}$ is a sequence of non-negative reals, α and β are ratios of odd positive integers. We obtain some new sufficient conditions for the oscillation of all solutions of equation (*). Examples are inserted to illustrate the importance of our results.

Key words: oscillations, fourth order, quasilinear, difference equations.

1. Introduction and basic notions

In this paper, we are concerned with the oscillatory behavior of fourth order quasilinear difference equations of the form

$$(1) \qquad \Delta^3 \left(p_{n-1} \left(\Delta x_{n-1}\right)^{\alpha}\right) + q_n x_n^{\beta} = 0, n = 1, 2, \cdots$$

where $\{p_n\}$ is a nondecreasing sequence of positive reals such that $\sum_{n=1}^{\infty} \frac{1}{p_n^{\frac{1}{\alpha}}} = \infty$, $\{q_n\}$ is a sequence of non-negative reals α and β are ratios of odd positive integers. By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined and satisfies equation (1) for all $n \geq 1$. A solution $\{x_n\}$ of equation (1) is said to be oscillatory if for every positive integer $N > 1$, there exists an integer $n \geq N$ such that $x_n x_{n+1} \leqslant 0$; otherwise it is said to be nonoscillatory. In recent years there has been an increasing interest in the study of oscillatory behavior of solutions of difference equations,see for example [1, 2, 4, 5, 6] and the references cited there in. Numerous results exist for first and second order difference equations,but the results dealing with fourth order equations are relatively scarce though such equations arise in the mathematical biology,bending of beams and other areas of mathematics in which discrete models are used. Therefore in this paper, we study the oscillatory behavior of equation (1) and obtain some new sufficient conditions for the oscillations of all solutions of equation (1). Examples are included to dwell upon the importance of our results.

2. Some preliminary lemmas

In this section we state some lemmas which are needed in the sequel to prove our main results.

Throughout this paper, we use the factorial function $(t)^{(r)}$ defined as follows:

$$(t)^{(r)} = \begin{cases} t(t-1)\cdots(t-r+1) & \text{if } r = 1,2,3,\cdots \\ 0 & \text{if } r = 0 \text{ or } r > t \end{cases}$$

Lemma 1. *Let $\{y_n\}$ be a sequence of real numbers in $\mathbb{N} = \{1,2,3,\cdots\}$. Let $\{y_n\}$ and $\Delta^3 y_n$ be of constant sign with $\Delta^3 y_n$ not being identically zero on any subset $\{n, n+1, \cdots\}$ of $\mathbb{N}$. If*

$$y_n \Delta^3 y_n \leqslant 0,$$

then there exists an integer $n_2 \geq n_1$ such that either

$$\operatorname{sgn} y_n = \operatorname{sgn} \Delta y_n = \operatorname{sgn} \Delta^2 y_n, \quad \textit{for} \quad n \geqslant n_2, \tag{2}$$

or

$$\operatorname{sgn} y_n = \operatorname{sgn} \Delta^2 y_n \neq \operatorname{sgn} \Delta y_n, \quad \textit{for} \quad n \geqslant n_2. \tag{3}$$

Proof. The proof follows immediately from the Discrete Kneser's Theorem [1, Theorem 1.7.11]. ■

Lemma 2. *Let $\{y_n\}$ be as defined in Lemma 1 and such that $y_n > 0$ and* (2) *holds for all $n \geq n_1$. Then*

$$y_n \geqslant \frac{1}{2}\left(\frac{n}{2}\right)^{(2)} \Delta^2 y_n, \quad \textit{for} \quad n \geqslant n_2. \tag{4}$$

Proof. From (2), we have

$$y_n > 0, \;\; \Delta y_n > 0, \;\; \Delta^2 y_n > 0 \quad \text{for} \quad \Delta^3 y_n \leqslant 0 \quad \text{for} \quad n \geqslant n_1,$$

and hence

$$\Delta y_n = \Delta y_{n_1} + \sum_{s=n_1}^{n-1} \Delta^2 y_s \geqslant (n - n_1)\, \Delta^2 y_n.$$

Summing the last inequality from n_1 to $n-1$, we have

$$y_n \geqslant y_{n_1} + \Delta^2 y_n \sum_{s=n_1}^{n-1} (s-n_1) \geqslant \frac{1}{2}\left(\frac{n}{2}\right)^{(2)} \Delta^2 y_n \quad \text{for} \quad n \geqslant n_2 = 2n_1,$$

and the proof is complete. ■

Lemma 3. *Let $\{z_n\}$ be be a positive sequence such that $\Delta z_n > 0$, $\Delta^2 z_n < 0$, $\Delta^3 z_n > 0$ and $\Delta^4 z_n \leqslant 0$ for $n \geqslant n_1$. Then*

$$z_n \geqslant \frac{1}{6}\left(\frac{n}{2}\right)^{(3)} \Delta^3 z_n, \quad n \geqslant n_2 \geqslant 2n_1. \tag{5}$$

Proof. We know

$$\sum_{s=n_1}^{n-1} (s-n_1)^{(3)} \Delta^4 z_{s-3} = (n-n_1)^{(3)} \Delta^3 z_{n-3} - 3(n-n_1)^{(2)} \Delta^2 z_{n-2} + 6(n-n_1)\Delta z_{n-1} - 6z_n + 6z_{n_1}.$$

Then using the hypothesis,we obtain

$$\begin{aligned} z_n &\geqslant \frac{1}{6}(n-n_1)^{(3)} \Delta^3 z_{n-3} \geqslant \frac{1}{6}(n-n_1)^{(3)} \Delta^3 z_n \\ &\geqslant \frac{1}{6}\left(\frac{n}{2}\right)^{(3)} \Delta^3 z_n, \quad \text{for} \quad n \geqslant n_2 = 2n_1. \end{aligned}$$

This completes the proof of Lemma 3. ■

Lemma 4. *Let $\{z_n\}$ be a positive sequence such that*

$$\Delta z_n > 0, \ \Delta\left((\Delta z_n)^{\frac{\alpha}{\beta}}\right) < 0, \ \Delta^2\left((\Delta z_n)^{\frac{\alpha}{\beta}}\right) > 0 \ \text{and} \ \Delta^3\left((\Delta z_n)^{\frac{\alpha}{\beta}}\right) \leqslant 0$$

for $n \geq n_1$. Then

$$z_n^{\frac{\alpha}{\beta}} \geqslant \frac{1}{6}\left(\frac{n}{2}\right)^{(3)} \Delta^2\left((\Delta z_n)^{\frac{\alpha}{\beta}}\right) \ \text{for} \ n \geqslant n_2 \geqslant 2n_1.$$

Proof. The proof is similar to that of Lemma 3 and hence the details are omitted. ■

Lemma 5. *Suppose $F_n \geq 0$ and $Q_n \geq 0$ for all $n \geq n_0 \in \mathbb{N}$. If there exists a positive sequence $\{W_n\}$ such that*

$$W_{n+1} - Q_n W_n + F_n \leqslant 0, \quad n \geqslant n_0$$

then

$$\sum_{n=n_0}^{\infty} F_n \exp\left(\sum_{t=n_0}^{n} Q_t\right) < \infty.$$

Proof. For the proof of Lemma 5, see [8]. ■

Lemma 6. *Assume that* $\{x_n\}$ *is an eventually positive solution of equation* (1)*. Let*

$$y_n = p_{n-1}(\Delta x_{n-1})^{\alpha}. \tag{6}$$

Then $y_n > 0$ *eventually.*

Proof. Let $x_{n-1} > 0$ for all $n \geq n_1 \in \mathbb{N}$. Then by the equation (1), we have

$$\Delta^3 y_n = -q_n x_n^{\beta} \leqslant 0,$$

for all $n \geq n_1$, which implies that $\{\Delta y_n\}$, $\{\Delta^2 y_n\}$ are monotonic and either

$$\Delta^2 y_n < 0$$

or

$$\Delta^2 y_n > 0.$$

We claim that $\Delta^2 y_n > 0$. Suppose $\Delta^2 y_n < 0$ for $n \geq n_1$. Then there is a constant $d > 0$ and an integer $n_2 \in \mathbb{N}$ such that

$$y_n \leq -d \text{ for } n \geqslant n_2.$$

Therefore, we have

$$\Delta x_{n-1} \leqslant -\left(\frac{d}{p_{n-1}}\right)^{\frac{1}{\alpha}}$$

for all $n \geq n_2$. Summing the last inequality from n_2 to n and then taking $n \to \infty$, we see that $x_n \to -\infty$ as $n \to \infty$. This is a contradiction and hence $\Delta^2 y_n > 0$ for $n \geq n_1$.

Next we consider the following three possible cases:

Case 1. For $n \geq n_1$, we have

$$y_n < 0 \quad \text{and} \quad \Delta y_n < 0.$$

Case 2. For $n \geq n_1$, we have

$$y_n > 0, \;\; \Delta y_n > 0 \quad \text{and} \quad \Delta^2 y_n > 0.$$

Case 3. For $n \geq n_1$, we have

$$y_n < 0, \;\; \Delta y_n < 0 \quad \text{and} \quad \Delta^2 y_n > 0.$$

For Cases 1 and 2, using a similar method to the above, we can obtain a contradiction and so Cases 1 and 3 are impossible. For Case 2, we see that $y_n > 0$ eventually. This completes the proof. ■

3. Oscillation results

In this section, we derive some new sufficient conditions for the oscillation of all solutions of equation (1). We begin with the case $\alpha = \beta$.

Theorem 1. *Assume $\alpha = \beta$ and $\sigma_n = \min\left\{\left(\frac{n}{8}\right)^{(3)}, \frac{1}{6}\left(\frac{n}{2}\right)^{(3)}\right\}$. If*

$$Q_n = \frac{p_n - \delta\sigma_n q_n}{p_n} \geqslant 0 \quad \text{where} \quad \delta = 1 \quad \text{if} \quad \alpha \geqslant 1 \quad \text{and}$$

$\delta = \alpha$ if $\alpha < 1$ and

$$\sum_{n=n_0}^{\infty} q_n \exp\left(\sum_{i=n_0}^{n} Q_i\right) = \infty, \tag{7}$$

then every solution of equation (1) oscillates.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1). Assume without loss of generality that $x_{n-1} > 0$ for all $n \geq n_1 \in \mathbb{N}$. Let y_n be as in Lemma 6. Then by (1) and (6) we have

$$\Delta^3 y_n = -q_n x_n^{\alpha} \leqslant 0, \quad n \geqslant n_1 \tag{8}$$

and therefore $\{y_n\}$, $\{\Delta y_n\}$, $\{\Delta^2 y_n\}$ are strictly monotonic. By Lemma 6, $y_n > 0$ eventually. From Lemma 1 we have either

$$y_n > 0, \quad \Delta y_n > 0 \quad \text{and} \quad \Delta^2 y_n > 0 \tag{9}$$

or

$$y_n > 0, \quad \Delta y_n < 0 \quad \text{and} \quad \Delta^2 y_n > 0 \quad \text{for} \quad n \geqslant n_2 \geqslant n_1. \tag{10}$$

From Lemma 6, we have $\Delta x_{n-1} > 0$. Hence there exists a constant $M > 0$ and a positive integer $n_3 \geq n_2$ such that

$$x_n \geq M \quad \text{for all} \quad n \geq n_3. \tag{11}$$

In the case of (9), we have from Lemma 2

$$y_n \geqslant \frac{1}{2}\left(\frac{n}{4}\right)^{(2)} \Delta^2 y_n \quad \text{for all} \quad n \geqslant N_0. \tag{12}$$

Let ℓ be a positive integer such that $N_0+\ell<n\leq N_0+\ell+1$ for $n\geq N_1=N_0+\ell+1$. Then we have

$$n>\frac{1}{2}(n+\ell)\quad\text{for}\quad n\geqslant N_1.$$

From (12) we obtain

$$y_n>\frac{1}{2}\left(\frac{n+\ell}{8}\right)^{(2)}\Delta^2 y_n \tag{13}$$

for $n\geqslant N_1$.

From (6) we have

$$x_n^\alpha=\left(\left(\frac{y_n}{p_{n-1}}\right)^{\frac{1}{\alpha}}+x_{n-1}\right)^\alpha\geqslant\delta\left(\frac{y_n}{p_{n-1}}+x_{n-1}^\alpha\right) \tag{14}$$

where δ is already defined in the hypothesis.

In view of (13) and (14) we obtain

$$\begin{aligned} x_n^\alpha &\geqslant \frac{\delta}{p_n}\sum_{j=0}^{\ell-1}y_{n-j}+\delta^{\ell-1}x_{n-\ell}\\ &\geqslant \frac{\delta}{p_n}\sum_{j=0}^{\ell-1}\frac{1}{2}\left(\frac{n+\ell-j}{8}\right)^{(2)}\Delta^2 y_{n-j}+\delta^{\ell-1}M^\alpha\\ &\geqslant \frac{\delta}{2p_n}\left(\frac{n}{8}\right)^{(2)}\sum_{j=0}^{\ell-1}\Delta^2 y_{n-j}+M_1,\quad\text{for}\quad n\geqslant N_1 \end{aligned} \tag{15}$$

where $M_1=\delta^{\ell-1}M^\alpha$. Since $\{\Delta^2 y_n\}$ is decreasing and $\Delta^2 y_n>0$, we have

$$\sum_{j=0}^{\ell-1}\Delta^2 y_{n-j}\geqslant\sum_{i=N_2}^{n-1}\Delta^2 y_i, \tag{16}$$

where $N_2=N_1+1$.

Substituting (15) and (16) in (8), we obtain

$$\Delta^3 y_n+\frac{\delta}{2}\frac{q_n}{p_n}\left(\frac{n}{8}\right)^{(2)}\sum_{i=N_2}^{n-1}\Delta^2 y_i+M_1 q_n\leqslant 0. \tag{17}$$

Set $V_n=\sum_{i=N_2}^{n-1}\Delta^2 y_i$, $n\geqslant N_2$. Then, by (17),we get

$$\Delta^2 V_n+\frac{\delta}{2}\frac{q_n}{p_n}\left(\frac{n}{8}\right)^{(2)}V_n+M_1 q_n\leqslant 0. \tag{18}$$

From (9) and (18), we have

$$V_n > 0, \;\; \Delta V_n > 0, \;\; \Delta^2 V_n \leqslant 0$$

and by Lemma 4.1 of Hooker and Patula [3] there exists an integer $N_3 \geq N_2$ such that

$$V_n \geqslant \left(\frac{n}{2}\right) \Delta V_n, \quad n \geqslant N_3. \tag{19}$$

From (18) and (19) we obtain

$$\Delta^2 V_n + \frac{\delta}{2} \frac{q_n}{p_n} \left(\frac{n}{8}\right)^{(2)} \left(\frac{n}{2}\right) \Delta V_n + M_1 q_n \leqslant 0$$

for $n \geq N_3$. Since $\left(\frac{n}{8}\right)^{(2)} \left(\frac{n}{4}\right) \geqslant \left(\frac{n}{8}\right)^{(3)}$, we have from the last inequality

$$\Delta^2 V_n + \delta \left(\frac{n}{8}\right)^{(3)} \frac{q_n}{p_n} \Delta V_n + M_1 q_n \leqslant 0, \quad n \geqslant N_3. \tag{20}$$

Next assume (10) holds. Then we have

$$x_n^{\alpha} \geqslant \frac{\delta}{p_n} \sum_{j=0}^{\ell-1} y_{n-j} + M_1. \tag{21}$$

Since $y_n > 0$ and decreasing ,we have

$$\sum_{j=0}^{\ell-1} y_{n-j} \geqslant \sum_{i=N_2}^{n-1} y_i. \tag{22}$$

From (8),(21) and (22), we obtain

$$\Delta^3 y_n + \delta \frac{q_n}{p_n} \sum_{i=N_2}^{n-1} y_i + M_1 q_n \leqslant 0. \tag{23}$$

Set $u_n = \sum_{i=N_2}^{n-1} y_i$, $n \geqslant N_2$. Then, by (23), we get

$$\Delta^4 u_n + \delta \frac{q_n}{p_n} u_n + M_1 q_n \leqslant 0. \tag{24}$$

From (10), we know

$$u_n > 0, \;\; \Delta u_n > 0, \;\; \Delta^2 u_n < 0, \;\; \Delta^3 u_n > 0$$

and by Lemma 3, there exists an integer $N_4 \geq N_2$ such that

$$u_n \geqslant \frac{1}{6} \left(\frac{n}{2}\right)^{(3)} \Delta^3 u_n, \quad n \geqslant N_4. \tag{25}$$

Using (25) in (24) we obtain

$$\Delta^4 u_n + \frac{\delta}{6}\left(\frac{n}{2}\right)^{(3)} \frac{q_n}{p_n}\,\Delta^3 u_n + M_1 q_n \leqslant 0, \quad n \geq N_4. \tag{26}$$

Let $\mathbf{W}_n = \Delta V_n$ for (20) and $\mathbf{W}_n = \Delta^3 u_n$ for (26). Then we have $\mathbf{W}_n > 0$ for $n \geq N_5 = \max(N_3, N_4)$.

From (20) and (26) we obtain

$$\Delta \mathbf{W}_n + \delta\sigma_n \frac{q_n}{p_n}\,\mathbf{W}_n + M_1 q_n \leqslant 0, \quad n \geqslant N_5.$$

By Lemma 5,we have

$$M_1 \sum_{n=N_5}^{\infty} q_n \exp\left(\sum_{i=N_5}^{n} Q_i\right) < \infty,$$

which is a contradiction and the proof is complete. ■

Theorem 2. *Assume that all solutions of inequality*

$$\Delta W_n + B_n W_n^{\frac{\beta}{\alpha}} + M q_n \leqslant 0 \quad \textit{for} \quad n \geqslant n_0, \tag{27}$$

where M is any positive constant and

$$B_n = \min\left\{\delta q_n \left(\frac{1}{6p_n}\left(\frac{n}{2}\right)^{(3)}\right)^{\frac{\beta}{\alpha}}, \delta\frac{n}{2} q_n \left(\frac{1}{2p_n}\left(\frac{n}{8}\right)^{(2)}\right)^{\frac{\beta}{\alpha}}\right\},$$

are oscillatory for $\alpha = \beta$ and for $\alpha \neq \beta$. Then every solution of equation (1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 1, we obtain

$$\Delta^3 y_n = -q_n x_n^{\beta} \leqslant 0, \quad n \geqslant n_1. \tag{28}$$

Again proceeding as in the proof of Theorem 1, we have from (28)

$$\Delta^3 y_n + \delta q_n \left(\frac{\left(\frac{n}{8}\right)^{(2)}}{2p_n}\right)^{\frac{\beta}{\alpha}} \sum_{i=N_2}^{n-1} \left(\Delta^2 y_i\right)^{\frac{\beta}{\alpha}} + M_1 q_n \leqslant 0. \tag{29}$$

Let $V_n = \sum_{i=N_2}^{n-1} \left(\Delta^2 y_i\right)^{\frac{\beta}{\alpha}}, n \geqslant N_2$. Then from (29) and (19) we have

$$\Delta\left((\Delta V_n)^{\frac{\alpha}{\beta}}\right) + \delta\,\frac{n}{2}\,q_n \left(\frac{\left(\frac{n}{8}\right)}{2p_n}\right)^{\frac{\beta}{\alpha}} \Delta V_n + M_1 q_n \leqslant 0 \tag{30}$$

where $\Delta V_n > 0$. Once again proceeding as in the proof of Theorem 1 and using Lemma 4, we obtain

$$\Delta^3\left((\Delta u_n)^{\frac{\alpha}{\beta}}\right) + \delta\left(\frac{\frac{1}{6}\left(\frac{n}{2}\right)^{(3)}}{p_n}\right)^{\frac{\beta}{\alpha}} q_n\,\Delta^2\left((\Delta u_n)^{\frac{\alpha}{\beta}}\right) + M_1 q_n \leqslant 0. \tag{31}$$

Let $\mathbf{W}_n = (\Delta V_n)^{\frac{\alpha}{\beta}}$, if (30) holds and $\mathbf{W}_n = \Delta^2\left((\Delta u_n)^{\frac{\alpha}{\beta}}\right)$ if (31) holds. Then in both the cases $\{\mathbf{W}_n\}$ is a positive solution of either

$$\Delta \mathbf{W}_n + B_n W_n^{\frac{\beta}{\alpha}} + M_1 q_n \leqslant 0$$

or

$$\Delta \mathbf{W}_n + B_n W_n + M_1 q_n \leqslant 0$$

which is a contradiction. This competes the proof. ■

Finally we give a easily verifiable condition for the oscillation of all solutions of equation (1).

Theorem 3. *Assume that*

$$\sum_{n=n_0}^{\infty} A_n q_n = \infty \tag{32}$$

where

$$A_n = \max\left\{ n^{(2)}, \left(\frac{n^{(2)}}{p_n}\right)^{\frac{\beta}{\alpha}} \right\}.$$

Then every solution of equation (1) *oscillates.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality,we may assume that $x_{n-1} > 0$ for all $n \geq n_1$. Then $\{y_n\}$ be as in Lemma 6. Then by Lemma 1, we have either (9) or (10) holds. If (10) holds,then there exists a constant $M > 0$ and $n_2 \geq n_1$ such that $x_n^{\beta} \geqslant M$ for $n \geq n_2$. From (1), we have

$$\Delta^3 y_n \leqslant -M q_n. \tag{33}$$

Multiplying both sides of (33) by $n^{(2)}$ and summing,we get

$$\begin{aligned} n^{(2)}\Delta^2 y_{n+1} - n_2^{(2)}\Delta^2 y_{n_2+1} - 2n\Delta y_{n+2} + 2n_1\Delta y_{n_2+2} \\ + 2y_{n+3} - 2y_{n_2+3} = -\sum_{s=n_2}^{n-1} s^{(2)} q_s. \end{aligned} \tag{34}$$

It is easy to see that, in view of (10) we have from (34)

$$\sum_{s=n_2}^{n-1} s^{(2)} q_s \;<\; n_2^{(2)} \Delta^2 y_{n_2+1} - 2n_1 \Delta y_{n_2+2} + 2y_{n_2+3}$$

which contradicts (32) as $n \longrightarrow \infty$. If (9) holds, then by Discrete Taylor's formula [1], we have

$$\begin{aligned} y_n \;&=\; y_{n_1} + (n-n_1)\,\Delta y_1 + \frac{1}{2}\sum_{j=n_1}^{n-2} (n-j-1)^{(2)}\,\Delta^2 y_j \\ &\geqslant\; (n-n_1)\,\Delta y_1 \geqslant \left(\frac{n}{2}\right)\Delta y_1, \quad \text{for} \quad n_2 \geqslant 2n_1. \end{aligned}$$

Hence, $y_n \geq M_1 n$ and therefore by (6),

$$\Delta x_{n-1} \;\geqslant\; M_1^{\frac{1}{\alpha}} \left(\frac{n}{p_{n-1}}\right)^{\frac{1}{\alpha}}, \quad n \geqslant n_2.$$

Summing the last inequality from n_2 to n, we obtain

$$x_n \;\geqslant\; M_1^{\frac{1}{\alpha}} \sum_{s=n_2}^{n} \frac{s^{\frac{1}{\alpha}}}{(p_{s-1})^{\frac{1}{\alpha}}}$$

or

$$\begin{aligned} x_n^{\alpha} \;&\geqslant\; \delta M_1^{\alpha} \sum_{s=n_2}^{n-1} \frac{s}{p_{s-1}} \;\geqslant\; \delta\,\frac{M_1^{\alpha}}{2p_n}\,(n-n_2)^{(2)} \\ &\geqslant\; \frac{\delta}{2}\,M_1^{\alpha}\,\frac{\left(\frac{n}{2}\right)^{(2)}}{p_n}, \quad n \geqslant n_3 \geqslant 2n_2. \end{aligned}$$

Then for all $n \geq n_3$, we have

$$x_n^{\beta} \;\geqslant\; \left(\frac{\delta}{2}\right)^{\frac{\beta}{\alpha}} M_1^{\beta} \left(\frac{\left(\frac{n}{2}\right)^{(2)}}{p_n}\right)^{\frac{\beta}{\alpha}}, \quad n \geqslant n_2. \tag{35}$$

Using (35) in equation (1) and summing, we obtain

$$\sum_{s=n_3}^{n-1} \left(\frac{\left(\frac{s}{2}\right)^{(2)}}{p_s}\right)^{\frac{\beta}{\alpha}} q_s \;<\; \infty,$$

which contradicts (32). This completes the proof of the theorem. ■

We conclude this paper with the following examples.

Example 1. Consider the difference equation

$$\Delta^3\left((\Delta x_{n-1})^5\right) + \frac{1}{n^{10}} x_n^5 = 0. \tag{36}$$

Then it is easy to see that all the conditions of Theorem 1 are satisfied and hence all solutions of equation (36) are oscillatory.

Example 2. Consider the difference equation

$$\Delta^3\left((n-1)(\Delta x_{n-1})^3\right) + \frac{1}{n^{\frac{4}{3}}} x_n^5 = 0. \tag{37}$$

Then all the conditions of Theorem 3 are satisfied and hence all solutions of equation (37) are oscillatory.

References

[1] AGARWAL R.P., Difference Equations and Inequalities, *Marcel Dekker, Second Edition,* NewYork, 2000.

[2] ANDERSON D.R., A fourth order nonlinear difference equations, *J. Diff. Eqn. Appl.,* 9(2003), 161–169.

[3] HOOKER J.W., PATULA W.T., A second order nonlinear difference equation, oscillation and asymptotic behavior, *J. Math. Anal. Appl.,* 91(1983), 9–24.

[4] HOOKER J.W., PATULA W.T., Growth and oscillation properties of solutions of fourth order linear difference equations, *J. Austral. Math. Soc.Ser. B,* 26(1985), 310–328..

[5] POPENDA J., SCHMEIDAL E., On the solutions of fourth order difference equations, *Rocky Mountain J. Math.,* 25(1995), 1485–1499.

[6] GRAEF J.R., THANDAPANI E., Oscillatory and asymptotic behavior of fourth order nonlinear delay difference equations, *Fasci. Math.,* 31(2001), 23–36.

[7] THANDAPANI E., MAHALINGAM K., Oscillatory properties of third order neutral delay difference equations, *Demon. Math.,* 35(2002), 325–337.

[8] ZHANG G., CHENGS.S., Elementary non-existence criteria for a recurrence relation, *Chinese J. Math.,* 24(1996), 229-235.

ETHIRAJU THANDAPANI
DEPARTMENT OF MATHEMATICS, PERIYAR UNIVERSITY
SALEM-636011, TAMILNADU, INDIA
e-mail: ethandapani@yahoo.co.in

BOTHIAH SELVARAJ
DEPARTMENT OF MATHEMATICS, PERIYAR UNIVERSITY
SALEM-636011, TAMILNADU, INDIA

Received on 22.02.2004 and, in revised form, on 29.10.2005.