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CONVERGENCE AND PERIODIC PROPERTIES OF SOLUTIONS FOR A CLASS OF DELAY DIFFERENCE EQUATION

Abstract: We propose a class of delay difference equation with piecewise constant nonlinearity. The convergence of solutions and the existence of globally asymptotically stable periodic solutions are investigated for such a class of difference equation.

Key words: difference equation, convergence, periodic solution.

1. Introduction

The qualitative behavior of solutions of differential equations with piecewise constant argument and delay difference equations have been the subject of many recent investigations. See, for example, [1-15] and the references cited therein. As mentioned in papers by Cook and Wiener [3, 4] and Shah and Wiener [10], the strong interest in differential equations with piecewise constant argument is motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations. In this paper, we consider the following delay difference equation.

$$x_n = ax_{n-1} + (1-a)f(x_{n-k}), \quad \text{for } n = 1, 2, ..., \tag{1.1}$$

where $a \in (0,1)$, k is a positive integer and $f : R \to R$ is a signal transmission function of the piecewise constant nonlinearity:

$$f(\xi) = \begin{cases} 1, & \xi \in (-b, b] \\ 0, & \xi \in (-\infty, -b] \cup (b, \infty), \end{cases} \tag{1.2}$$

for some constant $b \in (0, \infty)$.

Equation (1.1) can be derived from the following delay differential equation with a piecewise constant argument

$$\dot{x} = -\mu x + \beta f(x[t-l]), \tag{1.3}$$

where $\dot{x} = \frac{dx}{dt}$, $\mu > 0$ and $\beta > 0$ are given constants, l is an nonnegative integer, $f : R \to R$ is a signal transmission function defined by (1.2), and $[\cdot]$ denotes the greatest integer function.

As we know, equation (1.3) has also wide applications in certain biomedical models. For some background on (1.3) and some other systems of differential equations involving piecewise constant argument, we refer to [1− 4, 11−12]. It is easy to convert (1.3) into a difference equation (1.1). In fact, we may rewrite (1.3) in the following form

$$\frac{d}{dt}(x(t)e^{\mu t} = e^{\mu t}\beta f(x([t-l])). \tag{1.4}$$

Let n be a positive integer and $k = l+1$. Then we integrate (1.4) from $n-1$ to $t \in [n-1, n)$ to obtain

$$x(t)e^{\mu t} - x(n-1)e^{\mu(n-1)} = \frac{\beta}{\mu}(e^{\mu t} - e^{\mu(n-1)})f(x(n-k)). \tag{1.5}$$

Letting $t \to n$ and simplifying, we get from (1.5)

$$x(n) = e^{-\mu}x(n-1) + \frac{\beta}{\mu}(1-e^{-\mu})f(x(n-k)). \tag{1.6}$$

Set $x_n^* = \frac{\mu}{\beta}x(n)$ for any nonnegative integer n, $f^*(u) = f\left(\frac{\beta}{\mu}u\right)$, $b^* = \frac{\mu}{\beta}b$, $a = e^{-\mu}$, and then drop $*$ to obtain (1.1).

Our goal is to discuss the convergence and periodic properties of delay difference equation (1.1) with (1.2).

For the sake of simplicity, let N denotes the set of all nonnegative integers. For any $a, b \in N$, define $N(a) = \{a, a+1, \ldots, \}$ and $N(a, b) = \{a, a+1, \ldots, b\}$ whenever $a \le b$. In particular, $N = N(0)$. By a solution of (1.1), we mean a sequence $\{x_n\}$ of points in R that is defined for all $n \in N(-k)$ and satisfies (1.1) for $n \in N$. Let X denote the set of mappings from $N(-k, -1)$ to R. Clearly, for any $\varphi \in X$, equation (1.1) has an unique solution x_n satisfying the initial conditions

$$x_i = \varphi(i) \quad \text{for } i \in N(-k, -1). \tag{1.7}$$

We shall concentrate on the case where $\varphi+b$ and $\varphi-b$ have no sign changes on $N(-k, -1)$. More precisely, we consider those $\varphi \in X_1 \cup X_2 \cup X_3 = X_b \subset X$, where

$$X_1 = \{\varphi; \varphi : N(-k, -1) \to R \text{ and } \varphi(i) \le -b \text{ for } i \in N(-k, -1)\},$$

$$X_2 = \{\varphi; \varphi : N(-k, -1) \to R \text{ and } -b < \varphi(i) \le b \text{ for } i \in N(-k, -1)\}$$

and

$$X_3 = \{\varphi; \varphi : N(-k,-1) \to R \text{ and } \varphi(i) > b \text{ for } i \in N(-k,-1)\}.$$

2. Convergence of solutions

In this section, we consider the convergence of solutions of (1.1).

Theorem 2.1. *Let $b > 1$, then $x_n \to 1$ as $n \to \infty$.*

Proof. In view of (1.1), we have

$$ax_{n-1} \le x_n \le ax_{n-1} + (1-a) \quad \text{for } n \in N. \tag{2.1}$$

By induction, it follows that

$$a^{n+1}\varphi(-1) \le x_n \le (\varphi(-1)-1)a^{n+1} + 1 \quad \text{for } n \in N, \tag{2.2}$$

which implies that there exists a positive integer m_1 such that $-b < x_n \le b$ for $n \in N(m_1)$. Thus, we have

$$x_n = ax_{n-1} + (1-a) \quad \text{for } n \in N(m_1+k). \tag{2.3}$$

Therefore,

$$x_n = (x_{m_1+k-1} - 1)a^{n-m_1-k+1} + 1 \quad \text{for } n \in N(m_1+k), \tag{2.4}$$

which implies that $x_n \to 1$ as $n \to \infty$. ■

Theorem 2.2. *Let $b = 1$, then $x_n \to 1$ as $n \to \infty$.*

Proof. We distinguish several cases.

Case 1. Let $\varphi \in X_2$.

By using (1.1) and (1.2), we can see that

$$x_n = (\varphi(-1)-1)a^{n+1} + 1 \quad \text{for } n \in N(0,k-1). \tag{2.5}$$

This, together with the fact that $a \in (0,1)$ and $\varphi \in X_2$, implies that $x_n \in X_2$ for $n \in N(0,k-1)$. By induction, it is easy to verify that $x_n \in X_2$ for $n \in N$. Therefore,

$$x_n = (\varphi(-1)-1)a^{n+1} + 1 \quad \text{for } n \in N,$$

it follows that $x_n \to 1$ as $n \to \infty$.

Case 2. Let $\varphi \in X_3$.

By (1.1) and (1.2), we show that

$$x_n = \varphi(-1)a^{n+1} \quad \text{for } n \in N(0,k-1). \tag{2.6}$$

Let m_1 be the least nonnegative integer such that

$$x_{m_1-1} > 1, \qquad x_{m_1} \geq 1, \tag{2.7}$$

then (2.6) holds for $n \in N(0, m_1 + k - 1)$.

By (2.6) and (2.7), we have

$$-1 < x_{m_1+i} = \varphi(-1)a^{m_1+i+1} < \varphi(-1)a^{m_1+1} = x_{m_1} \leq 1 \text{ for } i \in N(0, k-1),$$

which implies $x_{m_1+i} \in X_2$ for $i \in N(0, k-1)$. By Casel, we have $x_n \to 1$ as $n \to \infty$.

Case 3. Let $\varphi \in X_1$.

The proof of Case 3 is similar to that of Case 2, and thus is omitted. ■

Remark 2.1. Theorem 2.1 and Theorem 2.2 indicate that the unique equilibrium 1 is the global attractor of equation (1.1) when $b \geq 1$.

3. Existence and attraction of periodic solutions

The following Lemma is helpful for discussing existence and attraction of periodic solutions.

Lemma 3.1. *Let* $0 < b < 1$. *For any solution* $\{x_n\}$ *of (1.1) with initial value* $\varphi \in X_b$, *then there exists integers* m_1 *and* $m \in N(-1)$ *with* $m - m_1 \geq k$ *such that* $x_n \in X_2$ *for* $n \in N(m_1, m)$ *and* $x_{m+1} \in (b, 1)$.

Proof. We distinguish several cases. ■

Case 1. Let $\varphi \in X_2$. We shall show that there exists an $n_0 \in N(-1)$ such that $x_n \in X_2$ for $n \in N(-k, n_0)$ and $x_{n_0+1} \notin X_2$. Otherwise, we have $x_n \in X_2$ for any $n \in N(-k)$. It follows from (1.1) and (1.2) that

$$x_n = (\varphi(-1) - 1)a^{n+1} + 1 \quad \text{for } n \in N, \tag{3.1}$$

which implies that $x_n \to 1 > b$ as $n \to \infty$. It contradicts the fact that $x_n \in X_2$ for $n \in N(-k)$.

Note that

$$x_{n_0+1} = ax_{n_0} + (1 - a) > -b,$$

which implies that $x_{n_0+1} \in X_3$.

Moreover,

$$x_{n_0+1} = ax_{n_0} + (1 - a) \leq ab + (1 - a) < 1,$$

so, the conclusion of Lemma 3.1 holds, where $m = n_0$ and $m_1 = -k$.

Case 2. Let $\varphi \in X_3$. For this case, by the similar argument as Case 2 of Theorem 2.2, it is easy to verify that there exists some $n_1 \in N(-1)$ such that $x_n\ in X_2$ for $n \in N(n_1, n_1+k-1)$. Hence, Case 2 can be proved by using Case 1.

Case 3. Let $\varphi \in X_1$. The proof of Case 3 is similar to that of Case 2, and thus is omitted.

Remark 3.1. From the proof of Lemma 3.1, we see that to study the limiting behavior of solutions with initial values in X_b for $b \in (0,1)$, it suffices to restrict initial condition $\varphi \in X_2$ and the first iteration $x_0 \in D_0 = (b,1)$.

Theorem 3.1. *Let $b \in I_1(p,q) \cap I_2(p,q)$ for some $p,q \in N$, where $I_1(p,q) = [\,a^{p+1}, \frac{a^p(1-a^{k-1})}{1-a^{k+p-1}})$ and $I_2(p,q) = [\,1-a^q+a^{p+q+k}, 1-\frac{a^{q+1}(1-a^{k+p})}{1-a^{2k+p+q}})$. Then the equation (1.1) exists a globally asymptotically stable periodic solution $\{x'_n\}$ with initial condition $\varphi \in X_b$, whose minimal period is $2k+p+q$.*

Proof. From the Remark 3.1, it suffices to consider $\varphi \in X_2$ and the first iteration $x_0 \in D_0 = (b,1)$.

Note that the iteration of the linear map

$$g_1(u) = au + 1 - a \tag{3.2}$$

satisfies

$$g_1^{(n)}(u) = a^n u + 1 - a^n \tag{3.3}$$

and that the iteration of the linear map

$$g_2(u) = au \tag{3.4}$$

satisfies

$$g_2^{(n)}(u) = a^n u. \tag{3.5}$$

Let $g_1^{(n)}(D_0) = D_n$ for $n \in N(1, k-1)$.

Since $\varphi \in X_2$ and $x_0 \in D_0 = (b,1)$, it is clear that the solution $\{x_n\}$ of (1.1) with (1.2) satisfies

$$x_n = g_1^{(n)}(x_0) \quad \text{for } n \in N(1, k-1). \tag{3.6}$$

Moreover, it is easy to prove that

$$D_n = (g_1^{(n)}(b), g_1^{(n)}(1)) \quad \text{for } n \in N(1, k-1). \tag{3.7}$$

In view of $u \in (0,1)$, we have

$$1 > g_1^{(k)}(u) > g_1^{(k-1)}(u) > \ldots > g_1^{(0)}(u) = u. \tag{3.8}$$

Recall that $b \in (0,1)$, from (3.7) and (3.8), it follows that $D_n \subset X_3$ holds for all $n \in N(0, k-1)$.

Let n_1 be the largest integer such that $x_n \in X_3$ for $n \in N(0, n_1 + k - 1)$. Then, from (1.1) and (1.2), we can obtain

$$x_{n+k-1} = g_2^{(n)}(x_{k-1}) = g_2^{(n)} g_1^{(k-1)}(x_0) \quad \text{for } n \in N(1, n_1 + k), \tag{3.9}$$

which implies that $x_{n+k-1} \in D_{n+k-1}$ for $n \in N(1, n_1 + k)$ where $D_{n+k-1} = g_2^{(n)} g_1^{(k-1)}(D_0)$ for $n \in N(1, n_1 + k)$. Furthermore, it follows from (3.7) that

$$\begin{aligned} D_{n+k-1} &= (g_2^{(n)} g_1^{(k-1)}(b), g_2^{(n)} g_1^{(k-1)}(1)) \\ &= (a^{n+k-1}b - a^{n+k-1} + a^n, a^n) \ \text{ for } n \in N(1, n_1 + k). \end{aligned} \tag{3.10}$$

Since $b \in I_1(p, q)$, from (3.10), it is easy to verify that

$$D_{n+k-1} \subset X_3 \quad \text{for } n \in N(0, p), \tag{3.11}$$

which leads to $n_1 \geq p$ and

$$x_{n+k-1} \in D_{n+k-1} \subset X_2 \quad \text{for } n \in N(p+1, p+k). \tag{3.12}$$

Thus, it is easy to see $n_1 = p$. Taking $n = p + k$ in (3.9), we have

$$x_{2k+p-1} = g_2^{(k+p)} g_1^{(k-1)}(x_0) = a^{2k+p-1}x_0 - a^{2k+p-1} + a^{k+p}. \tag{3.13}$$

Let n_2 be the largest integer such that $x_{n+2k+p-1} \in X_2$ for $n \in N(0, n_2)$. Then, from (1.1) and (3.13), we get

$$\begin{aligned} x_{n+2k+p-1} &= g_1^{(n)}(x_{2k+p-1}) = (x_{2k+p-1} - 1)a^n + 1 = a^{n+2k+p-1}x_0 \\ &\quad -a^{n+2k+p-1} + a^{n+k+p} - a^n + 1 \quad \text{for } n \in N(1, n_2 + k). \end{aligned} \tag{3.14}$$

This implies that $x_{n+2k+p-1} \in D_{n+2k+p-1}$ for $n \in N(1, n_2 + k)$ where

$$D_{n+2k+p-1} = g_1^{(n)}(D_{2k+p-1}) = g_1^{(n)} g_2^{(k+p)} g_1^{(k-1)}(D_0). \tag{3.15}$$

Substituting (3.10) with $n_1 = p$ into (3.15), we calculate $D_{n+2k+p-1}$ to be

$$\begin{aligned} D_{n+2k+p-1} &= (g_1^{(n)} g_2^{(k+p)} g_1^{(k-1)}(b), g_1^{(n)} g_2^{(k+p)} g_1^{(k-1)}(1)) \\ &= (a^{n+2k+p-1}b - a^{n+2k+p-1} + a^{n+k+p} - a^n + 1, a^{n+k+p} \\ &\quad -a^n + 1) \quad \text{for } n \in N(1, n_2 + k). \end{aligned} \tag{3.16}$$

Since $b \in I_2(p, q)$, from (3.16), it follows that

$$x_{n+2k+p-1} \in D_{n+2k+p-1} \subset X_2 \quad \text{for } n \in N(0, q), \tag{3.17}$$

and

$$x_{n+2k+p-1} \in D_{n+2k+p-1} \subset D_0 \subset X_3 \quad \text{for } n \in N(q+1, q+k), \tag{3.18}$$

which implies that $n_2 = p$.

Taking $n = q+1$, from (3.18), we have

$$x_{2k+p+q} \in D_{2k+p+q} \subset D_0.$$

From the above facts, we can construct the mapping $g(x) : D_0 \to D_0$ as follows

$$\begin{aligned} g(x) &= g_1^{(q+1)} g_2^{(k+2)} g_1^{(k-1)}(x) \\ &= a^{2k+p+q}x - a^{2k+p+q} + a^{k+p+q+1} - a^{q+1} + 1. \end{aligned} \tag{3.19}$$

Obviously,

$$\lim_{n\to\infty} g^{(n)}(x) = 1 - a^{q+1}(1-a^{k+p})(1-a^{2k+p+k})^{-1} = x^*. \tag{3.20}$$

Notes that $b \in I_1(p,q) \cap I_2(p,q)$, it can be checked that $x^* \in D_0$. Hence, x^* is the unique fixed point of $g(x)$ in D_0. Clearly, the unique solution $\{x'_n\}$ with initial value $\varphi \in X_2$ and the first iteration $x'_0 = x^*$ is a periodic solution, whose minimal period is $2k+p+q$. From Lemma 3.1 and (3.20), we see that the solution $\{x'_n\}$ is a globally asymptotically stable periodic solution with initial value $\varphi \in X_b$. The proof is complete ■

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