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ON CONVERGENCE OF CONVOLUTION TYPE SINGULAR INTEGRAL OPERATORS DEPENDING ON TWO PARAMETERS

Abstract: Convolution type singular integral operators depending on two parameters are considered here of the form

$$T(f;x,\lambda)=\int_a^b f(t)\,K(t-x,\lambda)\,dt.$$

Here f belongs to the function space $L_1(a,b)$. In this paper three theorems are proved, one for existence of the operator $T(f;x,\lambda)$ and the others for its pointwise convergence to $f(x_0)$, as (x,λ) tends to (x_0,λ_0). In contrast to previous works, the kernel function $K(t,\lambda)$ of $T(f;x,\lambda)$ does not have to be $2\pi-$periodic, positive or even. Our result improves and extends the results of [1]-[3].

Key words: approximation, convolution type singular integral operators, Lebesgue point, generalized Lebesgue point, $\mu-$generalized Lebesgue point.

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1. Introduction

In papers [1] − [3] studied the pointwise convergence of integrable functions in $L_1(-\pi,\pi)$, by a two-parameter family of convolution type singular integral operators of the form

$$U(f;x,\lambda)=\int_{-\pi}^{\pi} f(t)\,K(t-x,\lambda)\,dt,\quad x\in(-\pi,\pi).$$

They considered four cases where x_0 is a continuous point, a Lebesgue point, a generalized Lebesgue point and a $\mu-$generalized Lebesgue point of f.

In the present paper, we investigate the pointwise convergence of $T(f;x,\lambda)$ to $f(x_0)$ in $L_1(a,b)$, by another two-parameter family of convolution type singular integral operators of the form

$$T(f;x,\lambda)=\int_a^b f(t)\,K(t-x,\lambda)\,dt,\quad x\in(a,b),\ \ \lambda\in\Lambda\subset R. \tag{1}$$

We note that taking $\lambda \in N$, $x = x_0$, and $K \geq 0$ this two-parameters family is reduced to the single parameter sequence in [4], [5] and [6].

First, we shall give conditions which provide the operator's existence.

Definition (**Class** $\mathcal{A}$). *We take a family $\mathcal{K} = (K_\lambda)_{\lambda\in\Lambda}$ of functions $K(t,\lambda) : R \times \Lambda \to R$. We will say that the function $K(t,\lambda)$ belongs to class $\mathcal{A}$, if the following conditions hold:*

a) As function of t, $K(t,\lambda)$ is defined on $(-\infty,\infty)$ and integrable for each fixed $\lambda \in \Lambda$ (Λ is a given set of numbers with an accumulation point λ_0).

b) $\lim\limits_{(x,\lambda)\to(x_0,\lambda_0)} \int_{-\infty}^{\infty} K(t-x,\lambda)\,dt = 1.$

c) $\lim\limits_{\lambda\to\lambda_0} \int_{|t|>\delta} |K(t,\lambda)|\,dt = 0$, *for every $\delta > 0$.*

d) $\lim\limits_{\lambda\to\lambda_0} [\sup\limits_{|t|\geq\delta} |K(t,\lambda)|] = 0$, *for every $\delta > 0$.*

e) There exists a $\delta_0 > 0$ such that $|K(t,\lambda)|$ is non-decreasing on $(-\delta_0, 0]$ and non-increasing on $[0,\delta_0)$ as a function of t, for each $\lambda \in \Lambda$.

2. Existence of the operator

The proofs of the theorems are based on the following theorem.

Theorem 1. *Let $1 \leq p < \infty$, We assume that the kernel function $K(t,\lambda)$ satisfies a). If $f \in L_p < a,b >$, then the operators, which are defined in (1), $T(f) \in L_p < a,b >$ and*

$$\|T(f)\|_{L_p<a,b>} \;\leq\; \|K(.,\lambda)\|_{L_1(-\infty,\infty)}\,\|f\|_{L_p<a,b>}$$

for every $\lambda \in \Lambda$. This implies that $T(f;x,\lambda)$ defined a continuous transformation over $L_p< a,b >$, where $< a,b >$ is an arbitrary interval in R.

Proof. This kind of existence theorem is also valid in general functional spaces (see e.g. [7]). ∎

We denote $\widetilde{f} \in L_1(R)$ as

$$\widetilde{f}(t) := \begin{cases} f(t)\,, & t \in < a,b >,\\ 0\,, & t \notin < a,b >\,. \end{cases} \tag{2}$$

3. Main results

In the present section, we assume that $< a, b >$ is an arbitrary finite interval in R, such as $[a, b]$, $[a, b)$, $(a, b]$ or (a, b).

Theorem 2. *Suppose that the function $K(t, \lambda)$ belongs to class $\mathcal{A}$. Let x_0 be a $\mu-$generalized- Lebesgue point of function $f(x) \in L_1 < a, b >$, i.e., for some x_0 satisfy the condition*

$$\lim_{h \to 0^+} \frac{1}{\mu(h)} \int_0^h |f(x_0 + t) - f(x_0)|\, dt = 0, \tag{3}$$

where $\mu(t)$ defined on $[0, b-a]$, increasing, absolutely continuous and $\mu(0) = 0$. If (x, λ) tends to (x_0, λ_0) on any planar set Z on which the function

$$\int_{x_0-\delta}^{x_0+\delta} |K(t - x, \lambda)|\, \mu'(|t - x_0|)dt + 2\,|K(0, \lambda)|\, \mu(|x - x_0|), \ \ 0 < \delta < \delta_0 \tag{4}$$

is bounded, then

$$\lim_{(x,\lambda) \to (x_0,\lambda_0)} |T(f; x, \lambda) - f(x_0)| = 0. \tag{5}$$

Proof. Suppose that

$$x_0 + \delta < b, \ \ x_0 - \delta > a \ \text{ and } \ 0 < x_0 - x < \frac{\delta}{2}, \tag{6}$$

for any $0 < \delta < \delta_0$.

Set $|I(x, \lambda)| = |T(f; x, \lambda) - f(x_0)|$. According to condition $b)$ and (2), we shall write

$$\begin{aligned} |I(x, \lambda)| &= \left| \int_a^b f(t)K(t - x, \lambda)dt\, - f(x_0) \right| \\ &= \left| \int_{-\infty}^{\infty} \tilde{f}(t)K(t - x, \lambda)dt - f(x_0) \right| \\ &= \left| \int_{-\infty}^{\infty} \tilde{f}(t)K(t - x, \lambda)dt - \tilde{f}(x_0) \int_{-\infty}^{\infty} K(t - x, \lambda)dt \right. \\ &\quad \left. + \tilde{f}(x_0) \int_{-\infty}^{\infty} K(t - x, \lambda)dt - f(x_0) \right| \end{aligned}$$

Also according to (2) and (6), we have $\widetilde{f}(x_0) = f(x_0)$. Hence from (3), we can divide the last equality as follows:

$$\begin{aligned}
|I(x,\lambda)| &\leq \int_{-\infty}^{\infty} \left|\widetilde{f}(t) - \widetilde{f}(x_0)\right| |K(t-x,\lambda)|\, dt \\
&\quad + |f(x_0)| \left| \int_{-\infty}^{\infty} K(t-x,\lambda) dt - 1 \right| \\
&= \left\{ \int_{-\infty}^{a} + \int_{a}^{x_0-\delta} + \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} + \int_{x_0+\delta}^{b} + \int_{b}^{\infty} \right\} \left|\widetilde{f}(t) - \widetilde{f}(x_0)\right| \\
&\quad \times |K(t-x,\lambda)|\, dt + |f(x_0)| \left| \int_{-\infty}^{\infty} K(t-x,\lambda) dt - 1 \right| \\
&= I_0(x,\lambda) + I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) \\
&\quad + I_4(x,\lambda) + I_5(x,\lambda) + I_6(x,\lambda).
\end{aligned}$$

Since

$$\begin{aligned}
|I(x,\lambda)| \leq I_0(x,\lambda) + I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) \\
+ I_4(x,\lambda) + I_5(x,\lambda) + I_6(x,\lambda),
\end{aligned}$$

it is sufficient to show that the first four terms on the right hand side of the last inequality tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$ on Z.

Firstly we consider $I_1(x,\lambda)$ and $I_4(x,\lambda)$. From condition $e)$ and (2), it is seen that

$$\begin{aligned}
I_1(x,\lambda) &= \int_{a}^{x_0-\delta} |f(t) - f(x_0)|\, |K(t-x,\lambda)|\, dt \\
&\leq \sup_{t \in <a, x_0-\delta]} |K(t-x,\lambda)| \int_{a}^{x_0-\delta} |f(t) - f(x_0)|\, dt.
\end{aligned}$$

Since $x_0 - x < \frac{\delta}{2}$

$$\begin{aligned}
I_1(x,\lambda) &\leq \sup_{|u| > \frac{\delta}{2}} |K(u,\lambda)| \int_{a}^{b} |f(t) - f(x_0)|\, dt \\
&= \sup_{|u| > \frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\}.
\end{aligned} \tag{7}$$

In the same way, we find that

$$(8)\qquad \begin{aligned} I_4(x,\lambda) &= \int\limits_{x_0+\delta}^{b} |f(t)-f(x_0)|\,|K(t-x,\lambda)|\,dt \\ &\le \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \int\limits_{x_0+\delta}^{b} |f(t)-f(x_0)|\,dt \\ &\le \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\}. \end{aligned}$$

Next we consider $I_2(x,\lambda)$. From (3) and (2), for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(9)\qquad \int\limits_{x_0-h}^{x_0} |f(t)-f(x_0)|\,dt < \varepsilon\,\mu(h)$$

and

$$(10)\qquad \int\limits_{x_0}^{x_0+h} |f(t)-f(x_0)|\,dt < \varepsilon\,\mu(h)$$

for all $0 < h \le \delta$.

Let

$$(11)\qquad F(t) := \int\limits_{t}^{x_0} |f(y)-f(x_0)|\,dy,$$

then

$$(12)\qquad dF(t) = -\,|f(t)-f(x_0)|\,dt.$$

From (9) and (11), for all t satisfying the condition $x_0 - t \le \delta$, we have

$$(13)\qquad F(t) \le \varepsilon\,\mu(x_0-t).$$

Hence, now we can evaluate $I_2(x,\lambda)$.
Evaluation of $I_2(x,\lambda)$ can be done in the following way:

$$I_2(x,\lambda) = \int\limits_{x_0-\delta}^{x_0} |f(t)-f(x_0)|\,|K(t-x,\lambda)|\,dt.$$

By (11) and (12), we can rewrite $I_2(x,\lambda)$ as

$$|I_2(x,\lambda)| = \left| -\int_{x_0-\delta}^{x_0} |K(t-x,\lambda)|\, dF(t) \right|.$$

Integration by parts give us,

$$\begin{aligned} |I_2(x,\lambda)| &= \left| -F(x_0-\delta)\, |K(x_0-\delta-x,\lambda)| \right. \\ &\quad \left. + \int_{x_0-\delta}^{x_0} F(t)\, \frac{\partial}{\partial t} |K(t-x,\lambda)|\; dt \right| \\ &\le |F(x_0-\delta)|\, |K(x_0-\delta-x,\lambda)| \\ &\quad + \int_{x_0-\delta}^{x_0} |F(t)|\, \frac{\partial}{\partial t} |K(t-x,\lambda)|\; dt. \end{aligned}$$

It follows that from (13),

$$\begin{aligned} |I_2(x,\lambda)| &\le \varepsilon\, \mu(\delta)\; |K(x_0-\delta-x,\lambda)| \\ &\quad + \varepsilon \int_{x_0-\delta}^{x_0} \mu(x_0-t) \frac{\partial}{\partial t} |K(t-x,\lambda)|\; dt. \end{aligned}$$

Integration by parts again, then we get

$$\begin{aligned} |I_2(x,\lambda)| &\le \varepsilon\, \mu(\delta)\; |K(x_0-\delta-x,\lambda)| + \varepsilon \left\{ -\mu(\delta)\, |K(x_0-\delta-x,\lambda)| \right. \\ &\quad \left. - \int_{x_0-\delta}^{x_0} \mu'(x_0-t)\, |K(t-x,\lambda)|\; dt \right\} \\ &= \varepsilon \int_{x_0-\delta}^{x_0} \mu'(x_0-t)\, |K(t-x,\lambda)|\; dt. \end{aligned}$$

Let

$$I_{2,1}(x,\lambda) := \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-x-t)\, |K(t,\lambda)|\; dt. \tag{14}$$

Here we note that, if a function f is monotone on $[a,b]$, the

$$V[f;a;b] = \bigvee_a^b (f) = |f(b)-f(a)|\,.$$

According to condition e) and (6), for (14) we obtain the following estimate:

$$(15)\ I_{2,1}(x,\lambda) = \int\limits_{x_0-x-\delta}^{x_0-x} \left\{ \bigvee_{x_0-x-\delta}^{t} |K(s,\lambda)| + |K(x_0-\delta,\lambda)| \right\} \mu'(x_0-x-t)dt$$

$$= \int\limits_{x_0-x-\delta}^{0} \left[\bigvee_{x_0-x-\delta}^{t} |K(s,\lambda)| \right] \mu'(x_0-x-t)dt$$

$$+ \int\limits_{0}^{x_0-x} \left[\bigvee_{x_0-x-\delta}^{t} |K(s,\lambda)| \right] \mu'(x_0-x-t)dt$$

$$+ |K(x_0-\delta,\lambda)| \int\limits_{x_0-x-\delta}^{x_0-x} \mu'(x_0-x-t)dt$$

$$= \int\limits_{x_0-x-\delta}^{0} [\,|K(t,\lambda)| - |K(x_0-\delta-x,\lambda)|\,]\, \mu'(x_0-x-t)dt$$

$$+ \int\limits_{0}^{x_0-x} \{|K(0,\lambda)| - |K(x_0-\delta-x,\lambda)|\}\, \mu'(x_0-x-t)dt$$

$$+ \int\limits_{0}^{x_0-x} \{|K(0,\lambda)| - |K(t,\lambda)|\}\, \mu'(x_0-x-t)dt$$

$$+ |K(x_0-\delta,\lambda)| \int\limits_{x_0-x-\delta}^{x_0-x} \mu'(x_0-x-t)dt$$

$$= \int\limits_{x_0-x-\delta}^{0} |K(t,\lambda)|\, \mu'(x_0-x-t)dt + 2\,|K(0,\lambda)| \int\limits_{0}^{x_0-x} \mu'(x_0-x-t)dt$$

$$- |K(x_0-\delta-x,\lambda)| \left\{ \int\limits_{x_0-x-\delta}^{0} \mu'(x_0-x-t)dt + \int\limits_{0}^{x_0-x} \mu'(x_0-x-t)dt \right\}$$

$$- \int\limits_{0}^{x_0-x} |K(t,\lambda)|\, \mu'(x_0-x-t)dt + |K(x_0-\delta-x,\lambda)| \int\limits_{x_0-\delta}^{x_0} \mu'(x_0-t)dt$$

$$\leq \int_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\, \mu'(x_0-x-t)dt + 2\,|K(0,\lambda)| \int_0^{x_0-x} \mu'(x_0-x-t)dt$$
$$+\ |K(x_0-\delta-x,\lambda)| \left\{ \int_{x_0-\delta}^{x_0} \mu'(x_0-t)dt - \int_{x_0-\delta}^{x_0} \mu'(x_0-t)dt \right\}$$
$$= \int_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\, \mu'(x_0-x-t)dt + 2\,|K(0,\lambda)|\, \mu(|x-x_0|)$$
$$= \int_{x_0-\delta}^{x_0} |K(t-x,\lambda)|\, \mu'(x_0-t)dt + 2\,|K(0,\lambda)|\, \mu(|x-x_0|).$$

Consequently by (15),

$$|I_2(x,\lambda)| \leq \varepsilon \left\{ \int_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\, \mu'(x_0-t)dt + 2\,|K(0,\lambda)|\, \mu(|x-x_0|) \right\}. \tag{16}$$

We can use similar method for evaluating $I_3(x,\lambda)$.

Let

$$G(t) \ := \int_{x_0}^{t} |f(y)-f(x_0)|\, dy. \tag{17}$$

Then

$$dG(t) \ = \ |f(t)-f(x_0)|\, dt. \tag{18}$$

Moreover, for all t satisfying the condition $t-x_0 \leq \delta$, we have from (10) and (17)

$$G(t) \ \leq \ \varepsilon\, \mu(t-x_0). \tag{19}$$

According to (17) and (18), we can rewrite $I_3(x,\lambda)$ as

$$|I_3(x,\lambda)| = \left| \int_{x_0}^{x_0+\delta} |K(t-x,\lambda)|\, dG(t) \right|.$$

Integration by parts, we find that

$$\begin{aligned} |I_3(x,\lambda)| &= \Bigg| G(x_0+\delta)\ |K(x_0+\delta-x,\lambda)| \\ &\quad + \int_{x_0}^{x_0+\delta} G(t)\,\frac{\partial}{\partial t}\left(-|K(t-x,\lambda)|\right)\,dt \Bigg| \\ &\le |G(x_0+\delta)|\,|K(x_0+\delta-x,\lambda| \\ &\quad + \int_{x_0}^{x_0+\delta} |G(t)|\,\frac{\partial}{\partial t}\left(-|K(t-x,\lambda)|\right)dt. \end{aligned}$$

While $t\in[x_0,x_0+\delta)$, according to condition $e)$ and (6), $\frac{\partial}{\partial t}\left(-|K(t-x,\lambda)|\right)$ is positive. Therefore, from (19) we can write the following inequality.

$$\begin{aligned} |I_3(x,\lambda)| &\le \varepsilon\,\mu(\delta)\ |K(x_0+\delta-x,\lambda)| \\ &\quad + \varepsilon \int_{x_0}^{x_0+\delta} \mu\,(t-x_0)\frac{\partial}{\partial t}\left(-|K(t-x,\lambda)|\right)dt. \end{aligned} \tag{20}$$

If we using integration by parts, the second term of the right hand side in (20), then

$$\begin{aligned} |I_3(x,\lambda)| &\le \varepsilon\,\mu(\delta)\ |K(x_0+\delta-x,\lambda)| \\ &\quad + \varepsilon\left\{-\mu(\delta)\,|K(x_0+\delta-x,\lambda)| + \int_{x_0}^{x_0+\delta}\mu'(t-x_0)\,|K(t-x,\lambda)\,|\,dt\right\} \\ &= \varepsilon\int_{x_0}^{x_0+\delta}\mu'(t-x_0)\,|K(t-x,\lambda)|\ dt. \end{aligned}$$

Here we denote

$$I_{3,1}(x,\lambda) := \int_{x_0-x}^{x_0-x+\delta}\mu'(t+x-x_0)\,|K(t,\lambda)\,|\,dt. \tag{21}$$

While $t\in[x_0,x_0+\delta)$, according to condition $e)$ and (6), for (21) we obtain the following estimate:

$$I_{3,1}(x,\lambda) = \int\limits_{x_0}^{x_0+\delta} \left\{ \bigvee_t^{x_0+\delta} |K(s-x,\lambda)| + |K(x_0+\delta-x,\lambda)| \right\} \mu'(t-x_0)dt \tag{22}$$

$$= \int\limits_{x_0}^{x_0+\delta} \left[\bigvee_t^{x_0+\delta} |K(s-x,\lambda)| \right] \mu'(t-x_0)dt$$

$$+ \ |K(x_0+\delta-x,\lambda)| \int\limits_{x_0}^{x_0+\delta} \mu'(t-x_0)dt$$

$$= \int\limits_{x_0}^{x_0+\delta} [\,|K(t-x,\lambda)| - |K(x_0+\delta-x,\lambda)|\,]\, \mu'(t-x_0)dt$$

$$+ \ |K(x_0+\delta-x,\lambda)| \int\limits_{x_0}^{x_0+\delta} \mu'(t-x_0)dt$$

$$= \int\limits_{x_0}^{x_0+\delta} |K(t-x,\lambda)|\, \mu'(t-x_0)dt.$$

From (22), we find that

$$|I_3(x,\lambda)| \ \le \ \varepsilon \int\limits_{x_0}^{x_0+\delta} |K(t-x,\lambda)|\, \mu'(t-x_0)dt. \tag{23}$$

Finally, let us consider the integrals $I_0(x,\lambda)$ and $I_5(x,\lambda)$. According to (2) we write

$$\begin{aligned} I_0(x,\lambda) + I_5(x,\lambda) &= \int\limits_{t \notin <a,b>} \left| \widetilde{f}(t) - f(x_0) \right| |K(t-x,\lambda)|\, dt \\ &= \int\limits_{t \notin <a,b>} |f(x_0)|\, |K(t-x,\lambda)|\, dt \\ &= |f(x_0)| \int\limits_{t \notin <a,b>} |K(t-x,\lambda)|\, dt. \end{aligned}$$

According to (6), when $t \notin [a,b]$, we have $t < a$ or $t > b$. If $t < a$ then

$$t - x < a - x < x_0 - x - \delta < -\frac{\delta}{2} < 0,$$

while if $t > b$

$$t - x > b - x > x_0 - x + \delta > \delta > 0.$$

This implies that, for any $\delta > 0$

$$\int\limits_{t \notin <a,b>} |K(t-x,\lambda)|\,dt \le \int\limits_{|t-x|>\frac{\delta}{2}} |K(t-x,\lambda)|\,dt = \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\,du.$$

Thus we get

$$I_0(x,\lambda) + I_5(x,\lambda) \le |f(x_0)| \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\,du, \tag{24}$$

which in view of condition c) tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Combining (7), (8), (16), (23) and (24), we get

$$\begin{aligned}
|I(x,\lambda)| &\le 2 \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\} \\
&\quad + \varepsilon \int\limits_{x_0}^{x_0+\delta} |K(t-x,\lambda)|\,\mu'(t-x_0)dt \\
&\quad + \varepsilon \left\{ \int\limits_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\,\mu'(t-x_0-x-t)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|) \right\} \\
&\quad + |f(x_0)| \left| \int\limits_a^b K(t-x,\lambda)dt - 1 \right| \\
&\le 2 \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\} \\
&\quad + \varepsilon \left\{ \int\limits_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\,\mu'(|t-x_0|)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|) \right\} \\
&\quad + |f(x_0)| \left| \int\limits_a^b K(t-x,\lambda)dt - 1 \right|.
\end{aligned}$$

This inequality is shown to be also valid for $-\frac{\delta}{2} < x_0 - x < 0$.

Therefore our theorem now follows, in view of conditions b), c), d) and (4).

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} T(f;x,\lambda) \;=\; f(x_0)$$

and this proves (5). ■

If we consider the case $< a, b >= R$, we have the following theorem.

Theorem 3. *Suppose that the function $K(t,\lambda)$ belongs to class $\mathcal{A}$. If (x,λ) tends to (x_0,λ_0) on any planar set Z on which the function*

$$\int_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\,\mu'(|t-x_0|)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|), \quad 0<\delta<\delta_0$$

is bounded. Then at each point x_0 for which (3) holds, we have for $f(x)\in L_1(-\infty,\infty)$

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T(f;x,\lambda)-f(x_0)| \;=\; 0.$$

Proof. We can divide $|I(x,\lambda)| := |T(f;x,\lambda)-f(x_0)|$ four parts as follows:

$$\begin{aligned}|I(x,\lambda)| &\le \left\{\int_{-\infty}^{x_0-\delta} + \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} + \int_{x_0+\delta}^{\infty}\right\} |f(t)-f(x_0)|\,|K(t-x,\lambda)|\,dt \\ &\quad + |f(x_0)|\left|\int_{-\infty}^{\infty} K(t-x,\lambda)dt - 1\right| \\ &= I_1(x,\lambda)+I_2(x,\lambda)+I_3(x,\lambda)+I_4(x,\lambda)+I_6(x,\lambda).\end{aligned}$$

Analogously to the proof of Theorem 2 we can show that $I_2(x,\lambda)\to 0$, $I_3(x,\lambda)\to 0$ and $I_6(x,\lambda)\to 0$ as $(x,\lambda)\to(x_0,\lambda_0)$. Hence, it is sufficient to show that

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} I_1(x,\lambda) \;=\; 0$$

and

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} I_4(x,\lambda) \;=\; 0.$$

According to (6), when $t\notin[x_0-\delta,x_0+\delta]$, we have $t<x_0-\delta$ or $t>x_0+\delta$. If $t<x_0-\delta$ then

$$t-x < x_0-x-\delta < -\frac{\delta}{2} < 0,$$

while if $t>x_0+\delta$

$$t-x > x_0-x+\delta > \delta > \frac{\delta}{2} > 0.$$

Therefore, we can write $I_1(x,\lambda) + I_4(x,\lambda)$ as follows:

$$I_1(x,\lambda) + I_4(x,\lambda) = \int\limits_{t \notin [x_0-\delta, x_0+\delta]} |f(t) - f(x_0)|\, |K(t-x,\lambda)|\, dt$$

$$\leq \int\limits_{t \notin [x_0-\delta, x_0+\delta]} |f(t)|\, |K(t-x,\lambda)|\, dt + |f(x_0)| \int\limits_{t \notin [x_0-\delta, x_0+\delta]} |K(t-x,\lambda)|\, dt$$

$$\leq \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \int\limits_{|t-x|>\frac{\delta}{2}} |f(t)|\, dt + |f(x_0)| \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\, du$$

$$\leq \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\, \|f\|_{L_1(-\infty,\infty)} + |f(x_0)| \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\, du,$$

for any $\delta > 0$. Hence according to conditions $c)$ and $d)$, $I_1(x,\lambda) + I_4(x,\lambda)$ tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Thus, the proof of the theorem is completed. ■

Example. We consider the function

$$K(t,\lambda) = \begin{cases} 2\lambda\,, & t \in [0, \frac{1}{\lambda}]\,, \\ -\lambda\,, & t \in [-\frac{1}{\lambda}, 0)\,, \\ 0\,, & t \notin [-\frac{1}{\lambda}, \frac{1}{\lambda}]\,, \end{cases} \tag{25}$$

where $\Lambda = [1,\infty)$ is a set of indices with natural topology and $\lambda_0 = \infty$ is an accumulation point of Λ in this topology.

From (25), one has

$$\int\limits_R K(t-x,\lambda)dt \;=\; -\int\limits_{[-\frac{1}{\lambda},0)} \lambda dt + \int\limits_{[0,\frac{1}{\lambda}]} 2\lambda dt \;=\; 1 < \infty.$$

Furthermore, it is easy to see that

$$\lim_{\lambda\to\infty} \int\limits_{|t|\geq\delta} |K(t,\lambda)| dt \;=\; 0,$$

and

$$\lim_{\lambda\to\infty} [\, \sup_{|t|\geq\delta} |K(t,\lambda)|\,] \;=\; 0,$$

for every $\delta > 0$.

In addition, by (25) $|K(t,\lambda)|$ is non-decreasing on $(-\infty, 0]$ and non-increasing on $[0,\infty)$ as a function of t, for each $\lambda \in \Lambda$.

This implies that the kernel function $K(t,\lambda)$ belongs to **Class** $\mathcal{A}$.

We assume that $\mu(t)=t$. If we use (25) in Theorem 2 and Theorem 3, we find

$$\int_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\,\mu'(|t-x_0|)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|)$$

$$= \int_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\; dt + 4\,\lambda\;|x-x_0|\,.$$

The first part of the right hand side of this equality is finite. Furthermore

$$\lim_{(x,\lambda)\to(x_0,\infty)} \lambda\;|x-x_0| \;=\; M<\infty,$$

if and only if the rates of convergence of $\lambda\to\infty$ and $x\to x_0$ are equivalent.

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