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OSCILLATION AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

Abstract. Sufficient conditions in terms of the coefficient functions for the oscillation and asymptotic behavior of nonoscillatory solutions of a class of second order nonlinear neutral differential equations have been obtained. The results improve some earlier results.

Key words: oscillatory solution, nonoscillatory solution.

AMS Mathematics Subject Classification: 34C10, 34K15.

1. Introduction

In this paper, we consider the oscillation and asymptotic behavior of nonoscillatory solutions of the second order neutral differential equations of the form

$$
\begin{aligned}
[x(t) - \sum_{i=1}^{l} c_i(t)x(t-\tau_i)]'' &+ \sum_{i=1}^{m} p_i(t)G(x(t-\delta_i)) \\
- \sum_{i=1}^{n} q_i(t)G(x(t-\sigma_i)) \ &= \ f(t)
\end{aligned}
\tag{1}
$$

where $m \geq n$ and $\tau_1 \cdots \tau_l$, $\delta_1 \cdots \delta_m$, and $\sigma_1 \cdots \sigma_n$ are positive reals,$c_i, f \in C([0,\infty), R)$, $i = 1, ..., l$, $p_i \in C([0,\infty),\ [0,\infty))$ for $i = 1, \cdots, m$ and $q_i \in C([0,\infty), [0,\infty))$ for $i = 1, \cdots, n$, $G \in C(R,R)$, G is nondecreasing with $xG(x) > 0$ for $x \neq 0$. We assume that there exists a continuous function $F(t) \in C^2([0,\infty), R)$ such that $F''(t) = f(t)$ and $\lim_{t\to\infty} F(t) = 0$.

By a solution of (1), we mean a continuous function $x(t)$ which is defined for $t \geq t_0 - \rho$ such that $x(t) - \sum_{i=1}^{l} c_i(t)x(t-\tau_i) \in C^2([t_0,\infty), R)$ and (1) is satisfied for $t \geq t_0$ where $\rho = \max\{\tau_1 \cdots \tau_l, \delta_1 \cdots \delta_m, \sigma_1 \cdots \sigma_n\}$. A solution

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of (1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

Throughout this work we assume that $p_i(t) - q_i(t - \delta_i + \sigma_i) \geq 0$ for $i = 1, \cdots, n$.

Sufficient conditions for oscillation of solutions of first order neutral delay differential equations with positive and negative coefficients have been studied by many authors, see ([1],[2],[6], [7],[8]). It is recently, that second order neutral differential equations with positive and negative coefficients have been given a serious study. In a recent paper, Parhi and Chand [5] studied (1) when $G(x) = x$ and they obtained various sufficient conditions for the oscillation of all bounded solutions of the linear homogeneous equation. Further, Manojlovic et al. [4] studied Eq.(1) with$G(x) = x$ where they have assumed one additional condition $q_i(t) \leq q_i(t-\sigma_i)$ for every $i = 1, \cdots, n$. In this paper, we study Eq.(1) on various ranges on $\sum_{i=1}^{l} c_i(t)$ and improve the results of [5] by removing not only on the boundedness on the solutions but also relaxing other conditions as well. Our results improve the results of [4] also, where we remove the condition $q_i(t) \leq Q_i(t-\sigma_i)$ for every $i = 1, \cdots, n$.

We consider the following ranges on $\sum_{i=1}^{l} c_i(t)$:

(A_1) $\quad 0 \leq \sum\limits_{i=1}^{l} c_i(t) \leq c < 1$

(A_2) $\quad -1 \leq c_1 \leq \sum\limits_{i=1}^{l} c_i(t) \leq 0$

(A_3) $\quad -c_3 \leq \sum\limits_{i=1}^{l} c_i(t) \leq -c_2 < -1$

(A_4) $\quad 1 \leq c_4 \leq \sum\limits_{i=1}^{l} c_i(t) \leq c_5$

(A_5) $\quad -c_6 \leq \sum\limits_{i=1}^{l} c_i(t) \leq -c_7 \leq 0$

where $c_1, \cdots, c_7$ are positive constants.

The following assumptions are needed for use in the sequel:

(H_1) $\quad \liminf\limits_{|u|\to\infty} \frac{G(u)}{u} \leq \beta$, where $\beta > 0$ is a real number.

(H_2) $\quad \lim\limits_{t\to\infty} \sum\limits_{i=1}^{n} \int\limits_{t_0}^{t} [p_i(s) - q_i(s - \delta_i + \sigma_i)]\, ds = \infty.$

(H_3) $\quad \lim\limits_{t\to\infty} \frac{k}{t} \int\limits_{t_0}^{t} s\{\sum\limits_{i=1}^{n} [p_i(s) - q_i(s - \delta_i + \sigma_i)]\}\, ds > 1$
for any positive constant k.

(H_4) $\quad F(t)$ is oscillatory.

(H_5) $\quad \beta \sum\limits_{i=1}^{n} \int\limits_{s-\delta_i+\sigma_i}^{\infty} q_i(\theta)\, d\theta\, ds < 1$ when $\delta_i \geq \sigma_i$

(H_6) $\quad c + \beta \sum\limits_{i=1}^{n} \int\limits_{s-\delta_i+\sigma_i}^{\infty} q_i(\theta)\, d\theta\, ds < 1$

(H_7) $\quad \delta_i \geq \sigma_i$ for every $i = 1, \cdots, n$.

(H_8) $\quad \sigma_i \geq \delta_i$ for every $i = 1, \cdots, n$.

(H_9) $\quad \beta \sum\limits_{i=1}^{n} \int\limits_{s-\delta_i+\sigma_i}^{\infty} q_i(\theta)\, d\theta\, ds < 1 + c_7$

The following result will be needed for our use (see Lemma 1.5.4 in [3]).

Lemma 1. *Let $a \in (-\infty, 0), \tau \in (0, \infty), t_0 \in R$ and suppose that a function $x \in C[[t_0 - \tau, \infty), R]$ satisfy the inequality*

$$x(t) \;\leq\; a + \max_{t-\tau \leq s \leq t} x(s)$$

for $t \geq t_0$. Then $x(t)$ cannot be a nonnegative function.

2. Main results - the case when $\delta_i \geq \sigma_i$, $i = 1, \cdots, n$.

In this section, we consider Eq. (1) when $\delta_i \geq \sigma_i$, $i = 1, \cdots, n$. We shall obtain sufficient conditions under which a solution of the equation is either oscillatory or tends to zero as $t \to \infty$. We observe that the results hold when G is either linear or sublinear. This is mainly due to the assumption (H_1).

Theorem 1. *Let $c_i(t), i = 1, \cdots, l$ be as in (A_1). If (H_1), (H_6), (H_7) and either of (H_2) or (H_3) are satisfied, then every solution of (1) is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a solution of (1). If $x(t)$ is oscillatory, then there is nothing to prove. Let $x(t)$ be nonoscillatory. Assume that $x(t) > 0$ eventually. There exists a $t_1 \geq t_0 + \rho > 0$ such that $x(t) > 0$ and $x(t-\rho) > 0$ for $t \geq t_1$. Setting

$$w(t) = x(t) - \sum_{i=1}^{l} c_i(t) x(t - \tau_i) - \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s-\delta_i+\sigma_i}^{s} q_i(\theta) G(x(\theta - \sigma_i))\, d\theta\, ds \;-\; F(t), \tag{2}$$

then Eq. (1) can be written as

$$w''(t) + \sum_{i=1}^{n} \{p_i(t) - q_i(t - \delta_i + \sigma_i)\} G(x(t - \delta_i)) \leq 0 \tag{3}$$

for $t \geq t_1$.Hence $w''(t) \leq 0$ for $t \geq t_1$. Thus there exists a $t_2 \geq t_1$ such that $w'(t) > 0$ or < 0 for $t \geq t_2$. Let $w'(t) < 0$ for $t \geq t_2$. This in turn implies that $w(t) < 0$ for $t \geq t_3 \geq t_2$ and $\lim_{t\to\infty} w(t) = -\infty$. Then there exist $t_4 > t_3$, $\epsilon > 0$ and $\lambda > 0$ such that $0 < \epsilon < \lambda$, $w(t) < -\lambda$ and $F(t) < \epsilon$ for $t - \rho > t_4$. Hence from (2),

$$\begin{aligned} x(t) &= w(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i) \\ &\quad + \sum_{i=1}^{n} \int_{t_0}^{t}\int_{s-\delta_i+\sigma_i}^{s} q_i(\theta)G(x(\theta-\sigma_i))\,d\theta\,ds + F(t) \\ &\leq -\lambda + [\sum_{i=1}^{l} c_i(t) + \beta\sum_{i=1}^{n}\int_{t_0}^{t}\int_{s-\delta_i+\sigma_i}^{s} q_i(\theta)\,d\theta\,ds] \max_{t-\rho\leq s\leq t} x(s) + \epsilon \\ &\leq -(\lambda-\epsilon) + \max_{t-\rho\leq s\leq t} x(s). \end{aligned}$$

Then by Lemma 1, it follows that $x(t)$ cannot be nonnegative, a contradiction. Hence $w'(t) < 0$ is not possible.

Next, suppose that $w'(t) > 0$ for $t \geq t_2$. Then $w(t) > 0$ or < 0 for large t, say for $t \geq t_5 \geq t_2$. First, suppose that $w(t) < 0$ for $t \geq t_5$. Then $w(t)$ is bounded and

$$x(t) - \sum_{i=1}^{l} c_i(t)x(t-\tau_i) - \sum_{i=1}^{n}\int_{t_0}^{t}\int_{s-\delta_i+\sigma_i}^{s} q_i(\theta)G(x(\theta-\sigma_i))\,d\theta\,ds < F(t). \tag{4}$$

We claim that $x(t)$ is bounded. If not, then there exists a sequence $\{T_k\}_{k=1}^{\infty}$, $T_k > t_5$ for every k such that $T_k \to \infty$ and $x(T_k) \to \infty$ as $k \to \infty$. In particular, for $t = T_k$, (4) gives

$$x(T_k)[1 - c - \beta\sum_{i=1}^{n}\int_{t_0}^{T_k}\int_{s-\delta_i+\sigma_i}^{s} q_i(\theta)\,d\theta\,ds] < F(T_k).$$

Letting $k \to \infty$, we obtain a contradiction. Hence our claim holds. Further, if $\limsup_{t\to\infty} x(t) = \lambda > 0$, then integration of (3) form t_5 to t yields a contradiction, because G is nondecreasing and (H_2) or (H_3) holds. Hence $x(t) \to 0$ as $t \to \infty$.

Finally, suppose that $w(t) > 0$ for $t \geq t_5$. From the increasingness of $w(t)$ and the assumptions on $F(t)$, it follows that there exists a real $\beta_0 > 0$ such that $w(t) + F(t) > \beta_0$ for large t, that is

$$\begin{aligned} \phi(t) = x(t) &- \sum_{i=1}^{l} c_i(t)x(t-\tau_i) \\ &- \sum_{i=1}^{n}\int_{t_0}^{t}\int_{s-\delta_i+\sigma_i}^{s} q_i(\theta)G(x(\theta-\sigma_i))\,d\theta\,ds > \beta_0 \end{aligned} \tag{5}$$

for $t \geq t_6 > t_5$. This in turn implies that there exists a positive number β_1 such that

$$\phi(t) \ \geq \ \beta_1 w(t) \tag{6}$$

for $t \geq t_7 \geq t_6$.If this is not true, then there exists a sequence $\{T''_k\}$, $T''_k \to \infty$ as $k \to \infty$ such that

$$\phi(T''_k) \ \leq \ \frac{1}{k} w(T''_k) \tag{7}$$

or

$$w(T''_k) + T(T''_k) \ \leq \ \frac{1}{k} w(T''_k)$$

or

$$(1 - \frac{1}{k}) w(T''_k) + F(T''_k) \ \leq \ 0.$$

If $w(T''_k) \to \infty$, then $F(T''_k) \to -\infty$, a contradiction to the boundedness of $F(t)$. If $w(T''_k)$ tends to a constant, then from (7), we have $\phi(T''_k) \to 0$ as $k \to \infty$ a contradiction to (5). Hence (6) holds. consequently, $x(t) \geq \beta_1 w(t)$ for $t \geq t_7$. Then from (3)

$$w''(t) + \sum_{i=1}^{n} \{p_i(t) - q_i(t - \delta_i + \sigma_i)\} G(\beta_1 w(t - \delta_i)) \ \leq \ 0 \tag{8}$$

for $t \geq t_8 \geq t_7$.

Let (H_2) hold. Since $w(t) > \mu$ for some $\mu > 0$, then integrating (8) from t_8 to t and letting $t \to \infty$, we obtain a contradiction.

Next, suppose that (H_3) holds. Set $r(t) = -w'(t)$. Then $r'(t) \ = \ -w''(t)$. Then $r(t) < 0$, nondecreasing and

$$t r'(t) \geq G(\beta_1 \mu) t \sum_{i=1}^{n} \{p_i(t) - q_i(t - \delta_i + \sigma_i)\}$$

for $t \geq t_8$. Integrating the above inequality from t_8 to t gives

$$t r(t) - t_8 r(t_8) - \int_{t_8}^{t} r(s)\, ds \geq G(\beta_1 \mu) \int_{t_8}^{t} s \sum_{i=1}^{n} \{p_i(s) - q_i(s - \delta_i + \sigma_i)\}\, ds.$$

Since $r(t)$ is nondecreasing, then the above integral inequality gives

$$1 \ \geq \ \frac{G(\beta_1 \mu)}{t(-r(t_8))} \int_{t_8}^{t} s \sum_{i=1}^{n} \{p_i(s) - q_i(s - \delta_i + \sigma_i)\}\, ds,$$

a contradiction. Hence $w(t) > 0$ is not possible for large t.

If $x(t) < 0$ for large t, then one may proceed as above to prove the theorem. This completes the proof of the theorem. ■

Theorem 2. *Let $\sum_{i=1}^{l} c_i(t)$ be in the range (A_5). If (H_1), (H_2), (H_7) and (H_9) are satisfied, then every solution of (1) is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Assume that $x(t) > 0$ and $x(t-\rho) > 0$ for $t \geq t_1 \geq t_0 + \rho > 0$. Setting $w(t)$ as in (2), we obtain (3). Hence $w''(t) \leq 0$ for $t \geq t_1$. Then $w'(t) > 0$ or < 0 for some $t \geq t_2 \geq t_1$.

Let $w'(t) > 0$ for $t \geq t_2$. Then integration of (3) from t_2 to t gives

$$w'(t_1) \;\geq\; \sum_{i=1}^{n} \int_{t_2}^{t} \{p_i(s) - q_i(s - \delta_i + \sigma_i)\} G(x(s - \delta_i))\, ds.$$

Letting $t \to \infty$, the above inequality, in view of (H_2), yields $G(x(t)) \to 0$ as $t \to \infty$. Hence $x(t) \to 0$ as $t \to \infty$.

Next, suppose that $w'(t) < 0$ for $t \geq t_2$. Thus there exists a $t_3 \geq t_2$ such that $w(t) < 0$ for $t \geq t_3$ and $\lim_{t\to\infty} w(t) = -\infty$. We claim that $x(t)$ is bounded. If not, there exists a sequence $\{T_k\}_{k=1}^{\infty}$ such that $T_k \geq t_3$ for every k, $T_k \to \infty$ as $k \to \infty$, $w(T_k) \to \infty$ and $x(T_k) \to \infty$ as $k \to \infty$ and $\max_{t_3 \leq t \leq T_k} x(t) = x(T_k)$. Then we have

$$\begin{aligned} w(T_k) \;&=\; x(T_k) - \sum_{i=1}^{l} c_i(T_k) x(T_k - \tau_i) \\ &\quad - \sum_{i=1}^{n} \int_{t_0}^{T_k} \int_{s-\delta_i+\sigma_i}^{s} q_i(\theta) G(x(\theta - \sigma_i))\, d\theta\, ds - F(T_k) \\ &\geq\; x(T_k)[1 - \sum_{i=1}^{l} c_i(T_k) - \beta \sum_{i=1}^{n} \int_{t_0}^{T_k} \int_{s-\delta_i+\sigma_i}^{s} q_i(\theta)\, d\theta\, ds] - F(T_k). \end{aligned}$$

Letting $k \to \infty$, in view of (H_9), we obtain $w(T_k) \to \infty$ as $k \to \infty$, a contradiction. Hence our claim holds, that is, $x(t)$ is bounded. Consequently, $w(t)$ is bounded, a contradiction.

If $x(t) < 0$, the proof of the theorem may be treated similarly. The theorem is proved. ∎

Remark 1. Theorem 2 improves Theorem 3 due to Manojlovic et al. [4].

In the following, we give a stronger condition than (H_2) under which every solution of (1) oscillates when (H_7) holds.

Theorem 3. *Let $\sum_{i=1}^{l} c_i(t)$ be in the range (A_5). If $(H_1), (H_4), (H_7)$ and (H_9) and*

$$(H_{10}) \qquad \sum_{i=1}^{n} \{p_i(t) - q_i(t - \delta_i + \sigma_i)\} \geq b, \quad b \geq 0 \text{ is a constant}$$

hold, then every solution of (1) is oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1). Assume that $x(t) > 0$ and $x(t-\rho) > 0$ for $t \geq t_1 \geq t_0 + \rho > 0$. Then from (3), we have $w''(t) \leq 0$ for $t \geq t_1$ and hence $w'(t) > 0$ or < 0 for some $t \geq t_2 \geq t_1$.

If $w'(t) < 0$ for $t \geq t_2$, then $\lim_{t\to\infty} w(t) = -\infty$. Proceeding as in Theorem 2, one may show that $x(t)$ is bounded.Consequently, $w(t)$ is bounded, a contradiction.

Next, suppose that $w'(t) > 0$ for $t \geq t_2$. Then integrating (3) from t_2 to t, we obtain

$$\infty \ > \ w'(t_2) \ \geq \ b \int_{t_2}^{\infty} G(x(s-\delta_i))\, ds.$$

Therefore, $G(x(t)) \in L^1([t_2,\infty))$. Since $uG(u) > 0$ and G is nondecreasing, then $x(t) \in L^1([t_2,\infty))$. Hence

$$z(t) \ = \ x(t) - \sum_{i=1}^{l} c_i(t)x(t-\tau_i) \in L^1([t_2,\infty)).$$

Setting $\phi(t) = z(t) - F(t)$, we see that

$$\phi'(t) \ = \ w'(t) + \sum_{i=1}^{n} \int_{t-\delta_i+\sigma_i}^{t} q_i(s)G(x(s-\sigma_i))\, ds \ \geq \ 0. \tag{9}$$

Hence $\phi(t)$ is nondecreasing. Further, since(H_4) holds, then $\phi(t) > 0$ for large t. Hence

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} (z(t) - F(t)) = \lim_{t\to\infty} \phi(t) = \mu, \quad \mu \in (0,\infty).$$

Thus there exists a $t_3 \geq t_2$ and $0 < \epsilon < \mu$ such that $z(t) > \mu - \epsilon$ for $t \geq t_3$. Hence $z(t) \notin L^1([t_2,\infty))$, a contradiction. Hence $x(t) \not> 0$ for large t.

In a similar way one may show that $x(t) \not< 0$ for large t. This completes the proof of the theorem. ■

Corollary 1. *Suppose that all the conditions of Theorem 3 are satisfied except the condition (H_4). Then every solution of (1) is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the lines of Theorem 3, one may arrive at $x(t) \in L^1([t_2,\infty))$ and (9). Since $\phi(t)$ is nondereasing, then

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} (z(t) - F(t)) = \lim_{t\to\infty} \phi(t) = \mu, \quad \mu \in [0,\infty).$$

If $\mu > 0$, then we obtain a contradiction as in the proof of Theorem 3. If $\mu = 0$, then $x(t) < z(t)$ implies that $x(t) \to 0$ as $t \to \infty$. The proof is complete. ■

Proceeding as in the lines of Theorem 1, one may prove the following theorem.

Theorem 4. *Let $c_i(t), i = 1, \cdots, l$ be in the range (A_2) or (A_3). Further assume that $(H_1), (H_5)$ and (H_7) hold. If either (H_2) or (H_3) holds, then every solution of (1) is oscillatory or tends to zero as $t \to \infty$.*

Theorem 5. *Let $c_i(t)$, $i = 1, \cdots, l$ be in the range (A_4). Let (H_1), (H_7) and*

$$(H_{11}) \qquad \sum_{i=1}^{n} \int_{t_0}^{\infty} \int_{s-\delta_i+\sigma_i}^{s} q_i(\theta)\, d\theta\, ds \;<\; \infty$$

hold. If either (H_2) or (H_3) is satisfied, then every bounded solution of (1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Since $x(t)$ is bounded, then $\limsup_{t\to\infty} x(t) > 0$ implies that $w'(t) \to -\infty$ as $t \to \infty$ and hence $w(t) \to -\infty$ as $t \to \infty$. On the other hand, since $x(t)$ is bounded ,and (H_1) and (H_{11}) hold, then (2) yields that $x(t) \to \infty$ as $t \to \infty$, a contradiction. Thus the theorem is proved. ■

Remark 2. In the above results, the condition (H_7) forces us to assume (H_1). The above results remain true when G is linear or sublinear. The prototype of G satisfying the hypothesis of the above results is $G(u) = |u|^\gamma \text{sgn} u$, $\gamma \leq 1$.

3. Main results - the case when $\sigma_i \geq \delta_i$, $i = 1, \cdots, n$.

In the following, we shall replace the assumption (H_7) by (H_8). Hence the following results remains true for all types of G.

Theorem 6. *Let $c_i(t), i = 1, \cdots, l$ be in the range (A_2) or (A_3) or (A_5). Further, suppose that (H_2) and (H_8) hold. Then every solution of (1) is oscillatory or tends to zero as $t \to \infty$.*

Proof. let $x(t)$ be a eventually positive solution of (1). Then $w(t) > 0$ or < 0 for large t. If $w(t) < 0$ for large t, then $x(t) < F(t)$ for large t and hence $x(t) \to 0$ as $t \to \infty$. If $w(t) > 0$ for large t, then $w'(t) > 0$ for large t, say for $t \geq t_2$. Integration (3) from t_2 to t and using (H_2) and the nondecreasing property of G, we see that $x(t) \to 0$ as $t \to \infty$. The above line holds when $x(t) < 0$ for large t. The proof is complete. ■

Theorem 7. *Suppose that $c_i(t), i = 1, \cdots, l$ be in the range (A_1). If (H_2) and (H_8) hold, then every solution of (1) is oscillatory or tends to zero as $t \to \infty$.*

Proof. If $x(t)$ is an eventually positive solution of (1). Setting $w(t)$ as in (2), we obtain (3) for large t. Hence $w(t) > 0$ or < 0 for large t. If $w(t) > 0$

for large t, then $w'(t) > 0$ eventually. Then integration of (3) from t_1, t_1 large enough, to ∞, in view of (H_2) and the nondecreasing property of G, we see that $x(t) \to 0$ as $t \to \infty$. Let $w(t) < 0$ for large t. then

$$x(t) \ < \ F(t) + w(t) + \sum_{i=1}^{l} c_i(t)x(t-\tau_i). \tag{10}$$

If $\lim_{t\to\infty} w(t) = -\lambda$, $\lambda > 0$, then there exists a $\epsilon > 0$ such that for $0 < \epsilon < \lambda$, we obtain, for large t

$$\limsup_{t\to\infty} x(t) \ < \ -(\lambda - \epsilon) + c \limsup_{t\to\infty} x(t)$$

or,

$$(1-c) \limsup_{t\to\infty} x(t) \ < \ -(\lambda - \epsilon) < 0$$

a contradiction to the fact that $x(t) > 0$ eventually. If $\lim_{t\to\infty} w(t) = 0$, then taking lim sup both sides in (10) we have

$$\limsup_{t\to\infty} x(t) \ < \ c \limsup_{t\to\infty} x(t),$$

which ultimately yields that $x(t) \to 0$ as $t \to \infty$. The proof of the theorem is same if $x(t) < 0$ eventually. This completes the proof of the theorem. ■

Theorem 8. *Let $c_i(t)$, $i = 1, \cdots, l$ be in the range (A_4). If (H_2), (H_8) and*

$$\sum_{i=1}^{n} \int_{t_0}^{\infty} \int_{s}^{s-\delta_i+\sigma_i} q_i(\theta)\, d\theta\, ds \ < \ 1, \tag{H_{12}}$$

then every bounded solution of (1) is oscillatory or tend to zero as $t \to \infty$.

Proof. If $x(t) > 0$ for large t, and bounded, then (H_{12}) implies that $w(t)$ is bounded. If $\limsup_{t\to\infty} x(t) > 0$, then integration of (3) from t_2 to ∞, t_2 large enough, we have $w'(t) \to -\infty$ a contradiction to the boundedness of $w(t)$. Hence $x(t) \to 0$ as $t \to \infty$. The proof of the theorem may be treated similarly if we assume $x(t) < 0$ for large t. The proof is complete. ■

Remark 3. From the above results, it follows that when $G(u) = u$, that is for the linear case, the assumption $\alpha_i \geq \sigma_i$ or $\alpha_i \leq \sigma_i$ is not required though the authors have assumed (see [4], [5]). It would be interesting if one removes the restriction (H_1) on G for $\delta_i \geq \sigma_i$, $i = 1, \cdots, n$ (see Section 2).

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