F A S C I C U L I   M A T H E M A T I C I

Nr 38 2007

Songlin Yang

A NOTE ON SEQUENCE-COVERING IMAGES OF METRIC SPACES

Abstract. In this paper, we prove that every topological space is a sequence-covering image of a metric space, which answers a question on pseudo-sequence-covering images of metric spaces.

Key words: sequence-covering mapping, pseudo-sequence-covering mapping, sequentially-quotient mapping, network.

AMS Mathematics Subject Class: 54E35, 54E40, 54D80.

1. Introduction

Sequence-covering mappings, pseudo-sequence-covering mappings and sequentially-quotient mappings play an important role in study of images of metric spaces. It is well known that every sequence-covering mapping is pseudo-sequence-covering, and if the domain is metric, every pseudo-sequence-covering mapping is sequentially-quotient[4]. But none of these implications can be reversed. This leads us to investigate images of metric spaces under these mappings. In [10], S. Lin proved the following theorem [10, Corollary 1.3.9] (see [5, Corollary 3.3], for example).

Theorem 1. *A topological space is a pseudo-sequence-covering, s-image of a metric space iff it is a sequentially-quotient, s-image of a metric space.*

By viewing the above result, Y. Ge [5] raised the following question.

Question 1. *Can "s-" in Theorem 1 be omitted? More precisely, is every sequentially-quotient image of a metric space a pseudo-sequence-covering image of a metric space?*

In this paper, we answer the above question affirmatively. Throughout this paper, all topological spaces are assumed to be Hausdorff and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers, $\{x_n\}$ denotes a sequence, where the n-th term is x_n. For a sequence $L = \{x_n\}$, $f(L)$ denotes the sequence $\{f(x_n)\}$. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if

$\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$. Let $\mathcal{P}$ be a family of subsets of X and let $x \in X$. $\bigcup \mathcal{P}$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$ respectively. For a sequence $\{P_n : n \in \mathbb{N}\}$ of subsets of a space X, we abbreviate $\{P_n : n \in \mathbb{N}\}$ to $\{P_n\}$. A point $b = (\beta_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n).

Definition 1. *Let $f : X \longrightarrow Y$ be a mapping.*

(1) f is called a sequence-covering mapping [13] if for any convergent sequence S in Y there exists a convergent sequence L in X such that $f(L) = S$.

(2) f is called a pseudo-sequence-covering mapping [8], if for any convergent sequence S converging to y in Y, there exists a compact subset K of X such that $f(K) = S \bigcup \{y\}$.

(3) f is called a sequentially-quotient mapping [1], if for any convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

Remark 1. (1) "Pseudo-sequence-covering mapping" in Definition 1(2) was also called "sequence-covering mapping" by G.Gruenhage, E.Michael and Y.Tanaka in [7].

(2) Sequence-covering mapping $\Longrightarrow$ pseudo-sequence-covering mapping $\Longrightarrow$ (if the domain is metric) sequentially-quotient mapping [5, Remark 2.4(2)].

Definition 2. *Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a cover of a topological space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ is called a network of X [12], if for every $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.*

Lemma 1. *Let $f : X \longrightarrow Y$ be a mapping, and $\{y_n\}$ be a sequence converging to y in Y. If $\{B_k\}$ is a decreasing network at some $x \in f^{-1}(y)$ in X, and $\{y_n\}$ is eventually in $f(B_k)$ for every $k \in \mathbb{N}$, then there is a sequence $\{x_n\}$ converging to x such that $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$.*

Proof. Let $\{B_k\}$ be a decreasing network at some $x \in f^{-1}(y)$ in X, and let $\{y_n\}$ be eventually in $f(B_k)$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $y_n \in f(B_k)$ for every $n > n_k$, so $f^{-1}(y_n) \bigcap B_k \neq \emptyset$ for every $n > n_k$. Without loss of generality, we can assume $1 < n_k < n_{k+1}$ for each $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, pick

$$x_n \in \begin{cases} f^{-1}(y_n)\,, & n < n_1 \\ & \\ f^{-1}(y_n) \bigcap B_k\,, & n_k \le n < n_{k+1}, \end{cases}$$

then $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$. It suffices to prove that $\{x_n\}$ converges to x.

Let U be an open neighborhood of x. There exists $k \in \mathbb{N}$ such that $x \in B_k \subset U$. For each $n > n_k$, there exists $k' \geq k$ such that $n_{k'} \leq n < n_{k'+1}$, so $x_n \in B_{k'} \subset B_k \subset U$. This proves that $\{x_n\}$ converges to x. ■

Now we give the main theorem of this paper, which gives an affirmative answer for Question 1.

Theorem 2. *The following are equivalent.*
(1) X is a sequence-covering image of a metric space.
(2) X is a pseudo-sequence-covering image of a metric space.
(3) X is a sequentially-quotient image of a metric space.
(4) X is a topological space.

Proof. It is clear that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). We only need to prove that (4) $\Longrightarrow$ (1).

Let X be a topological space. For every $x \in X$ and every sequence $S = \{x_n\}$ converging to x, put $P_{S,i} = \{x_n : n > i\} \bigcup \{x\}$ for every $i \in \mathbb{N}$ and $\mathcal{P}_S = \{P_{S,i} : i \in \mathbb{N}\}$. Put $\mathcal{P}_x = \bigcup\{\mathcal{P}_S : S$ is a sequence converging to $x\}$ and $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$. It is clear that $\{x\} \in \mathcal{P}$ for every $x \in X$. We construct a metric space as follows. Let $\mathcal{P} = \{P_\beta : \beta \in \Lambda\}$. For every $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endow Λ_n a discrete topology. Put $M = \{b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Lambda_n : \{P_{\beta_n}\}$ is a network at some point x_b in $X\}$. It suffices to prove the following four facts.

Fact 1. M is a metric space:

In fact, Λ_n, as a discrete space, is a metric space for every $n \in \mathbb{N}$. So M, which is a subspace of the Tychonoff-product space $\Pi_{n\in\mathbb{N}}\Lambda_n$, is a metric space.

Fact 2. Let $b = (\beta_n) \in M$. Then there exists unique x_b such that $\{P_{\beta_n}\}$ is a network at x_b in X:

The existence comes from the construction of M, we only need to prove the uniqueness. Let $\{P_{\beta_n}\}$ be a network at both x_b and x'_b in X, then $\{x_b, x'_b\} \subset P_{\beta_n}$ for every $n \in \mathbb{N}$. If $x_b \neq x'_b$, then there exists an open neighborhood U of x_b such that $x'_b \notin U$. Because $\{P_{\beta_n}\}$ is a network at x_b in X, there exists $n \in \mathbb{N}$ such that $x_b \in P_{\beta_n} \subset U$, thus $x'_b \notin P_{\beta_n}$, a contradiction. This proves the uniqueness.

By Fact 2, for every $b = (\beta_n) \in M$, there exists unique x_b such that $\{P_{\beta_n}\}$ is a network at x_b in X. Define $f(b) = x_b$. Thus we construct a correspondence $f : M \longrightarrow X$.

Fact 3. f is continuous and onto, so f is a mapping:

Firstly, for every $x \in X$, $\{x\} \in \mathcal{P} = \{P_\beta : \beta \in \Lambda\}$, so for every $n \in \mathbb{N}$, there exists $\beta_n \in \Lambda_n$ such that $\{x\} = P_{\beta_n}$. Thus $\{P_{\beta_n}\}$ is a network at x in X. Put $b = (\beta_n)$, then $b \in M$ and $f(b) = x$. So f is onto. Secondly, let $b = (\beta_n) \in M$ and let $f(b) = x_b$. If U is an open neighborhood of x, then there exists $k \in \mathbb{N}$ such that $x_b \in P_{\beta_k} \subset U$ because $\{P_{\beta_n}\}$ is a network at x_b in X. Put $V = \{c = (\gamma_n) \in M : \gamma_k = \beta_k\}$, then U is an open neighborhood of b. It is easy to see that $f(V) \subset P_{\beta_k} \subset U$. So f is continuous.

Fact 4. f is sequence-covering:

Let $S = \{x_n\}$ be a sequence converging to x in X. It is clear that $\{x_n\}$ is eventually in $P_{S,i}$ for every $i \in \mathbb{N}$, and so $\{x_n\}$ is eventually in $\bigcap_{i \leq k} P_{S,i}$ for every $k \in \mathbb{N}$. For every $i \in \mathbb{N}$, since $P_{S,i} \in \mathcal{P}$, there exists $\beta_i \in \Lambda_i$ such that $P_{S,i} = P_{\beta_i}$. It is clear that $\{P_{\beta_i}\}$ is a network at x in X. Put $b = (\beta_i)$, then $b \in f^{-1}(x)$. For every $k \in \mathbb{N}$, put $B_k = \{(\gamma_i) \in M : \gamma_i = \beta_i \text{ for } i \leq k\}$. Then $\{B_k\}$ is a decreasing neighborhood base at b in M. It is not difficulty to prove that $f(B_k) = \bigcap_{i \leq k} P_{\beta_i}$. In fact, let $c = (\gamma_i) \in B_k$, then $\{P_{\gamma_i}\}$ is a network at $f(c)$ in X. So $f(c) \in \bigcap_{i \in \mathbb{N}} P_{\gamma_i} \subset \bigcap_{i \leq k} P_{\gamma_i} = \bigcap_{i \leq k} P_{\beta_i}$, Thus $f(B_k) \subset \bigcap_{i \leq k} P_{\beta_i}$. On the other hand, let $y \in \bigcap_{i \leq k} P_{\beta_i}$. By Fact 3, there exists $c' = (\gamma'_i) \in M$ such that $f(c') = y$, so $\{P_{\gamma'_i}\}$ is a network at y in X. For every $i \in \mathbb{N}$, put $\gamma_i = \beta_i$ if $i \leq k$, and $\gamma_i = \gamma'_{i-k}$ if $i > k$. Put $c = (\gamma_i)$. It is easy to see that $c \in B_k$. Note that $\{P_{\gamma_i}\}$ is a network at y in X, so $y = f(c) \in f(B_k)$. Thus $\bigcap_{i \leq k} P_{\beta_i} \subset f(B_k)$. So $f(B_k) = \bigcap_{i \leq k} P_{\beta_i}$. Because $\{x_n\}$ is eventually in $\bigcap_{i \leq k} P_{\beta_i} = f(B_k)$ for every $k \in \mathbb{N}$, by Lemma 1, there exists a sequence $\{b_n\}$ converging to b in M such that $b_n \in f^{-1}(x_n)$ for every $n \in \mathbb{N}$. This proves that f is sequence-covering.

By the above Fact 1, Fact 3 and Fact 4, X is a sequence-covering image of a metric space. ■

Remark 2. A mapping $f : X \longrightarrow Y$ is 1-sequence-covering ([11]) if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$. It is clear that 1-sequence-covering mapping $\Longrightarrow$ sequence-covering mapping. Note that a topological space need not to be a 1-sequence-covering image of a metric space (see [10, Theorem 2.4.11], for example). So "sequence-covering" in Theorem 2(1) can not be replaced by "1-sequence-covering".

References

[1] Boone J.R., Siwiec F., Sequentially quotient mappings, *Czech. Math. J.,* 26(1976), 174-182.

[2] Engelking R., *General Topology,* Sigma Series in Pure Mathematics 6, (Heldermann, Berlin, revised ed.) 1989.

[3] Ge Y., Characterizations of sn-metrizable spaces, *Publ. Inst. Math. Nouv. Ser.,* 74(88)(2003), 121-128.

[4] Ge Y., On pseudo-sequence coverings π-images of metric spaces, *Matematicki Vesnik,* 57(2005), 113-120.

[5] Ge Y., Mappings in Ponomarev-systems, *Topology Proceedings,* 29(2005), 141-153.

[6] GRUENHAGE G., *Generalized metric spaces,* In: K.Kumen and J.E.Vaughan eds, Handbook of Set-Theoretic Topology, Amsterdan: North-Holland, 423-501, 1984.
[7] GRUENHAGE G., MICHAEL E., TANAKA Y., Spaces determined by point-countable covers, *Pacific J. Math.,* 113(1984), 303-332.
[8] IKEDA Y., LIU C., TANAKA Y., Quotient compact images of metric spaces and related matters, *Topology Appl.,* 122(2002), 237-252.
[9] LIN S., A note on the Arens' spaces and sequential fan, *Topology Appl.,* 81(1997), 185-196.
[10] LIN S., *Point-Countable Covers and Sequence-Covering Mappings,* Chinese Science Press, (Chinese) Beijing, 2002.
[11] LIN S., YAN P., Sequence-covering maps of metric spaces, *Topology Appl.,* 109(2001), 301-314.
[12] MICHAEL E., $\aleph_0$-spaces, *J. Math. Mech.,* 15(1966), 983-1002.
[13] SIWIEC F., Sequence-covering and countably bi-quotient mappings, *General Topology Appl.,* 1(1971), 143-154.

SONGLIN YANG
DEPARTMENT OF MATHEMATICS, SUZHOU UNIVERSITY
SUZHOU, 215006, P.R.CHINA
e-mail: songlin@public1.sz.js.cn

Received on 02.07.2006 and, in revised form, on 09.02.2007.