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STRONG MAXIMUM PRINCIPLES FOR INFINITE SYSTEMS OF PARABOLIC DIFFERENTIAL-FUNCTIONAL INEQUALITIES WITH NONSTANDARD INITIAL INEQUALITIES

$$(u^j(x,t_0) - K^j) + \sum_i h_i(x)(u^j(x,T_i) - K^j) \le 0, \quad j \in N$$

Abstract. The aim of this paper is to present strong maximum principles for infinite systems of parabolic differential-functional inequalities with nonstandard initial inequalities with sums in relatively arbitrary (n+1)-dimensional time-space sets more general than the cylindrical domain.

Key words: infinite systems, nonstandard initial inequalities, strong maximum principle.

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1. Introduction

We shall consider an infinite system of parabolic type differential-functional inequalities of the following form

$$u^i_t(x,t) \le F_i\left(x,t,u^i(x,t), u^i_x(x,t), u^i_{xx}(x,t), u\right) \quad (i \in \mathbf{N}), \tag{1}$$

where $x = (x_1, ..., x_n)$, $(x,t) \in D$ and $D \subset \mathbf{R}^n \times (t_0, t_0+T]$.

The symbol u denotes the mapping

$$u : \mathbf{N} \times \widetilde{D} \ni (i,x,t) \longrightarrow u^i(x,t) \in \mathbf{R},$$

where $\widetilde{D}$ is an arbitrary set such that

$$\overline{D} \subset \widetilde{D} \subset \mathbf{R}^n \times (-\infty, t_0+T].$$

The right-hand sides F_i $(i \in \mathbf{N})$ of system (1) are functionals of u, $u^i_x(x,t) = grad_x u^i(x,t)$ and $u^i_{xx}(x,t)$ denote the matrices of second order derivatives with respect to x of $u^i(x,t)$ $(i \in \mathbf{N})$.

In this paper we give theorems on strong maximum principles for problems with inequalities (1) and with the nonstandard inequalities

$$\left(u^j\left(x,t_0\right)-K^j\right)+\sum_{i\in I_*}h_i\left(x\right)\left(u^j\left(x,T_i\right)-K^j\right)\le 0 \text{ for } x\in S_{t_0}\ (j\in\mathbf{N}),$$

where K^i $(i\in\mathbf{N})$ are constant functions such that $(K^1,K^2,...)\in l^\infty$ and $S_{t_0}=int\{x\in\mathbf{R}^n:(x,t_0)\in\overline{D}\}$.

The results obtained in this paper are generalization of some thesis from publications: J. Chabrowski [4] and L. Byszewski [2].

2. Preliminaries

We shall use the following notations: $\mathbf{R}=(-\infty,+\infty)$, $\mathbf{R}_-=(-\infty,0]$, $\mathbf{N}=\{1,2,...\}$, $x=(x_1,...,x_n)\in\mathbf{R}^n$ $(n\in\mathbf{N})$.

By l^∞ we denote the Banach space of real sequences $\xi=(\xi^1,\xi^2,...)$ such that $\sup\{\left|\xi^j\right|:j=1,2,...\}<\infty$ and $\|\xi\|_{l^\infty}=\sup\{\left|\xi^j\right|:j=1,2,...\}$. For $\xi=(\xi^1,\xi^2,...)$, $\eta=(\eta^1,\eta^2,...)\in l^\infty$ we write $\xi\le\eta$ in the sense $\xi^i\le\eta^i$ $(i\in\mathbf{N})$.

By $M_{n\times n}(\mathbf{R})$ we denote the space of real square symmetric matrices $r=[r_{jk}]_{n\times n}$.

We write $r\ge 0$ if

$$\sum_{j,k=1}^{n}r_{jk}\lambda_j\lambda_k\ge 0$$

for all $(\lambda_1,...,\lambda_n)\in\mathbf{R}^n$.

Let t_0 be an arbitrary real finite number and let $T\in(0,\infty)$.

A set $D\subset\{(x,t):x\in\mathbf{R}^n,t_0<t\le t_0+T\}$ is called *a set of type (P)* if:

(a) The projection of the interior of set D on the t-axis is the interval (t_0,t_0+T).

(b) For every $\left(\widetilde{x},\widetilde{t}\right)\in D$ there exists a positive number $r=r\left(\widetilde{x},\widetilde{t}\right)$ such that

$$\left\{(x,t):\sum_{i=1}^{n}(x_i-\widetilde{x}_i)^2+\left(t-\widetilde{t}\right)^2<r,\quad t<\widetilde{t}\right\}\subset D.$$

(c) All the boundary points $\left(\widetilde{x},\widetilde{t}\right)$ of D for which there is a positive number $r=r\left(\widetilde{x},\widetilde{t}\right)$ such that

$$\left\{(x,t):\sum_{i=1}^{n}(x_i-\widetilde{x}_i)^2+\left(t-\widetilde{t}\right)^2<r,\quad t\le\widetilde{t}\right\}\subset D$$

belong to D.

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} \operatorname{int}\{x \in \mathbf{R}^n : (x, t_0) \in \overline{D}\} & \text{for} \quad t = t_0, \\ \{x \in \mathbf{R}^n : (x, t) \in D\} & \text{for} \quad t \neq t_0, \end{cases}$$

$$\sigma_t = \begin{cases} \operatorname{int}[\overline{D} \cap (\mathbf{R}^n \times \{t_0\})] & \text{for} \quad t = t_0, \\ D \cap (\mathbf{R}^n \times \{t\}) & \text{for} \quad t \neq t_0, \end{cases}$$

Let $\widetilde{D}$ be an arbitrary set such that

$$\overline{D} \subset \widetilde{D} \subset \mathbf{R}^n \times (-\infty, t_0 + T].$$

We introduce the following sets:

$$\partial_p D := \widetilde{D} \backslash D \quad \text{and} \quad \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\widetilde{x}, \widetilde{t}) \in D$, we denote by $S^- (\widetilde{x}, \widetilde{t})$ the set of points $(x, t) \in D$, that can be joined to $(\widetilde{x}, \widetilde{t})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (x, t) to $(\widetilde{x}, \widetilde{t})$.

Let $Z_\infty(\widetilde{D})$ denote the linear space of mappings

$$w : \mathbf{N} \times \widetilde{D} \ni (i, x, t) \longrightarrow w^i (x, t) \in \mathbf{R},$$

where functions

$$w^i : \widetilde{D} \ni (x, t) \longrightarrow w^i (x, t) \in \mathbf{R}$$

are continuous in $\overline{D}$ and

$$\sup\{\left|w^i (x, t)\right| : (x, t) \in \widetilde{D},\ i \in \mathbf{N}\} < \infty.$$

In the set of mappings w belonging to $Z_\infty(\widetilde{D})$ we define the functional $[\cdot]_{t,\infty}$ by the formula

$$[w]_{t,\infty} = \sup\{0, w^i (x, \widetilde{t}) : (x, \widetilde{t}) \in \widetilde{D},\ \widetilde{t} \leq t,\ i \in \mathbf{N}\},$$

where $t \leq t_0 + T$.

By $Z^{2,1}_\infty(\widetilde{D})$ we denote the linear subspace of $Z_\infty(\widetilde{D})$. A mapping w belongs to $Z^{2,1}_\infty(\widetilde{D})$ if w^i_t, $w^i_x = (w^i_{x_1}, ..., w^i_{x_n})$, $w^i_{xx} = [w^i_{x_j x_k}]_{n \times n}$ $(i \in \mathbf{N})$ are continuous in D.

For each $i \in \mathbf{N}$ by F_i we denote the mapping

$$F_i : D \times \mathbf{R} \times \mathbf{R}^n \times M_{n \times n} (\mathbf{R}) \times Z_\infty(\widetilde{D}) \ni (x, t, z, q, r, w) \\ \longrightarrow F_i (x, t, z, q, r, w) \in \mathbf{R} \quad (i \in \mathbf{N}),$$

where $q = (q_1, ..., q_n)$ and $r = [r_{jk}]$.

By P_i $(i \in \mathbf{N})$ we denote an operator given by the formula

$$(P_i w)(x,t) := w_t^i(x,t) - F_i\left(x,t,w^i(x,t), w_x^i(x,t), w_{xx}^i(x,t), w\right) \quad (i \in \mathbf{N}),$$

for $w \in Z_\infty^{2,1}(\widetilde{D})$ and $(x,t) \in D$.

A function $u \in Z_\infty^{2,1}(\widetilde{D})$ is called *a solution of the system of the functional-differential inequalities*

$$(P_i u)(x,t) \underset{(\geq)}{\leq} 0 \qquad (i \in \mathbf{N})$$

in D, if they satisfy the system for all $(x,t) \in D$.

For a given subset $E \subset D$, and a given mapping $w \in Z_\infty^{2,1}(\widetilde{D})$ and a fixed index $i \in \mathbf{N}$ the function F_i is called *uniformly parabolic with respect to* w *in* E, if there is a constant $\varkappa > 0$ (depending on E) such that for any two matrices $r = [r_{jk}] \in M_{n\times n}(\mathbf{R})$, $\widetilde{r} = [\widetilde{r}_{jk}] \in M_{n\times n}(\mathbf{R})$ and for $(x,t) \in E$ we have

$$(2) \qquad r \leq \widetilde{r} \Longrightarrow F_i\left(x,t,w^i(x,t), w_x^i(x,t), \widetilde{r}, w\right) - F_i\left(x,t,w^i(x,t), w_x^i(x,t), r, w\right) \geq \varkappa \sum_{j=1}^{n} (\widetilde{r}_{jj} - r_{jj}).$$

If (2) is satisfied for $\varkappa = 0$ and $r = w_{xx}^i(x,t)$, where $(x,t) \in E$, and for $\widetilde{r} = w_{xx}^i(x,t) + \widehat{r}$, where $(x,t) \in E$ and $\widehat{r} \geq 0$, then F_i is called *parabolic with respect to* w in E.

Let $I = \mathbf{N}$ or I is a finite set of mutually different natural numbers.

Let us define the following set:

$$Z = \bigcup_{i\in I} \sigma_{T_i},$$

where, in case if $I = \mathbf{N}$, the following conditions are satisfied:

(i) $t_0 < T_i \leq t_0 + T$ for $i \in I$ and $T_i \neq T_j$ for $i, j \in I, i \neq j$;

(ii) $T_0 := \inf\limits_{i\in I} T_i > t_0$;

(iii) $S_{t_0} \subset S_{T_i}$ for $i \in I$;

(iv) $S_{t_0} \subset S_t$ for every $t \in [T_0, t_0 + T]$,

and in case if I is a finite set of mutually different natural numbers, the conditions (i), (iii) are satisfied.

An unbounded set D of type (P) is called *a set of type* $(P_{Z\Gamma})$, if:

(a) $Z \neq \emptyset$,

(b) $\Gamma \cap \overline{\sigma}_{t_0} \neq \emptyset$.

Let Z_* denote a non-empty subset of Z. We define the following set:

$$I_* = \{i \in I : \sigma_{T_i} \subset Z_*\}.$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type $(P_{Z\Gamma})$ is called *a set of type* (P_{ZB}).

Observe,that if D is a set of type (P_{ZB}), then D satisfies condition (b) of definition of a set of type $(P_{Z\Gamma})$. Observe also, that if D_0 is an arbitrary bounded subset of $\mathbf{R}^n$, then $D = D_0 \times (t_0, t_0 + T]$ is a set of type (P_{ZB}). In the case, if D_0 is an arbitrary unbounded proper subset of $\mathbf{R}^n$, then $D = D_0 \times (t_0, t_0 + T]$ is a set of type $(P_{Z\Gamma})$.

For every set $A \subset \widetilde{D}$ and for each function $w \in Z_\infty(\widetilde{D})$ we apply the notation:

$$\max_{(x,t)\in A} w(x,t) := \left(\max_{(x,t)\in A} w^1(x,t), \max_{(x,t)\in A} w^2(x,t), ... \right).$$

3. Strong maximum principles with nonstandard inequalities with sums in sets of types $(P_{Z\Gamma})$ and (P_{ZB})

Theorem 1. *Assume that:*

(a) $D \subset \mathbf{R}^n \times (t_0, t_0 + T]$ *is a set of type* $(P_{Z\Gamma})$ *or* (P_{ZB}).

(b) F_i $(i \in \mathbf{N})$ *are the mappings as in Section 2 and there exist constant* $L > 0$ *such that*

$$F_i(x,t,z,q,r,w) - F_i(x,t,\widetilde{z},\widetilde{q},\widetilde{r},\widetilde{w}) \\ \le L\left(|z-\widetilde{z}| + |x| \sum_{j=1}^{n} |q_j - \widetilde{q}_j| + |x|^2 \sum_{j,k=1}^{n} |r_{jk} - \widetilde{r}_{jk}| + [w-\widetilde{w}]_{t,\infty} \right),$$

$i \in \mathbf{N}$ *for all* $(x,t) \in D$, $z,\widetilde{z} \in \mathbf{R}$, $q,\widetilde{q} \in \mathbf{R}^n$, $r,\widetilde{r} \in M_{n\times n}(\mathbf{R})$, $w,\widetilde{w} \in Z_\infty(\widetilde{D})$.

(c) $u \in Z^{2,1}_\infty(\widetilde{D})$ *and the maximum of function* u *on* Γ *is attained. Moreover,*

$$K^i := \max_{(x,t)\in\Gamma} u^i(x,t) \quad (i \in \mathbf{N}), \tag{3}$$

and $K \in l^\infty$ *is defined by formulae*

$$K : \mathbf{N} \times \widetilde{D} \ni (i,x,t) \longrightarrow K^i.$$

(d) The following inequalities hold

$$\left(u^j(x,t_0) - K^j\right) + \sum_{i\in I_*} h_i(x)\left(u^j(x,T_i) - K^j\right) \le 0 \; for \; x \in S_{t_0} \; (j \in \mathbf{N}), \tag{4}$$

where $h_i : S_{t_0} \longrightarrow R_-$ $(i \in I_*)$ *are given functions such that*

$$-1 \leq \sum_{i \in I_*} h_i(x) \leq 0 \qquad for \quad x \in S_{t_0}$$

and, additionally, if $card I_* = \aleph_0$*, then the series* $\sum_{i \in I_*} h_i(x)\, u^j(x, T_i)$ $(j \in N)$ *are convergent for* $x \in S_{t_0}$*.*

(e) *There exists a point* $(x^*, t^*) \in \widetilde{D}$ *such that*

$$u(x^*, t^*) = \max_{(x,t) \in \widetilde{D}} u(x,t).$$

Moreover,

$$M^i := u^i(x^*, t^*) \quad (i \in \mathbf{N}) \tag{5}$$

and $M \in l^\infty$ *is defined by*

$$M : \mathbf{N} \times \widetilde{D} \ni (i, x, t) \longrightarrow M^i.$$

(f) $F_i\left(x, t, M^i, 0, 0, M\right) \leq 0$ *for* $(x,t) \in D$ $(i \in \mathbf{N})$.

(g) *The function* u *is a solution of system*

$$(P_i u)(x,t) \leq 0 \qquad for \quad (x,t) \in D \;\; (i \in \mathbf{N}).$$

(h) *The mappings* F_i $(i \in \mathbf{N})$ *are parabolic with respect to* $u \in D$ *and uniformly parabolic with respect to* M *in any compact subset of* D*.*

Then

$$\max_{(x,t) \in \widetilde{D}} u(x,t) = \max_{(x,t) \in \Gamma} u(x,t). \tag{6}$$

Moreover, if there is a point $\left(\widetilde{x}, \widetilde{t}\right) \in D$ *such that*

$$u\left(\widetilde{x}, \widetilde{t}\right) = \max_{(x,t) \in \widetilde{D}} u(x,t),$$

then

$$u(x,t) = \max_{(x,t) \in \Gamma} u(x,t) \quad for \quad (x,t) \in S^-\left(\widetilde{x}, \widetilde{t}\right).$$

Proof. We shall prove Theorem 1 for a set of type $(P_{Z\Gamma})$. The proof for a set of type (P_{ZB}) is analogous.

It is obvious that a set of type $(P_{Z\Gamma})$ is a set of type (P_Γ) from [3], hence,in the case where $\sum_{i \in I_*} h_i(x) = 0$ for $x \in S_{t_0}$, Theorem 1 is a consequence of

Theorem 4.1 from [3]. Therefore, we shall give the proof of Theorem 1 only in the case where

$$-1 \leq \sum_{i \in I_*} h_i(x) < 0 \qquad for \quad x \in S_{t_0}. \tag{7}$$

We shall argue by contradiction. Suppose that (7) is satisfied and suppose that

$$M \neq K. \tag{8}$$

Then, from (3) and (5) we obtain

$$K^i \leq M^i \quad (i \in \mathbf{N}). \tag{9}$$

Consequently, (8) and (9) implies:

$$\text{There is } \ l \in \mathbf{N} \ \text{ such that } \ K^l < M^l. \tag{10}$$

It is easy to see, by assumption (e), that:

$$\begin{gathered}\text{There is a point } (x^*, t^*) \in \widetilde{D} \text{ such that} \\ u(x^*, t^*) = M := \max_{(x,t) \in \widetilde{D}} u(x,t).\end{gathered} \tag{11}$$

By (11), by assumption (c) and by (10) we have

$$(x^*, t^*) \in \widetilde{D} \setminus \Gamma = D \cup \sigma_{t_0}. \tag{12}$$

An argument analogous to the proof of Theorem 4.1 from [3] yields

$$(x^*, t^*) \notin D.$$

Hence

$$(x^*, t^*) \in \sigma_{t_0}. \tag{13}$$

On the other hand, by the definition of sets I and I_*, we must consider the following cases:

(A) I_* is a finite set, i.e., without loss generality there is a number $p \in \mathbf{N}$ such that $I_* = \{1, ..., p\}$.

(B) $card I_* = \aleph_0$.

First we shall consider case (A). By (4) and by the inequalities

$$u(x^*, T_i) < u(x^*, t_0) \quad (i = 1, ..., p),$$

which are consequences of (11) and (13) and of conditions (a)(i), (a)(iii) of the definition of a set of type $(P_{Z\Gamma})$, we have

$$0 \geq \left(u^j(x^*,t_0)-K^j\right)+\sum_{i=1}^{p} h_i(x^*)\left(u^j(x^*,T_i)-K^j\right) \\ \geq \left(u^j(x^*,t_0)-K^j\right)(1+\sum_{i=1}^{p} h_i(x^*)) \qquad (j\in\mathbf{N}).$$

From the last inequality we have

$$u(x^*,t_0)\leq K, \quad \text{if} \quad 1+\sum_{i=1}^{p} h_i(x^*)>0. \tag{14}$$

Hence, from (10) and (13), we obtain a contradiction of (14) with (11). Assume now that

$$\sum_{i=1}^{p} h_i(x^*)=-1.$$

We observe that for every $j\in\mathbf{N}$ there is a number $l_j\in\{1,...,p\}$ such that

$$u^j\left(x^*,T_{l_j}\right)=\max_{i=1,\dots,p} u^j(x^*,T_i).$$

Hence, by (4), we have

$$u^j(x^*,t_0)-u^j\left(x^*,T_{l_j}\right)=\left(u^j(x^*,t_0)-K^j\right)-\left(u^j\left(x^*,T_{l_j}\right)-K^j\right) \\ =\left(u^j(x^*,t_0)-K^j\right)+\sum_{i=1}^{p} h_i(x^*)\left(u^j\left(x^*,T_{l_j}\right)-K^j\right) \\ \leq\left(u^j(x^*,t_0)-K^j\right)+\sum_{i=1}^{p} h_i(x^*)\left(u^j(x^*,T_i)-K^j\right) \;\leq\; 0 \quad (j\in\mathbf{N}).$$

Consequently

$$u^j(x^*,t_0)\leq u^j\left(x^*,T_{l_j}\right) \quad (j\in\mathbf{N}), \quad \text{if} \quad \sum_{i=1}^{p} h_i(x^*)=-1. \tag{15}$$

Since, condition (a)(i) of the definition of a set type $(P_{Z\Gamma})$ implies inequalities $T_{l_j}>t_0$ $(j\in\mathbf{N})$, then from (13) we see that (15) contradicts (11). This completes the proof of formulae (6) in case (A).

It remains to investigate case (B). Analogously to the proof of (6) in case (A), by assumption (d) and by the inequalities

$$u(x^*,T_i)<u(x^*,t_0) \qquad (i\in I_*),$$

we have

$$\begin{aligned}0 &\geq \left(u^j\left(x^*, t_0\right)-K^j\right)+\sum_{i \in I_*} h_i\left(x^*\right)\left(u^j\left(x^*, T_i\right)-K^j\right) \\ &\geq \left(u^j\left(x^*, t_0\right)-K^j\right)\left(1+\sum_{i \in I_*} h_i\left(x^*\right)\right) \qquad (j \in \mathbf{N}).\end{aligned}$$

It imply that

$$u\left(x^*, t_0\right) \leq K, \qquad \text{if} \qquad 1+\sum_{i \in I_*} h_i\left(x^*\right)>0. \tag{16}$$

From the last inequality, by (10) and by (13), we obtain a contradiction of (16) with (11). Assume now that $\sum\limits_{i \in I_*} h_i\left(x^*\right)=-1$ and let

$$T_0^*:=\inf_{i \in I_*} T_i.$$

Since $u^i \in C(\overline{D})$ $(i \in \mathbf{N})$ and since, by (a)(iv) of the definition of a set of type $(P_{Z\Gamma})$, $x^* \in S_t$ for every $t \in [T_0, t_0+T]$, if $card I=\aleph_0$, it follows that for every $j \in \mathbf{N}$ there is $\widehat{t}_j \in [T_0^*, t_0+T]$ such that

$$u^j\left(x^*, \widehat{t}_j\right) = \max_{t \in [T_0^*, t_0+T]} u^j\left(x^*, t\right).$$

Consequently, by assumption (d), we obtain

$$\begin{aligned}u^j\left(x^*, t_0\right)-u^j\left(x^*, \widehat{t}_j\right) &= \left(u^j\left(x^*, t_0\right)-K^j\right)-\left(u^j\left(x^*, \widehat{t}_j\right)-K^j\right) \\ &= \left(u^j\left(x^*, t_0\right)-K^j\right)+\sum_{i \in I_*} h_i\left(x^*\right)\left(u^j\left(x^*, \widehat{t}_j\right)-K^j\right) \\ &\leq \left(u^j\left(x^*, t_0\right)-K^j\right)+\sum_{i \in I_*} h_i\left(x^*\right)\left(u^j\left(x^*, T_i\right)-K^j\right) \leq 0 \quad (j \in \mathbf{N}).\end{aligned}$$

Hence

$$u^j\left(x^*, t_0\right) \leq u^j\left(x^*, \widehat{t}_j\right) \quad (j \in \mathbf{N}), \quad \text{if} \quad \sum_{i \in I_*} h_i\left(x^*\right)=-1. \tag{17}$$

Since, condition (a)(ii) of the definition of a set of type $(P_{Z\Gamma})$ implies inequalities $\widehat{t}_j>t_0$ $(j \in \mathbf{N})$, hence, we see from (13) that (17) contradicts (11). This completes the proof of formulae (6).

The second part of Theorem 1 is a consequence of (6) and Lemma 3.1 from [3]. Therefore, the proof of Theorem 1 is complete. ■

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