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Yuji Liu

ANTI-PERIODIC BOUNDARY VALUE PROBLEMS FOR NONLINEAR IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS*

Abstract. This paper is concerned with the anti-periodic boundary value problems for nonlinear impulsive functional differential equations

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t))), & a.e.\ t \in [0, T], \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, \cdots, m, \\ x(0) = -x(T). \end{cases}$$

The sufficient conditions for the existence of at least one solution to above problem are established. The results generalize and improve the known ones. Examples are presented to illustrate the main results.

Key words: Anti-Periodic boundary value problem; impulsive differential equation; fixed-point theorem; growth condition.

AMS Mathematics Subject Classification: 34B10, 34B15.

1. Introduction

In paper [1], Luo, Shen and Neito studied the anti-periodic problem of impulsive differential equation

$$\begin{cases} x'(t) = f(t, x(t)), \ \ t \in [0, T], \ \ t \neq t_k, \ \ k = 1, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, \cdots, m, \\ x(0) = -x(T). \end{cases}$$

Following results are obtained.

Theorem A. *Suppose* $\lambda > 0$. *Assume that there are a function* $\psi : [0, +\infty) \to (0, +\infty)$ *and a positive function* $\rho \in L^1([0, T])$ *with*

$$|f(t, x) + \lambda x| \ \leq \ \rho(t)\psi(|x|),$$

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and there exist $b_k \geq 0$ such that

$$|I_k(x)| \ \leq \ b_k|x| \quad \text{and} \quad \sum_{k=1}^{m} b_k \ < \ 1 + e^{-\lambda T}. \tag{1}$$

Furthermore, suppose

$$\sup_{c\geq 0} \frac{c}{\psi(c)} \ > \ \frac{||\rho||_{L^1}}{1 + e^{-\lambda T} - \sum_{k=1}^{m} b_k}. \tag{2}$$

Then $(*)$ has at least one solution.

Under the assumptions

$$f(t,u) - f(t,v) \ \geq \ -\lambda(u-v) + M(u-v) \tag{3}$$

and that there are a pair of coupled lower and upper solutions for $(*)$, and I_k are nondecreasing, and other assumptions, the existence result was also proved by authors in [1] using lower and upper solutions methods and monotone iterative technique.

We note that equation (2) or (3) implies that $f(t,x)$ is at most linear in x, (1) implies that I_k is a t most linear. So the problem have not been solved when $f(t,x)$ is super-linear in x. Furthermore, there no paper concerned with the solvability of anti-periodic problems for functional differential equations with nonlinear impulses effects which are not nondecreasing or do not satisfy $|I_k(x)| \leq b_k|x|$.

There exist other papers concerned with the solvability of anti-periodic boundary value problems for first order differential equations with impulses effects, see [8, 9, 10, 13, 14, 15] and the references therein, or for higher order differential equations, we may see [2-7, 11] and the references cited there, but the methods used are lower and upper solutions methods and monotone iterative technique. It seems that there is no paper discussed the solvability of anti-periodic boundary value problems for first order functional differential equations with impulses effects.

In this paper, we are concerned with the existence of solutions of the anti-periodic boundary value problems for nonlinear impulsive functional differential equations

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t))), \\ \qquad\qquad\qquad\qquad t \in [0,T], \quad t \neq t_k, \quad k = 1, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, \cdots, m, \\ x(0) = -x(T). \end{cases} \tag{4}$$

where $T > 0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ are constants, $\alpha_k \in C^1([0,T],[0,T])$ for all $k = 1, \cdots, n$, the inverse function of α_i is denoted by $\beta_i (i = 1, \cdots, n)$, f is an impulsive Carathedeodory function, I_k are continuous functions.

The purpose of this paper is to improve Theorem A by using new methods and we also present new results for the existence of solutions of problem (4). The existence results for solutions of (4) will be established when f is super-linear. We don't need the assumptions that f is at most linear and I_k are nondecreasing. Some examples are presented to illustrate our theorems.

2. Main results and proofs

In this section, we establish the main results. To define solutions of (4), we introduce the Banach space.

Let $u : J = [0, T] \to R$, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, for $k = 0, \cdots, m$, define the function $u_k : (t_k, t_{k+1}] \to R$ by $u_k(t) = u(t)$. We will use the following Banach space

$$X = \left\{ \begin{array}{l} u : J \to R, u_k \in C^0(t_k, t_{k+1}], \ \ k = 0, \cdots, m, \ \ \text{there exist the limits} \\ \lim\limits_{t \to t_k^+} u(t), \quad \lim\limits_{t \to 0^+} u(t) = u(0), \quad \lim\limits_{t \to T^-} u(t) = u(T) \end{array} \right\}$$

and

$$Y = X \times R^m$$

with the norms

$$||x|| = \sup_{t \in [0,T]} |x(t)|$$

for $x \in X$ and

$$||(y, a_1, \cdots, a_m)|| = \max \{||y||, \ |a_i|, \ k = 0, \cdots, m\}$$

for $(y, a_1, \cdots, a_m) \in Y$.

A function f is an impulsive Carathedeodory function if

$*$ $f(\bullet, u_0, u_1, \cdots, u_n) \in X$ for each $u = (u_0, \cdots, u_n) \in R^{n+1}$;

$*$ $f(t, \bullet, \cdots, \bullet)$ is continuous for a.e. $t \in J$;

$*$ for each $r > 0$ there is $h_r \in L^1(J)$ so that

$$|f(t, u_0, u_1, \cdots, u_n)| \le h_r(t), \ a.e. t \in J \setminus \{t_1, \cdots, t_m\}$$

and every u satisfying $||(u_0, u_1, \cdots, u_n)|| < r$.

By a solution of (4) we mean a function $u \in X$ satisfying (4).

Lemma 1. *For each $\sigma \in L^1(J)$, the linear problem*

$$\left\{ \begin{array}{l} x'(t) = \sigma(t), \ \ a.e. \ \ t \in J, \\ x(0) = -x(T). \end{array} \right. \tag{5}$$

has unique solution

$$x(t) = -\frac{1}{2}\int_t^T \sigma(s)ds + \frac{1}{2}\int_0^t \sigma(s)ds. \tag{6}$$

Proof. Integrating (5) from 0 to T and using $x(0) = -x(T)$, we get

$$x(0) = -\frac{1}{2}\int_0^T \sigma(s)ds.$$

Integrating (5) from 0 to t, we have (6). ■

Lemma 2. *For each $\sigma \in L^1(J)$, $\theta_k \in R$, then the linear problem*

$$\begin{cases} x'(t) = \sigma(t), \quad a.e. \ t \in J, \\ \Delta x(t_k) = \theta_k, \quad k = 1, \cdots, m, \\ x(0) = -x(T). \end{cases} \tag{7}$$

has unique solution

$$x(t) = \frac{1}{2}\int_0^t \sigma(s)ds - \frac{1}{2}\int_t^T \sigma(s)ds - \frac{1}{2}\sum_{t \le t_k < T} \sigma_k + \frac{1}{2}\sum_{0 < t_k < t} \theta_k. \tag{8}$$

Proof. Integrating (7) from 0 to T and using $x(0) = -x(T)$, we get

$$x(0) = -\frac{1}{2}\int_0^T \sigma(s)ds - \frac{1}{2}\sum_{k=1}^m \sigma_k.$$

Integrating (7) from 0 to t, we have (8). One may see the details in the references [16, 17].

Now, we define a linear operator $L : D(L) \subseteq X \to Y$ and a nonlinear operator $N : X \to Y$:

$$Lx(t) = \begin{pmatrix} x'(t) \\ \Delta x(t_1) \\ \cdot \\ \cdot \\ \cdot \\ \Delta x(t_m) \end{pmatrix} \quad \text{for} \quad x \in D(L)$$

where $D(L) = \{u \in X, \ u_k \in C^1(t_k, t_{k+1}], \ k = 0, 1, \cdots, m, \ x(0) = -x(T)\}$ and

$$Nx(t) = \begin{pmatrix} f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t))) \\ I_1(x(t_1)) \\ \cdot \\ \cdot \\ \cdot \\ I_m(x(t_m)) \end{pmatrix} \quad \text{for} \quad x \in X.$$

It is easy to see that L is a Fredholm operator of index zero with $\mathrm{Ker}L = \{0\}$, N is $L-$compact on any open bounded subset of X and that $x \in X$ is a solution of problem (4) if and only if x is a solution of the operator equation $Lx = Nx$. ■

We set the following assumptions which should be used in the main results.

(A_1). $I_k(x)(2x + I_k(x) \leq 0$ for all $x \in R$ and $k = 1, \cdots, m$.
(A_2). $x(x + I_k(x)) \geq 0$ for all $x \in R$ and $k = 1, \cdots, m$.
(A_3). $I_k(x)(2x + I_k(x) \geq 0$ for all $x \in R$ and $k = 1, \cdots, m$.
(A_4). $xI_k(x) \geq 0$ for all $x \in R$ and $k = 1, \cdots, m$.

(C_1). There exist impulsive Carathedeodory functions $h : [0, T] \times R^n \to R$, $r \in X$ and $g_i : [0, T] \times R \to R$ such that

(i) $f(t, x_0, \cdots, x_n) = h(t, x_0, \cdots, x_n) + \sum_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$.

(ii) There exist constants $q \geq 0$ and $\beta > 0$ such that

$$h(t, x_0, \cdots, x_n)x_0 \ \leq \ -\beta|x_0|^{q+1}$$

holds for all $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$.

(iii) $\lim_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{|g_i(t,x)|}{|x|^q} = r_i \in [0, +\infty)$ for $i = 0, \cdots, n$.

(C_2). There exist impulsive Carathedeodory functions $h : [0, T] \times R^n \to R$, $r \in X$ and $g_i : [0, T] \times R \to R$ so that

(i) $f(t, x_0, \cdots, x_n) = h(t, x_0, \cdots, x_n) + \sum_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$.

(ii) There are constants $q \geq 0$ and $\beta > 0$ so that

$$h(t, x_0, \cdots, x_n)x_0 \ \geq \ \beta|x_0|^{q+1}$$

holds for all $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$.

(iii) $\lim_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{|g_i(t,x)|}{|x|^q} = r_i \in [0, +\infty)$ for $i = 0, \cdots, n$.

(C_3). There exist exist impulsive Carathedeodory functions $h : [0, T] \times R^n \to R$, $r \in X$ and $g_i : [0, T] \times R \to R$ so that

(i) $f(t, x_0, \cdots, x_n) = h(t, x_0, \cdots, x_n) + \sum_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$.

(ii) For $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$, $h(t, x_0, \cdots, x_n)x_0 \leq 0$ holds.

(iii) $\lim_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{|g_i(t,x)|}{|x|} = r_i \in [0, +\infty)$ for $i = 0, \cdots, n$.

(C_4). There exist exist impulsive Carathedeodory functions $h : [0, T] \times R^n \to R$, $r \in X$ and $g_i : [0, T] \times R \to R$ so that

(i) $f(t, x_0, \cdots, x_n) = h(t, x_0, \cdots, x_n) + \sum_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$.

(ii) For $(t, x_0, \cdots, x_n) \in [0, T] \times R^{n+1}$, $h(t, x_0, \cdots, x_n)x_0 \geq 0$ holds.

(iii) $\lim_{|x| \to +\infty} \sup_{t \in [0,T]} \frac{|g_i(t,x)|}{|x|} = r_i \in [0, +\infty)$ for $i = 0, \cdots, n$.

Theorem 1. *Assume that there exist functions $\psi_i : [0, +\infty) \to (0, +\infty)$ and positive functions $\rho_i \in L^1([0, T])$ such that*

$$|f(t, x_0, \cdots, x_n)| \leq \sum_{i=0}^{n} \rho_i(t)\psi_i(|x|),$$

and there exist $b_k \geq 0$ such that

$$|I_k(x)| \leq b_k|x| \quad \text{and} \quad \sum_{k=1}^{m} b_k < 2, \quad k = 1, \cdots, m.$$

Furthermore, suppose

$$\sup_{c>0} \frac{c}{\int_0^T \rho_0(s)ds\psi_0(c) + \sum_{i=1}^{n} \int_0^T \rho_i(s)ds\psi_i(c)} > \frac{1}{2 - \sum_{k=1}^{m} b_k}. \tag{9}$$

Then (4) has at least one solution.

Proof. Suppose $\lambda \in (0, 1)$, consider the operator equation $Lx = \lambda Nx$, i.e.,

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t))), & t \in [0, T], \\ & t \neq t_k, \quad k = 1, \cdots, m, \\ \Delta x(t_k) = \lambda I_k(x(t_k)), \quad k = 1, \cdots, m, \\ x(0) = -x(T). \end{cases} \tag{10}$$

It follows from Lemma 2 that

$$\begin{aligned} x(t) &= \lambda \frac{1}{2} \int_0^t f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))ds \\ &\quad - \lambda \frac{1}{2} \int_t^T f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))ds \\ &\quad - \lambda \frac{1}{2} \sum_{t \leq t_k < T} I_k(x(t_k)) + \lambda \frac{1}{2} \sum_{0 < t_k < t} I_k(x(t_k)). \end{aligned}$$

Hence

$$\begin{aligned}
|x(t)| &\le \frac{1}{2}\int_0^T |f(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))|ds + \frac{1}{2}\sum_{k=1}^{m}|I_k(x(t_k))| \\
&\le \frac{1}{2}\int_0^T \rho_0(s)\psi_0(|x(s)|)ds + \frac{1}{2}\sum_{i=1}^{n}\int_0^T \rho_i(s)\psi_i(|x(\alpha_i(s))|)ds \\
&\quad + \frac{1}{2}\sum_{k=1}^{p} b_k||x||_\infty \le \frac{1}{2}\left(\int_0^T \rho_0(s)ds\psi_0(||x||_\infty)\right. \\
&\quad \left. + \sum_{i=1}^{n}\int_0^T \rho_i(s)ds\psi_i(||x||_\infty) + \sum_{k=1}^{p} b_k||x||_\infty\right).
\end{aligned}$$

Then we get

$$\left(2-\sum_{k=1}^{m} b_k\right)||x||_\infty \le \int_0^T \rho_0(s)ds\psi_0(||x||_\infty) + \sum_{i=1}^{n}\int_0^T \rho_i(s)ds\psi_i(||x||_\infty).$$

So

$$\frac{||x||_\infty}{\int\limits_0^T \rho_0(s)ds\psi_0(||x||_\infty) + \sum\limits_{i=1}^{n}\int\limits_0^T \rho_i(s)ds\psi_i(||x||_\infty)} \le \frac{1}{2-\sum\limits_{k=1}^{m} b_k}.$$

From (9), there exists a constant $c_0 > 0$ such that $||x|| < c_0$.

Let $\Omega_1 = \{x \in X : ||X|| < c_0\}$. It follows that $||x||_\infty \ne c_0$ for $x \in \{x \in X : Lx = \lambda Nx, \lambda \in [0,1]\}$.

Choose the operator $T = L^{-1}N$. Consider

$$(I-\lambda T)(x) = (I-\lambda L^{-1}N)(x) = 0, \quad \lambda \in [0,1]. \tag{11}$$

It follows that every solution of (11) satisfies $x \in \Omega_1$. Therefor, $(I-\lambda T)(x) \ne 0$ for all $\lambda \in [0,1]$ and $x \in \partial\Omega_1$. The degree is defined on the bounded, open set Ω_1 and we have, by the invariance of the degree under homotopy (see [12]),

$$d((I-\lambda T)(x),\Omega_1,0) = d((I-T)(x),\Omega_1,0) = d(I,\Omega_1,0) = 1(\ne 0)$$

since $0 \in \Omega_1$. Therefor T has a fixed point, and thus, there exists a solution x to $x = Tx$. Thus, $Lx = Nx$ has at least one solution $x \in D(L)\cap\overline{\Omega 1}$, So x is a solution of (4). The proof is complete. ■

Remark 1. We apply Theorem 1 to problem $(*)$, we get that if there exist a function $\psi : [0,+\infty) \to (0,+\infty)$ and a function $\rho \in L^1([0,T])$ with

$$|f(t,x)| \le \rho(t)\psi(|x|),$$

and there exist $b_k \geq 0$ such that

$$|I_k(x)| \leq b_k|x| \quad \text{and} \quad \sum_{k=1}^{m} b_k < 2.$$

Furthermore, suppose

$$\sup_{c\geq 0} \frac{c}{\int_0^T \rho(s)ds\psi(c)} > \frac{1}{2-\sum_{k=1}^m b_k}.$$

Then $(*)$ has at least one solution. Theorem 1 generalizes and improves Theorem A.

Theorem 2. *Suppose (A_1), (A_2) and (C_3) hold. Then (4) has at least one solution if*

$$r_0 + \sum_{k=1}^{n} r_k \; < \; \frac{1}{4T}. \tag{12}$$

Proof. Similar to the proof of Theorem 1, we get (10). Then

$$x'(t)x(t) = \lambda f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t)))x(t).$$

It follows from (A_2) that

$$\begin{aligned} x(t_k^+)x(t_k) &= x(t_k)(x(t_k) + \lambda I_k(x(t_k))) \\ &\geq \lambda x(t_k)(x(t_k) + I_k(x(t_k))) \geq 0 \quad \text{for} \quad k = 1, \cdots, m. \end{aligned}$$

Together with $x(0) = -x(T)$, there is $\xi \in [0, T]$ such that $x(\xi) = 0$. Hence for $t \geq \xi$, we have from (A_1) and (C_3) that

$$\begin{aligned} \frac{1}{2}x^2(t) &= \frac{1}{2}x^2(\xi) + \lambda \sum_{\xi\leq t_k<t} I_k(x(t_k)(2x(t_k) + \lambda I_k(x(t_k))) \\ &\quad + \lambda \int_\xi^t f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\ &\leq \frac{1}{2}x^2(\xi) + \lambda \sum_{\xi\leq t_k<t} I_k(x(t_k)(2x(t_k) + I_k(x(t_k))) \\ &\quad + \lambda \int_\xi^t f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\ &\leq \lambda \int_\xi^t h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\ &\quad + \lambda \int_\xi^t x(s)g_0(x(s))ds + \lambda \sum_{i=1}^{n} \int_\xi^t x(s)g_i(x(\alpha_i(s)))ds \\ &\quad + \lambda \int_\xi^t x(s)r(s)ds \end{aligned}$$

$$\begin{aligned} &\leq \int_0^T |x(s)g_0(x(s))|ds \\ &\quad + \sum_{i=1}^n \int_0^T |x(s)g_i(x(\alpha_i(s)))|ds + \int_0^T |x(s)||r(s)|ds. \end{aligned}$$

Let $\epsilon > 0$ satisfy that

$$(r_0+\epsilon) + \sum_{k=1}^n (r_k+\epsilon) \ < \ \frac{1}{4T}. \tag{13}$$

For such $\epsilon > 0$, there is $\delta > 0$ so that for every $i = 0, 1, \cdots, n$,

$$|g_i(t,x)| \ < \ (r_i+\epsilon)|x| \quad \text{uniformly for} \quad t \in [0,T] \quad \text{and} \quad |x| > \delta. \tag{14}$$

Let, for $i = 1, \cdots, n$, $\Delta_{1,i} = \{t : \ t \in [0,T], \ |x(\alpha_i(t))| \leq \delta\}$, $\Delta_{2,i} = \{t : t \in [0,T], \ |x(\alpha_i(t))| > \delta\}$, $g_{\delta,i} = \max_{t\in[0,T],|x|\leq\delta} |g_i(t,x)|$, and $\Delta_1 = \{t \in [0,T], |x(t)| \leq \delta\}$, $\Delta_2 = \{t \in [0,T], |x(t)| > \delta\}$. Then we get

$$\begin{aligned} \frac{1}{2}x^2(t) &= \int_{\Delta_1} |x(s)g_0(x(s))|ds + \int_{\Delta_2} |x(s)g_0(x(s))|ds \\ &\leq \sum_{i=1}^n \int_{\Delta_{1,i}} |x(s)g_i(x(\alpha_i(s)))|ds + \sum_{i=1}^n \int_{\Delta_{2,i}} |x(s)g_i(x(\alpha_i(s)))|ds \\ &\quad + \int_0^T |x(s)||r(s)|ds \\ &\leq \sum_{i=0}^n g_{\delta,i}||x||_\infty + \sum_{i=1}^n (r_i+\epsilon) \int_{\Delta_{2,i}} |x(s)x(\alpha_i(s))|ds \\ &\quad + (r_0+\epsilon) \int_{\Delta_2} |x(s)|^2 ds + ||x||_\infty \int_0^T |r(t)|dt \\ &\leq \sum_{i=0}^n g_{\delta,i}||x||_\infty + \sum_{i=1}^n (r_i+\epsilon)T||x||_\infty^2 \\ &\quad + (r_0+\epsilon)T||x||_\infty^2 + ||x||_\infty \int_0^T |r(t)|dt. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2}x^2(t) &\leq \sum_{i=0}^n g_{\delta,i}||x||_\infty + \sum_{i=1}^n (r_i+\epsilon)T||x||_\infty^2 \\ &\quad + (r_0+\epsilon)T||x||_\infty^2 + ||x||_\infty \int_0^T |r(t)|dt. \end{aligned}$$

We get

$$\frac{1}{2}x^2(0) = \frac{1}{2}x^2(T) \le \sum_{i=0}^{n} g_{\delta,i}||x||_\infty + \sum_{i=1}^{n}(r_i+\epsilon)T||x||_\infty^2$$
$$+ (r_0+\epsilon)T||x||_\infty^2 + ||x||_\infty \int_0^T |r(t)|dt.$$

Then, for $t \in [0,\xi]$, we have by a similar way that

$$\begin{aligned}\frac{1}{2}x^2(t) &= \frac{1}{2}x^2(0) + \lambda \sum_{0\le t_k<t} I_k(x(t_k)(2x(t_k)+\lambda I_k(x(t_k))) \\ &\quad + \lambda \int_0^t f(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds \\ &\le \frac{1}{2}x^2(0) + \lambda \int_0^t f(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds \\ &\le 2\sum_{i=0}^{n} g_{\delta,i}||x||_\infty + 2\sum_{i=1}^{n}(r_i+\epsilon)T||x||_\infty^2 \\ &\quad + 2(r_0+\epsilon)T||x||_\infty^2 + 2||x||_\infty \int_0^T |r(t)|dt.\end{aligned}$$

It follows that that

$$\frac{1}{2}||x||_\infty^2 \le 2\sum_{i=0}^{n} g_{\delta,i}||x||_\infty + 2\sum_{i=1}^{n}(r_i+\epsilon)T||x||_\infty^2$$
$$+ 2(r_0+\epsilon)T||x||_\infty^2 + 2||x||_\infty \int_0^T |r(t)|dt.$$

It follows from (13) that there exists a constant $c_0 > 0$ such that $||x||_\infty < c_0$. The remainder of the proof is similar to that of the proof of Theorem 1 and is omitted. ■

Theorem 3. *Suppose* (A_2), (A_3) *and* (C_4) *hold. Then (4) has at least one solution if (12) holds.*

Proof. Similar to the proof of Theorem 1, we get (10). Then

$$x'(t)x(t) = \lambda f(t,x(t),x(\alpha_1(t)),\cdots,x(\alpha_n(t)))x(t).$$

It follows from (A_2) that

$$\begin{aligned}x(t_k^+)x(t_k) &= x(t_k)(x(t_k)+\lambda I_k(x(t_k))) \\ &\ge \lambda x(t_k)(x(t_k)+I_k(x(t_k))) \ge 0 \quad \text{for} \quad k=1,\cdots,m.\end{aligned}$$

Together with $x(0) = -x(T)$, there is $\xi \in [0,T]$ such that $x(\xi) = 0$. Hence for $t \leq \xi$, we get from (A_3) and (C_4) that

$$\begin{aligned}
\frac{1}{2}x^2(t) &= \frac{1}{2}x^2(\xi) - \lambda \sum_{t \leq t_k < \xi} I_k(x(t_k)(2x(t_k) + \lambda I_k(x(t_k))) \\
&\quad -\lambda \int_t^\xi f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\
&\leq -\lambda \int_t^\xi f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\
&\leq -\lambda \int_t^\xi h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s)ds \\
&\quad -\lambda \int_t^\xi x(s)g_0(x(s))ds - \lambda \sum_{i=1}^n \int_t^\xi x(s)g_i(x(\alpha_i(s)))ds \\
&\quad - \lambda \int_t^\xi x(s)r(s)ds \\
&\leq -\lambda \int_t^\xi x(s)g_0(x(s))ds - \lambda \sum_{i=1}^n \int_t^\xi x(s)g_i(x(\alpha_i(s)))ds \\
&\quad - \lambda \int_t^\xi x(s)r(s)ds \\
&\leq \int_0^T |x(s)g_0(x(s))|ds + \sum_{i=1}^n \int_0^T |x(s)g_i(x(\alpha_i(s)))|ds \\
&\quad + \int_0^T |x(s)||r(s)|ds.
\end{aligned}$$

The remainder of the proof is similar to that of the proof of Theorem 2 and is omitted. ■

Theorem 4. *Suppose* (A_4) *and* (C_2) *hold. Then problem* (4) *has at least one solution if*

$$r_0 + \sum_{k=1}^n r_k ||\beta_k'||_\infty^{q/(q+1)} < \beta. \tag{15}$$

Proof. Let $\lambda \in (0,1)$. Suppose x is a solution of the system (10). We divide the remainder of the proof into two steps.

Step 1. Prove that there is a constant $M > 0$ so that $\int_0^T |x(s)|^{q+1}ds \leq M$.

Multiplying both sides of the equation of (10) by $x(t)$, integrating it from 0 to T, we get

$$\begin{aligned}
&\frac{1}{2}\left(x(T)\right)^2-\frac{1}{2}\left(x(0)\right)^2-\frac{1}{2}\sum_{k=1}^{m}\left[\left(x(t_k^+)\right)^2-\left(x(t_k^-)\right)^2\right]\\
&\qquad= \lambda\int_0^T f(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds\\
&\qquad= \lambda\Bigg(\int_0^T h(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds\\
&\qquad\quad+\int_0^T g_0(s,x(s))x(s)ds\\
&\qquad\quad+\sum_{i=1}^{n}\int_0^T g_i(s,x(\alpha_i(s))x(s)ds+\int_0^T r(s)x(s)ds\Bigg).
\end{aligned}$$

It follows from (A_4) that

$$\begin{aligned}
&\left(x(t_k^+)\right)^2-\left(x(t_k^-)\right)^2=\left(x(t_k^+)-x(t_k^-)\right)\left(x(t_k^+)+x(t_k^-)\right)\\
&\quad= \lambda\Delta x(t_k^-)\left(2x(t_k^-)+\lambda\Delta x(t_k^-)\right)=\lambda I_k(x(t_k^-))\left(2x(t_k^-)+\lambda I_k(x(t_k^-))\right)\\
&\quad\geq \lambda 2x(t_k^-)I_k(x(t_k^-))\geq 0.
\end{aligned}$$

We get

$$\begin{aligned}
&\int_0^T h(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds+\int_0^T g_0(s,x(s))x(s)ds\\
&\qquad+\sum_{i=1}^{n}\int_0^T g_i(s,x(\alpha_i(s))x(s)ds+\int_0^T r(s)x(s)ds\leq 0.
\end{aligned}$$

It follows from (C_2) that

$$\begin{aligned}
\beta\int_0^T |x(s)|^{q+1}ds &\leq -\int_0^T g_0(s,x(s))x(s)ds\\
&\quad-\sum_{i=1}^{n}\int_0^T g_i(s,x(\alpha_i(s))x(s)ds-\int_0^T r(s)x(s)ds\\
&\leq \int_0^T |g_0(s,x(s))|x(s)|ds+\sum_{i=1}^{n}\int_0^T |g_i(s,x(\alpha_i(s))||x(s)|ds\\
&\quad+\int_0^T |r(s)||x(s)|ds.
\end{aligned}$$

Let $\epsilon>0$ satisfy that

$$(r_0+\epsilon)+\sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||_\infty^{q/(q+1)} \ < \ \beta. \tag{16}$$

For such $\epsilon > 0$, there is $\delta > 0$ so that for every $i = 0, 1, \cdots, n$,

$$|g_i(t,x)| < (r_i + \epsilon)|x|^q \quad \text{uniformly for} \quad t \in [0,T] \quad \text{and} \quad |x| > \delta. \tag{17}$$

Let, for $i = 1, \cdots, n$, $\Delta_{1,i} = \{t : \ t \in [0,T],\ |x(\alpha_i(t))| \le \delta\}$, $\Delta_{2,i} = \{t : t \in [0,T],\ |x(\alpha_i(t))| > \delta\}$, $g_{\delta,i} = \max_{t\in[0,T],|x|\le\delta} |g_i(t,x)|$, and $\Delta_1 = \{t \in [0,T], |x(t)| \le \delta\}$, $\Delta_2 = \{t \in [0,T], |x(t)| > \delta\}$. Then we get

$$\begin{aligned}
&\beta \int_0^T |x(s)|^{q+1} ds \\
&\le (r_0+\epsilon)\int_0^T |x(s)|^{q+1} ds + \sum_{k=1}^n (r_k+\epsilon)\int_0^T |x(\alpha_i(s))|^q |x(s)| ds \\
&\quad + \int_0^T |r(s)||x(s)| ds + \delta \int_0^T |x(s)| ds + \delta \sum_{k=1}^n \int_0^T |x(s) ds \\
&\le (r_0+\epsilon)\int_0^T |x(s)|^{q+1} ds \\
&\quad + \sum_{k=1}^n (r_k+\epsilon)\left(\int_0^T |x(\alpha_i(s))|^{q+1} ds\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
&\quad + \left(\int_0^T |r(s)| ds\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
&\quad + (n+1)\delta \int_0^T |x(s)| ds \\
&= (r_0+\epsilon)\int_0^T |x(s)|^{q+1} ds + \sum_{k=1}^n (r_k+\epsilon) \\
&\quad \times \left(\int_{\alpha_k(0)}^{\alpha_k(T)} |x(u)|^{q+1} |\beta_k'(u)| du\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
&\quad + \left(\int_0^T |r(s)| ds\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
&\quad + (n+1)\delta T^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
&\le (r_0+\epsilon)\int_0^T |x(s)|^{q+1} ds + \sum_{k=1}^n (r_k+\epsilon)||\beta_k'||_\infty^{q/(q+1)} \\
&\quad \times \left(\int_0^T |x(u)|^{1+q} du\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
&\quad + \left(\int_0^T |r(s)| ds\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)}
\end{aligned}$$

$$
\begin{aligned}
&+ (n+1)\delta T^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
= &\left((r_0+\epsilon) + \sum_{k=1}^{n} (r_k+\epsilon)||\beta_k'||_\infty^{q/(q+1)}\right) \int_0^T |x(s)|^{q+1} ds \\
&+ \left(\int_0^T |r(s)| ds\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)} \\
&+ (n+1)\delta T^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1} ds\right)^{1/(q+1)}.
\end{aligned}
$$

It follows from (16) that there is a constant $M > 0$ so that $\int_0^T |x(s)|^{q+1} ds \le M$.

Step 2. Prove that there is a constant $M_1 > 0$ so that $||x||_\infty \le M_1$.
It follows from Step 1 that there is $\xi \in [0,T]$ so that $|x(\xi)| \le (M/T)^{1/(q+1)}$.

Case 1. If $t < \xi$, multiplying two sides of the equation of (11) by $x(t)$, integrating it from t to ξ, we get, using (A_2), that

$$
\begin{aligned}
\frac{1}{2}(x(t))^2 &= \frac{1}{2}(x(\xi))^2 - \frac{1}{2} \sum_{t \le t_k < \xi} \left[\left(x(t_k^+)\right)^2 - \left(x(t_k^-)\right)^2\right] \\
&\quad - \lambda \int_t^\xi f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s) ds \\
&\le \frac{1}{2}(M/T)^{2/(q+1)} - \lambda \int_t^\xi f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s) ds \\
&\le \frac{1}{2}(M/T)^{2/(q+1)} - \lambda \left(\int_t^\xi h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s)))x(s) ds\right. \\
&\quad + \int_t^\xi g_0(s, x(s))x(s) ds \\
&\quad \left. + \sum_{i=1}^{n} \int_t^\xi g_i(s, x(\alpha_i(s))x(s) ds + \int_t^\xi r(s)x(s) ds\right) \\
&\le \frac{1}{2}(M/T)^{2/(q+1)} - \int_t^\xi g_0(s, x(s))x(s) ds \\
&\quad - \sum_{i=1}^{n} \int_t^\xi g_i(s, x(\alpha_i(s))x(s) ds - \int_t^\xi r(s)x(s) ds \\
&\le \frac{1}{2}(M/T)^{2/(q+1)} + \int_0^T |g_0(s, x(s))||x(s)| ds \\
&\quad + \sum_{i=1}^{n} \int_0^T |g_i(s, x(\alpha_i(s))||x(s)| ds + \int_0^T |r(s)||x(s)| ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2}\,(M/T)^{2/(q+1)} + \Bigg[\Bigg((r_0+\epsilon) + \sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||_\infty^{q/(1+q)}\Bigg) \\
&\qquad \times \int_0^T |x(s)|^{q+1}ds \\
&\quad + \left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)}\Bigg] \\
&\quad + (n+1)\delta T^{q/(q+1)} \left(\int_0^T |x(s)|^{q+1}ds\right)^{1/(q+1)} \\
&\le \frac{1}{2}\,(M/T)^{2/(q+1)} + \Bigg[\Bigg((r_0+\epsilon) + \sum_{k=1}^{n}(r_k+\epsilon)||\beta_k'||_\infty^{q/(1+q)}\Bigg) M \\
&\quad + \left(\int_0^T |r(s)|^{(q+1)/q}ds\right)^{q/(q+1)} M^{1/(q+1)}\Bigg] \\
&\quad + (n+1)\delta T^{q/(q+1)} M^{1/(q+1)} =: M_2.
\end{aligned}
$$

Hence one sees that

$$x^2(t) \ \le\ 2M_2 \ =:\ M_3, \quad \text{for} \quad t \in [0,\xi].$$

This implies $x^2(0) \le M_3$. So $x^2(T) = x^2(0) \le M_3$. For $t \in [\xi, T]$, we have

$$
\begin{aligned}
\frac{1}{2}\,(x(t))^2 \ &=\ \frac{1}{2}(x(T))^2 - \frac{1}{2}\sum_{t\le t_k<T}\left[\left(x(t_k^+)\right)^2 - \left(x(t_k^-)\right)^2\right] \\
&\quad - \lambda\int_t^T f(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds.
\end{aligned}
$$

Similar to above discussion, we get that there exists $M_4 > 0$ so that $x^2(t) \le M_4$ for $t \in [\xi, T]$. All above discussion implies that there exists a constant $d_0 > 0$ such that $|x(t)| \le d_0 < d_0 + 1 = c_0$. Thus $||x||_\infty < c_0$. The remainder of the proof is similar to that of the proof of Theorem 1 and is omitted. ■

Theorem 5. *Suppose* (A_1) *and* (C_1) *hold. Then (4) has at least one solution if*

$$r_0 + \sum_{k=1}^{n} r_k||\beta_k'||_\infty^{q/(1+q)} \ <\ \beta. \tag{18}$$

Proof. The proof is similar to that of Theorem 4. We consider system (10). Multiplying both sides of the equation of (10) by $x(t)$, integrating it from 0 to T, we get

$$\begin{aligned}
&\frac{1}{2}\,(x(T))^2-\frac{1}{2}\,(x(0))^2-\frac{1}{2}\sum_{k=1}^{m}\left[\left(x(t_k^+)\right)^2-\left(x(t_k^-)\right)^2\right]\\
&=\lambda\int_0^T f(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds\\
&=\lambda\Bigg(\int_0^T h(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds\\
&\quad+\int_0^T g_0(s,x(s))x(s)ds\\
&\quad+\sum_{i=1}^{n}\int_0^T g_i(s,x(\alpha_i(s))x(s)ds+\int_0^T r(s)x(s)ds\Bigg).
\end{aligned}$$

It follows from (A_1) that

$$\begin{aligned}
&\left(x(t_k^+)\right)^2-\left(x(t_k^-)\right)^2=\left(x(t_k^+)-x(t_k^-)\right)\left(x(t_k^+)+x(t_k^-)\right)\\
&\quad=\ \lambda\Delta x(t_k^-)\left(2x(t_k^-)+\lambda\Delta x(t_k^-)\right)=\lambda I_k(x(t_k^-))\left(2x(t_k^-)+\lambda I_k(x(t_k^-))\right)\\
&\quad\le\ \lambda I_k(x(t_k^-))\left(2x(t_k^-)+I_k(x(t_k^-))\right)\le 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^T h(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds+\int_0^T g_0(s,x(s))x(s)ds\\
&\qquad+\sum_{i=1}^{n}\int_0^T g_i(s,x(\alpha_i(s))x(s)ds+\int_0^T r(s)x(s)ds\ge 0.
\end{aligned}$$

The remainder of the proof is similar to that of Theorem 4 and is omitted. ■

Remark 2. In Theorem 2, Theorem 3, Theorem 4 and Theorem 5, f and I_k need not be superlinear and I_k need not either satisfy $|I_k(x)|\le b_k|x|$ or be nondecreasing. Hence the results (Theorems 2-5) are different from those in known papers.

3. Examples

In this section, we give examples, which can not be solved by the results in known papers, to illustrate the main results.

Example 1. Consider the problem

$$\left\{\begin{array}{l}
x'(t))=\sum_{k=0}^{2m+1}a_kx^k(t)+r(t),\quad a.e.\ t\in[0,T],\\
\Delta x(t_k)=b_k[x(t_k)]^3,\quad k=1,\cdots,p,\\
x(0)=-x(T)
\end{array}\right.\tag{19}$$

where m is a positive integer, $T > 0$, $b_k \geq 0$ for all $k = 1, \cdots, p$, $a_{2m+1} > 0$ and $a_k \in R$ for all $k = 1, \cdots, 2m$, $r \in X$. Corresponding to Theorem 4, choose

$$\begin{aligned} I_k(x) &= b_k x^3, \\ f(t, x_0) &= \sum_{k=0}^{2m+1} a_k x_0^k + r(t), \\ h(t, x_0) &= a_{2m+1} x_0^{2m+1}, \\ g_0(t, x_0) &= \sum_{k=0}^{2m} a_k x_0^k. \end{aligned}$$

It follows from Theorem 4 that problem (19) has at least one solution.

Example 2. Consider the problem

$$(20) \qquad \begin{cases} x'(t) = \sum\limits_{k=0}^{2m+1} a_k x^k(t) + \sum\limits_{k=1}^{2m+1} c_k x^k \left(\frac{1}{k} t\right) + r(t), \quad a.e.\ t \in [0, T], \\ \Delta x(t_k) = b_k x(t_k), \quad k = 1, \cdots, p, \\ x(0) = -x(T) \end{cases}$$

where m is a positive integer, $T > 0$, $b_k(2 + b_k) \leq 0$ for all $k = 1, \cdots, p$, $a_{2m+1} < 0$ and $a_k, c_k \in R$ for all $k = 1, \cdots, 2m$, $a_{2m+1} < 0$, $c_{2m+1} \in R$, $r \in X$.

It follows from Theorem 5 that problem (20) has at least one solution if $|c_{2m+1}| < (2m + 1)^{2m+2} a_{2m+1}$.

Example 3. Consider the problem

$$(21) \qquad \begin{cases} x'(t) = ax(t) + \sum\limits_{k=1}^{2m+1} c_k x\left(\frac{1}{k} t\right) + r(t), \quad a.e.\ \ t \in [0, T], \\ \Delta x(t_k) = b_k x(t_k), \quad k = 1, \cdots, p, \\ x(0) = -x(T) \end{cases}$$

where m is a positive integer, $T > 0$, $\sum\limits_{k=1}^{m} |b_k| < 2$ for all $k = 1, \cdots, p$, and $a_k, c_k \in R$ for all $k = 1, \cdots, 2m$, $c_{2m+1} \in R$, $r \in X$.

It follows from Theorem 1 that problem (21) has at least one solution if

$$T \left(|a| + \sum_{k=1}^{2m+1} |c_k| \right) < 2 - \sum_{k=1}^{m} |b_k|.$$

Example 4. Consider the problem

$$(22)\qquad \begin{cases} x'(t) = -\frac{[x(t)]^{2m+1}}{1+\sum\limits_{i=1}^{n}[x(t/i)]^2} + \sum\limits_{i=1}^{n} a_i x(\frac{1}{i}t) + r(t), \quad a.e.\ t \in [0,T], \\ \Delta x(t_k) = b_k x(t_k), \quad k = 1, \cdots, p, \\ x(0) = -x(T) \end{cases}$$

where $a_i, b_i \in R$, m is a positive integer, $T > 0$.

Corresponding to Theorem 2,

$$h(t, x_0, \cdots, x_n) \;=\; -\frac{x_0^{2m+1}}{1 + \sum\limits_{i=1}^{n} x_i^2}.$$

It follows from Theorem 2 that (22) has at least one solution if $\sum\limits_{k=0}^{n} |a_i| < \frac{1}{4T}$.

Remark 3. Above examples can not be solved by know theorems in [1-11].

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Yuji Liu
Department of Mathematics
Guangdong University of Business Studies
Uangzhou 510320, P.R.China
e-mail: liuyuji888@sohu.com

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