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Valeriu Popa

A COMMON FIXED POINT THEOREM FOR TWO MULTIFUNCTIONS DEFINED ON CLOSED BALL

Abstract. We prove a common fixed point theorem for two multifunctions defined on a closed ball of a complete metric space with values in the set of all nonempty and closed subsets of this space, multifunctions which satisfy an implicit contractive relation of Latif-Beg type.

Key words: multifunction, fixed point, common fixed point, implicit relation, contractive mapping of Latif-Beg type.

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1. Introduction and preliminaries

Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of X. Let $T : X \to P(X)$ be a multifunction. We denote by $Fix(T)$ the set of fixed points of T, i. e. $Fix(T) = \{ x \in X \mid x \in T(x) \}$.

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Further on we shall use the notations: $\overline{B}(x_0, r) = \{ x \in X \mid d(x_0, x) \leq r \}$, $P_{cl}(X) = \{ Y \in P(X) \mid Y \text{ is a closed set} \}$.

Assuming that (X, d) is complete, M. Frigon and A. Granas [2] proved a fixed point theorem for a multifunction $T : \overline{B}(x_0, r) \to P_{cl}(X)$. A. Petruşel [4] generalized the result of M. Frigon and A. Granas for multifunctions which satisfy Reich type conditions. A fixed point theorem for a multifunction which satisfies a general condition was proved by R. P. Agarwal and D. O'Regan in [1]. Recently, A. Sîntămărian [7] proved a common fixed point theorem for two multifunctions $T_1, T_2 : \overline{B}(x_0, r) \to P_{cl}(X)$, which generalizes the results of M. Frigon and A. Granas, A. Petruşel and R. P. Agarwal and D. O'Regan.

Quite recently A. Sîntămărian [8] proved a common fixed point theorem for two multivalued mappings $T_1, T_2 : \overline{B}(x_0, r) \to P_{cl}(X)$, which satisfy a contractive condition of Latif-Beg type [3].

Theorem 1 ([8]). *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T_1, T_2 : \overline{B}(x_0, r) \to P_{cl}(X)$ two multifunctions. We suppose that*

(i_1) *there exist* $a_{11}, \dots, a_{15} \in \mathbb{R}_+$, *with* $a_{11} + a_{12} + a_{13} + 2a_{14} < 1$, *such that for each* $x \in \overline{B}(x_0, r)$, *any* $u_x \in T_1(x)$ *and for all* $y \in \overline{B}(x_0, r)$, *there exists* $u_y \in T_2(y)$ *so that*

$$d(u_x, u_y) \leq a_{11}d(x, y)+a_{12}d(x, u_x)+a_{13}d(y, u_y)+a_{14}d(x, u_y)+a_{15}d(y, u_x);$$

(i_2) *there exist* $a_{21}, \dots, a_{25} \in \mathbb{R}_+$, *with* $a_{21} + a_{22} + a_{23} + 2a_{24} < 1$, *such that for each* $x \in \overline{B}(x_0, r)$, *any* $u_x \in T_2(x)$ *and for all* $y \in \overline{B}(x_0, r)$, *there exists* $u_y \in T_1(y)$ *so that*

$$d(u_x, u_y) \leq a_{21}d(x, y)+a_{22}d(x, u_x)+a_{23}d(y, u_y)+a_{24}d(x, u_y)+a_{25}d(y, u_x);$$

(ii) *there exists* $y_0 \in T_1(x_0) \cup T_2(x_0)$ *so that*

$$d(x_0, y_0) \leq \left(1 - \max\left\{\frac{a_{11} + a_{12} + a_{14}}{1 - (a_{13} + a_{14})}, \frac{a_{21} + a_{22} + a_{24}}{1 - (a_{23} + a_{24})}\right\}\right) r.$$

Then $Fix(T_1) = Fix(T_2) \in P_{cl}(X)$.

In [5] and [6] is introduced the study of fixed points for mappings satisfying implicit relations. The purpose of this paper is to prove a common fixed point theorem for two multifunctions $T_1, T_2 : \overline{B}(x_0, r) \to P_{cl}(X)$, which satisfy a new type of implicit relation, which generalizes [8, Theorem 3.1]. Also, we give a fixed point theorem for a multifunction $T : \overline{B}(x_0, r) \to P_{cl}(X)$ satisfying an implicit relation, which generalizes [8, Theorem 3.2] .

2. Implicit relations

Let $\mathcal{F}$ be the set of all continuous multifunctions $F(t_1, \dots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is non-increasing in variables $t_3, \dots, t_6$;

(F_2) there exist $h \in [0, 1)$ and $g \geq 0$ such that for every $u, v, w \geq 0$ with $F(u, v, v + w, u + w, u + v + w, w) \leq 0$, we have $u \leq hv + gw$.

Example 1. $F(t_1, \dots, t_6) = t_1 - a_1t_2 - a_2t_3 - a_3t_4 - a_4t_5 - a_5t_6$, where $a_1, \dots, a_5 \in \mathbb{R}_+$, with $a_1 + a_2 + a_3 + 2\,a_4 < 1$.

(F_1) Obviously.

(F_2) Let $F(u, v, v+w, u+w, u+v+w, w) = u-a_1v-a_2(v+w)-a_3(u+w)-a_4(u+v+w)-a_5w \leq 0$. Then $u \leq hv+gw$, where $0 \leq h = \frac{a_1+a_2+a_4}{1-(a_3+a_4)} < 1$ and $g = \frac{a_2+a_3+a_4+a_5}{1-(a_3+a_4)} \geq 0$.

Example 2. $F(t_1, \dots, t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, where $0 \leq k < 1/2$.

(F_1) Obviously.

(F_2) Let $F(u,v,v+w,u+w,u+v+w,w) = u - k\max\{v, v+w, u+w, u+v+w, w\} = u - k(u+v+w) \leq 0$. Then $u \leq hv + gw$, where $0 \leq h = g = k/(1-k) < 1$.

Example 3. $F(t_1,\dots,t_6) = t_1^2 - k\max\{t_2^2, t_3t_4, t_5t_6\}$, where $0 \leq k < 1/4$.
(F_1) Obviously.
(F_2) Let $F(u,v,v+w,u+w,u+v+w,w) = u^2 - k\max\{v^2, (v+w)(u+w), (u+v+w)w\} \leq 0$. Then $u^2 - k(u+v+w)^2 \leq 0$ and $u \leq \sqrt{k}(u+v+w)$. Hence $u \leq hv + gw$, where $0 \leq h = g = \frac{\sqrt{k}}{1-\sqrt{k}} < 1$.

Example 4. $F(t_1,\dots,t_6) = t_1^3 + t_1^2 + \frac{t_1}{1+t_5t_6} - (at_2^2 + bt_3^2 + ct_4^2)$, where $a,b,c \geq 0$ and $a+b+c < 1/4$.
(F_1) Obviously.
(F_2) Let $F(u,v,v+w,u+w,u+v+w,w) = u^3 + u^2 + \frac{u}{1+w(u+v+w)} - [av^2 + b(v+w)^2 + c(u+w)^2] \leq 0$. Then $u^2 - [av^2 + b(v+w)^2 + c(u+w)^2] \leq 0$, which implies $u^2 \leq (a+b+c)(u+v+w)^2$. Hence $u \leq hv + gw$, where $0 \leq h = g = \frac{\sqrt{a+b+c}}{1-\sqrt{a+b+c}} < 1$.

3. Common fixed points

Theorem 2. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T_1, T_2 : \overline{B}(x_0,r) \to P_{cl}(X)$ two multifunctions such that*

(i) for each $x \in \overline{B}(x_0,r)$, any $u_x \in T_1(x)$ and for all $y \in \overline{B}(x_0,r)$, there exists $u_y \in T_2(y)$ so that

$$F_1(d(u_x,u_y), d(x,y), d(x,u_x), d(y,u_y), d(x,u_y), d(y,u_x)) \leq 0,$$

where $F_1 \in \mathcal{F}$;

(ii) for each $x \in \overline{B}(x_0,r)$, any $u_x \in T_2(x)$ and for all $y \in \overline{B}(x_0,r)$, there exists $u_y \in T_1(y)$ so that

$$F_2(d(u_x,u_y), d(x,y), d(x,u_x), d(y,u_y), d(x,u_y), d(y,u_x)) \leq 0,$$

where $F_2 \in \mathcal{F}$;

(iii) there exists $y_0 \in T_1(x_0) \cup T_2(x_0)$ such that

$$d(x_0,y_0) \leq (1 - \max\{h_1,h_2\})r,$$

where h_1, h_2 are from definition of $\mathcal{F}$.

Then $Fix(T_1) = Fix(T_2) \in P_{cl}(X)$.

Proof. First, we prove that $Fix(T_1) = Fix(T_2)$. Let $x \in T_2(x)$. By (ii), for $x = y = u_x$, there exists $u_y \in T_1(x)$ such that

$$F_2(d(x,u_y),0,0,d(x,u_y),d(x,u_y),0) \le 0.$$

Since $F_2 \in \mathcal{F}$, then $d(x,u_y) = 0$. Therefore $x = u_y \in T_1(x)$ and $x \in Fix(T_1)$, hence $Fix(T_2) \subseteq Fix(T_1)$.

Similarly, by (i), we obtain $Fix(T_1) \subseteq Fix(T_2)$.

We put $h = \max\{h_1, h_2\} < 1$ and we suppose, for example, that there exists $x_1 = y_0 \in T_1(x_0)$ such that $d(x_0,x_1) \le (1-h)r$. It is clear that $x_1 \in \overline{B}(x_0,r)$.

By (i) we have that there exists $x_2 \in T_2(x_1)$ such that

$$F_1(d(x_1,x_2),d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_0,x_2),0) \le 0,$$

which implies that

$$F_1(d(x_1,x_2),d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_0,x_1)+d(x_1,x_2),0) \le 0.$$

Since $F_1 \in \mathcal{F}$,

$$d(x_1,x_2) \le h_1\, d(x_0,x_1) \le h\, d(x_0,x_1) \le h(1-h)r.$$

Using the triangle inequality we obtain

$$d(x_0,x_2) \le d(x_0,x_1) + d(x_1,x_2) \le (1-h)r + h(1-h)r = (1-h^2)r \le r,$$

hence $x_2 \in \overline{B}(x_0,r)$.

By (ii) we have that there exists $x_3 \in T_1(x_2)$ such that

$$F_2(d(x_2,x_3),d(x_1,x_2),d(x_1,x_2),d(x_2,x_3),d(x_1,x_3),0) \le 0,$$

which implies that

$$F_2(d(x_2,x_3),d(x_1,x_2),d(x_1,x_2),d(x_2,x_3),d(x_1,x_2)+d(x_2,x_3),0) \le 0.$$

Since $F_2 \in \mathcal{F}$,

$$d(x_2,x_3) \le h_2\, d(x_1,x_2) \le h\, d(x_1,x_2) \le h^2(1-h)r.$$

Because

$$d(x_0,x_3) \le d(x_0,x_2)+d(x_2,x_3) \le (1-h)(1+h)r+h^2(1-h)r = (1-h^3)r \le r,$$

we have that $x_3 \in \overline{B}(x_0,r)$.

By induction, we obtain that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following properties:

$$\begin{aligned}
&x_{2n-1} \in T_1(x_{2n-2}), \quad x_{2n} \in T_2(x_{2n-1}),\\
&d(x_{n-1}, x_n) \leq h^{n-1}(1-h)r,\\
&d(x_0, x_n) \leq (1-h^n)r \leq r, \text{ which implies that } x_n \in \overline{B}(x_0, r),
\end{aligned}$$

for each $n \in \mathbb{N}^*$.

The inequality $d(x_{n-1}, x_n) \leq h^{n-1}(1-h)r$, which holds for each $n \in \mathbb{N}^*$, implies that $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence, because $h < 1$ and (X, d) is a complete metric space. Let $x^* = \lim\limits_{n\to\infty} x_n$. Obviously $x^* \in \overline{B}(x_0, r)$.

We shall prove that x^* is a fixed point of T_1, for example. From $x_{2n} \in T_2(x_{2n-1})$ we have by (ii) that there exists $u_n \in T_1(x^*)$ such that

$$\begin{aligned}
F_2(d(x_{2n}, u_n), d(x_{2n-1}, x^*), d(x_{2n-1}, x_{2n}), d(x^*, u_n),&\\
d(x_{2n-1}, u_n), d(x^*, x_{2n})) \leq 0,&
\end{aligned}$$

for each $n \in \mathbb{N}^*$.

Then we have

$$\begin{aligned}
F_2(&d(x_{2n}, u_n), d(x_{2n-1}, x^*), d(x_{2n-1}, x^*) + d(x^*, x_{2n}), d(x^*, x_{2n})\\
&+ d(x_{2n}, u_n), d(x_{2n-1}, x^*) + d(x^*, x_{2n}) + d(x_{2n}, u_n), d(x^*, x_{2n})) \leq 0.
\end{aligned}$$

Since $F_2 \in \mathcal{F}$, then $d(x_{2n}, u_n) \leq h_2\, d(x_{2n-1}, x^*) + g_2\, d(x^*, x_{2n})$. On the other hand, we have that

$$d(x^*, u_n) \leq d(x^*, x_{2n}) + d(x_{2n}, u_n) \leq d(x^*, x_{2n}) + h_2\, d(x_{2n-1}, x^*) + g_2\, d(x^*, x_{2n}).$$

Letting n tend to infinity we obtain that $x^* = \lim_{n\to\infty} u_n$. Since $u_n \in T_1(x^*)$, for all $n \in \mathbb{N}^*$ and $T_1(x^*)$ is closed, it follows that $x^* \in Fix(T_1) = Fix(T_2)$.

Let us prove that $Fix(T_1) = Fix(T_2) \in P_{cl}(X)$. For this purpose let $y_n \in Fix(T_1) = Fix(T_2)$, for each $n \in \mathbb{N}^*$, such that $y_n \to y^*$, as $n \to \infty$. Clearly $y^* \in \overline{B}(x_0, r)$. For example, for $y_n \in T_1(y_n)$ we have that there exists $v_n \in T_2(y^*)$ so that

$$F_1(d(y_n, v_n), d(y_n, y^*), 0, d(y^*, v_n), d(y_n, v_n), d(y^*, y_n)) \leq 0,$$

which implies that

$$\begin{aligned}
F_1(d(y_n, v_n), d(y_n, y^*), d(y_n, y^*) + d(y_n, y^*), d(y^*, y_n) + d(y_n, v_n),&\\
d(y_n, v_n) + d(y_n, y^*) + d(y_n, y^*), d(y^*, y_n)) \leq 0.&
\end{aligned}$$

Since $F_1 \in \mathcal{F}$, then $d(y_n, v_n) \leq h_1 \, d(y_n, y^*) + g_1 \, d(y_n, y^*)$. Letting n tend to infinity we obtain that $\lim_{n\to\infty} v_n = y^*$. Since $v_n \in T_2(y^*)$, for each $n \in \mathbb{N}^*$ and $T_2(y^*)$ is closed, it follows that $y^* \in T_2(y^*)$. Therefore $Fix(T_1) = Fix(T_2) \in P_{cl}(X)$. ■

Corollary 1. *Theorem 1.*

Proof. The proof follows from Theorem 2 and Example 1. ■

If $T_1 = T_2 = T$ in Theorem 2, then we obtain the following theorem.

Theorem 3. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \overline{B}(x_0, r) \to P_{cl}(X)$ a multifunction such that:*

(i) for each $x \in \overline{B}(x_0, r)$, any $u_x \in T(x)$ and for all $y \in \overline{B}(x_0, r)$, there exists $u_y \in T(y)$ so that

$$F(d(u_x, u_y), d(x, y), d(x, u_x), d(y, u_y), d(x, u_y), d(y, u_x)) \leq 0,$$

where $F \in \mathcal{F}$;

(ii) there exists $y_0 \in T(x_0)$ such that

$$d(x_0, y_0) \leq (1 - h)r,$$

where h is from definition of $\mathcal{F}$.

Then $Fix(T) \in P_{cl}(X)$.

Corollary 2. [8, Sîntămărian] *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \overline{B}(x_0, r) \to P_{cl}(X)$ a multifunction for which there exist $a_1, \ldots, a_5 \in \mathbb{R}_+$, with $a_1 + a_2 + a_3 + 2\,a_4 < 1$ such that:*

(i) for each $x \in \overline{B}(x_0, r)$, any $u_x \in T(x)$ and for all $y \in \overline{B}(x_0, r)$, there exists $u_y \in T(y)$ so that

$$d(u_x, u_y) \leq a_1 \, d(x, y) + a_2 \, d(x, u_x) + a_3 \, d(y, u_y) + a_4 \, d(x, u_y) + a_5 \, d(y, u_x);$$

(ii) there exists $y_0 \in T(x_0)$ such that

$$d(x_0, y_0) \leq [1 - \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)}]r.$$

Then $Fix(T) \in P_{cl}(X)$.

Proof. The proof follows from Theorem 3 and Example 1, where $h = \frac{a_1+a_2+a_4}{1-(a_3+a_4)}$. ■

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Valeriu Popa
Department of Mathematics
University of Bacău
Str. Spiru Haret nr. 8
600114 Bacău, Romania
e-mail: vpopa@ub.ro

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