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COMMON FIXED POINT THEOREMS FOR HYBRID PAIRS OF MAPPINGS WITH SOME WEAKER CONDITIONS

Abstract. In this paper, we prove a common fixed point theorem for hybrid pairs of set and single valued mappings without assuming compatibility and continuity of any mapping on noncomplete metric spaces. To prove the theorem, we use a noncompatible condition, that is, weak commutativity of type (KB). We show that completeness of the whole space is not necessary for the existence of common fixed point. Our result improves, extends and generalizes the results of Fisher [5], Sastry and Naidu [18]. We give an example to validate our result. We also prove a common fixed point theorem on compact metric spaces. At the end, we improve our theorem by omitting the assumption of compactness. We also improve and generalize the results of Ahmed [2] and Fisher [5].

Key words: coincidence point, common fixed point, noncompatible maps.

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1. Introduction

Fixed point theorems for hybrid pair of set and single valued mappings have numerous applications in science and engineering (e.g. [12], [20]).

Sessa [19] introduced the concept of weakly commuting maps. Junngck [7] defined the notion of compatible maps in order to generalize the concept of weak commutativity and showed that weakly commmuting mappings are compatible but the converse is not true. On the other hand, Jungck and Rhoades [8], [9] defined the concept of compatibility and weak compatibility between a set valued mapping and a single valued mapping.
Most of the fixed point theorems existing in the mathematical literature deal with compatible and continuous mappings. So it would be a natural question: what about the mappings which are not compatible and continuous? Banach fixed point theorem has many applications but suffers from one drawback, the definition requires the continuity of the function. It has

been known from the paper of Kannan [10] that there exist maps that have a discontinuity in the domain but have a fixed point. One such function is the Dirichlet function defined on R, that is, $f(x) = \begin{cases} 1, & x \in Q, \\ 0, & \text{otherwise.} \end{cases}$ The Dirichlet function is not continuous at any point in R but has $x = 1$ as a fixed point. Another example is the function $f(x) = \begin{cases} \frac{4}{4-x}, & x \leq 2, \\ -1, & x > 2 \end{cases}$ defined on R. This function is not continuous at $x = 2$ but has $x = 2$ as a fixed point. These observations motivated several authors of the field to prove fixed point theorems for noncompatible, discontinuous mappings.

Pant [13]-[16] initiated the study of noncompatible maps and introduced point wise R- weak commutativity of mappings in [13]. He also showed that point wise R-weak commutativity is a necessary hence minimal condition for the existence of a common fixed point of contractive type maps [14].

Pathak, Cho and Kang [17] introduced the concept of R- weakly commuting mappings of type A and showed that they are not compatible. Recently, Kubiaczyk and Deshpande [11] extended the notion of R-weakly commuting mappings of type A in the settings of hybrid pair of mappings and defined weakly commuting mappings of type (KB). Some common fixed point theorems have been proved by using this new concept of weakly commuting mappings of type (KB) ([3], [11]).

On the other hand, Aamri and Moutawakii [1] also studied noncompatible mappings and introduced the notion of property (E.A). Recently, Djoudi and Khemis [4] extended property (E.A) in the settings of hybrid pair of mappings and introduced D-mappings.

In this paper, we prove common fixed point theorems for hybrid pairs of set and single valued mappings by using a non compatible condition, that is, weak commutativity of type (KB) on metric spaces. We show that the completeness of the whole space can be replaced by a weaker condition. We also show that the continuity of any mapping is not necessary for the existence of common fixed point. We improve and generalize the results of Fisher [5], Sastry and Naidu [18]. We give an example to validate our result. We also prove a common fixed point theorem on compact metric spaces by using weak commutativity of type (KB), which generalizes the result of Fisher [5]. At the end, we improve our theorem by using property (E.A) for hybrid pairs of mappings without assuming compactness of metric space and continuity of any mapping. This theorem also improves and generalizes the result of Ahmed [2] and Fisher [5].

2. Preliminaries

In the sequel, (X,d) denotes a metric space and $B(X)$ is the set of all nonempty bounded subsets of A. As in [3], [6] we define-

$$\begin{aligned} \delta(A,B) &= \sup\{d(a,b) : a \in A,\, b \in B\}, \\ D(A,B) &= \inf\{d(a,b) : a \in A,\, b \in B\}, \\ H(A,B) &= \inf\{r > 0 : A_r \supset B,\, B_r \supset A\}, \text{ for all } A, B \in B(X), \text{ where} \\ & \quad A_r = \{x \in X : d(x,a) < r \text{ for some } a \in A\}, \\ & \quad B_r = \{y \in X : d(y,b) < r \text{ for some } b \in B\}. \end{aligned}$$

If $A = \{a\}$ for some $a \in A$ we denote $\delta(a,B)$, $D(a,B)$, $H(a,B)$ for $\delta(A,B)$, $D(A,B)$ and $H(A,B)$ respectively. If $A = \{a\}$ and $B = \{b\}$, one can deduce that $\delta(A,B) = D(A,B) = H(A,B) = d(a,b)$. It follows immediatly from the definition of $\delta(A,B)$ that, $\delta(A,B) = \delta(B,A) \geq 0$, $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$, $\delta(A,A) = diam\ A$, $\delta(A,B) = 0$ iff $A = B = \{a\}$ for all $A, B, C \in B(X)$.

Definition 1 ([6]). *A sequence $\{A_n\}$ of non empty subsets of X is said to be convergent to a subset A of X if*

(i) Each point $a \in A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for all $n \in N$.

(ii) For arbitrary $\epsilon > 0$, there exists an integer $m > 0$ such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points $x \in X$ for which there exists a point $a \in A$, depending on x, such that $d(x,a) < \epsilon$. A is said to be the limit of the sequence $\{A_n\}$.

Lemma 1 ([6]). *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B respectively in $B(X)$, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

Lemma 2 ([6]). *Let $\{A_n\}$ be a sequence in $B(X)$ and $y \in X$ such that $\delta(A_n, y) \longrightarrow 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

Definition 2 ([6]). *The mappings $F : X \longrightarrow B(X)$ and $I : X \longrightarrow X$ are said to be weakly commuting if $IFx \in B(X)$ and*

$$\delta(FIx, IFx) \leq \max\{\delta(Ix, Fx), diamIFx\} \quad \textit{for all} \quad x \in X.$$

Note that, if F is a single valued mapping then the set $\{IFx\}$ consists of a single point. Therefore, $diamIFx = 0$ for all $x \in X$ and above inequality reduces to the well known condition given by Sessa [15], that is, $d(FIx, IFx) \leq d(Ix, Fx)$ for all $x \in X$. Two commuting mappings F and I are weakly commuting but the converse is not true as shown in [6].

Definition 3 ([8]). *The mappings $F : X \longrightarrow B(X)$ and $I : X \longrightarrow X$ are δ-compatible if $\lim_{n\longrightarrow\infty} \delta(FIx, IFx) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X), Fx_n \longrightarrow \{t\}$, $Ix_n \longrightarrow t$ for some t in X.*

Definition 4 ([1]). *Let I and J be two self mappings of a metric space (X, d). We say that I and J satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\longrightarrow\infty} Ix_n = \lim_{n\longrightarrow\infty} Jx_n = t$ for some t in X.*

Remark 1. It is clear from the Jungck's [7] definition that two self mappings I and J of a metric space (X, d) will be noncompatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\longrightarrow\infty} Ix_n = \lim_{n\longrightarrow\infty} Jx_n = t$ for some t in X. But $\lim_{n\longrightarrow\infty} d(IJx, JIx)$ is either non zero or nonexistent. Therefore, two noncompatible self mappings of a metric space (X, d) satisfy property (E.A).

Definition 5 ([4]). *The mappings $F : X \longrightarrow B(X)$ and $I : X \longrightarrow X$ are said to be D-mappings if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\longrightarrow\infty} Ix_n = t$ and $\lim_{n\longrightarrow\infty} Fx_n = \{t\}$ for some t in X.*

Definition 6 ([17]). *The mappings $I, J : X \longrightarrow X$ are called R-weakly commuting of type A_I if there exists some positive real number R such that*

$$d(IIx, JIx) \leq Rd(Ix, Jx) \quad \textit{for all} \quad x \in X.$$

It is shown in [17] that R-weakly commuting mappings of type A_I are compatible but the converse is not true.

Definition 7 ([11]). *The mappings $F : X \longrightarrow B(X)$ and $I : X \longrightarrow X$ are said to be weakly commuting of type (KB) at x if there exists some positive real number R such that*

$$\delta(IIx, FIx) \leq \ R\delta(Ix, Fx).$$

Here I and F are weakly commuting of type (KB) on X if the above inequality holds for for all $x \in X$. If I is single valued self mapping of X, this definition of weak commutativity reduces to that of Pathak, Cho and Kang [17].

Example 1. Let $X = [0, 2]$ and d be the usual metric on X. Define $I : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ by

$$Ix = \begin{cases} x + 1, & 0 \leq x < 1, \\ x, & 1 \leq x \leq 2 \end{cases}$$

and

$$Fx = \begin{cases} \left[1, \dfrac{x+2}{2}\right], & 0 \le x \le 1, \\ [1, x], & 1 < x \le 2 \end{cases}$$

Let $x_n = \dfrac{2n^3+1}{4n^5+1}$, $n = 0, 1, 2, 3..........$, then

$$\lim_{n \longrightarrow \infty} Ix_n = 1, \quad \lim_{n \longrightarrow \infty} Fx_n = \{1\}, \quad IFx_n \in B(X)$$

and

$$\begin{aligned} \lim_{n \longrightarrow \infty} \delta(FIx_n, IFx_n) &= \lim_{n \longrightarrow \infty} \delta\left(\left[1, \frac{2n^3+1}{4n^5+1} + 1\right],\right. \\ &\qquad \left.\left[1, \frac{2n^3+1}{2(4n^5+1)} + 1\right]\right) = 0. \end{aligned}$$

Thus, I and F are δ-compatible mappings. On the other hand, If $x = 1$ then $\delta(IIx, FIx) \le R\delta(Ix, Fx)$ for $R \ge 1$. Therefore, I and F are weakly commuting of type (KB) for $x = 1$. Also I and F are D-mappings.

Example 2. Let $X = [1, \infty]$ and d be the usual metric on X. Define $I : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ by $Ix = 2x + 1$ and $Fx = [1, 1 + x]$ for all $x \in X$. Then we can see that $IIx = 4x + 3$ and $FIx = [1, 2x + 2]$. Also $\delta(IIx, FIx) \le R\delta(Ix, Fx)$ for $R \ge 3$ and for all $x \in X$. Thus I and F are weakly commuting of type (KB) on X.

Consider the sequence $\{x_n\}$ in X defined by $x_n = 1 + \dfrac{1}{n}$, $n = 1, 2, 3, \ldots$. Then the hybrid pair $\{F, I\}$ neither satisfy the condition of D-mappings nor satisfy δ-compatibility.

3. Main results

Theorem 1. *Let (X, d) be a metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying the following conditions:*

(1) $$\cup F(X) \subseteq J(X), \quad \cup G(X) \subseteq I(X)$$

(2) $$\begin{aligned} \delta(Fx, Gy) \le \alpha \max\{&d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\} \\ &+ (1-\alpha)\left[aD(Ix, Gy) + bD(Jy, Fx)\right] \end{aligned}$$

for all $x, y \in X$, where

(3) $$0 \le \alpha < 1, \;\; a + b < 1, \;\; a \ge 0, \;\; b \ge 0, \;\; \alpha\,|a - b| < 1 - (a + b).$$

Suppose that one of $I(X)$ and $J(X)$ is complete. If both pairs $\{F, I\}$ and $\{G, J\}$ are weakly commuting of type (KB) at coincidence points in X, then there exists a unique $z \in X$ such that $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.

Proof. Let $x_0 \in X$ be arbitrary. By (1), we choose points $x_1, x_2, \ldots \in X$ such that $Jx_1 \in Fx_0 = Z_0$, $Ix_2 \in Gx_1 = Z_1$ and so on. Continuing in this manner we can define a sequence as follows:

$$(4) \quad Jx_{2n+1} \in Fx_{2n} = Z_{2n}, \quad Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+1}, \ \ n = 0, 1, 2, \ldots .$$

For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$ for $n = 0, 1, 2, \ldots$. By (2),

$$\begin{aligned} V_{2n} &= \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \\ &\leq \alpha \max\{d(Ix_{2n}, Jx_{2n+1}), \delta(Ix_{2n}, Fx_{2n}), \delta(Jx_{2n+1}, Gx_{2n+1})\} \\ &\quad + (1-\alpha)\,[aD(Ix_{2n}, Gx_{2n+1}) + bD(Jx_{2n+1}, Fx_{2n})] \\ &\leq \alpha \max\{\delta(Gx_{2n-1}, Fx_{2n}), \delta(Fx_{2n}, Gx_{2n+1})\} \\ &\quad + (1-\alpha)[a\delta(Gx_{2n-1}, Gx_{2n+1})] \\ &\leq \alpha \max\{V_{2n-1}, V_{2n}\} + (1-\alpha)a(V_{2n-1} + V_{2n}) \\ &\leq \beta V_{2n-1} \quad \text{for} \ \ n = 1, 2, 3, \ldots, \end{aligned}$$

where $\beta = \max\left\{\dfrac{\alpha + (1-\alpha)a}{1-(1-\alpha)a}, \dfrac{a}{1-a}\right\}$. The last inequality above follows easily upon considering the cases

$$V_{2n} \leq V_{2n-1} \quad \text{and} \quad V_{2n-1} \leq V_{2n}.$$

Similarly,

$$V_{2n+1} \leq \gamma V_{2n}, \ \ n = 0, 1, 2, 3, \ldots, \ \ \text{where} \ \ \gamma = \max\left\{\frac{\alpha + (1-\alpha)b}{1-(1-\alpha)b}, \frac{b}{1-b}\right\}.$$

Let $c = \beta\gamma$. If $a, b \in [0, 1/2]$, then $\beta < 1$, $\gamma < 1$. Therefore $0 \leq c < 1$. If $\max\{a, b\} \geq 1/2$, then since

$$\frac{\alpha + (1-\alpha)x}{1-(1-\alpha)x} \leq \frac{x}{1-x} \Longleftrightarrow \frac{1}{2} \leq x \quad \text{for all} \quad x \in [0, 1),$$

by hypothesis (3), it is easily seen that $0 \leq c < 1$. Then we deduce that

$$(5) \qquad V_{2n} = \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \leq c^n \delta(Fx_0, Gx_1) = c^n V_0$$

and

$$(6) \qquad \begin{aligned} V_{2n+1} &= \delta(Z_{2n+1}, Z_{2n+2}) = \delta(Gx_{2n+1}, Fx_{2n+2}) \\ &\leq c^n \delta(Gx_1, Fx_2) = c^n V_1 \quad \text{for} \quad n = 0, 1, 2, 3, \ldots . \end{aligned}$$

We put

$$M = \max\{\delta(Fx_0, Gx_1), \delta(Gx_1, Fx_2)\} = \max\{V_0, V_1\}.$$

Then if $z_n \in Z_n$ is an arbitrary point for $n = 0, 1, 2, 3, ...$, it follows that

$$d(z_{2n+1}, z_{2n+2}) \leq \delta(Z_{2n+1}, Z_{2n+2}) \leq c^n M,$$

$$d(z_{2n+2}, z_{2n+3}) \leq \delta(Z_{2n+1}, Z_{2n+2}) \leq c^n M.$$

Suppose that $J(X)$ is complete. Let $\{x_n\}$ be the sequence defined by (4), then

$$d(Jx_{2m+1}, Jx_{2n+1}) \leq \delta(Z_{2m}, Z_{2n}) < \varepsilon \quad \text{for } m, n > n_0, \ \ n_0 = 1, 2, 3, \ldots.$$

Therefore by the above, the sequence $\{Jx_{2n+1}\}$ is Cauchy and hence

$$Jx_{2n+1} \longrightarrow z = Jv \in J(X) \quad \text{for some } v \in X.$$

But $Ix_{2n} \in Gx_{2n-1} = Z_{2n-1}$ by (4), so that we have

$$d(Ix_{2n}, Jx_{2n+1}) \leq \delta(Z_{2n-1}, Z_{2n}) = V_{2n-1} \longrightarrow 0$$

Consequently, $Ix_{2n} \longrightarrow z$. Moreover, we have for $n = 1, 2, 3, \ldots$.

$$\delta(Fx_{2n}, z) \leq \delta(Fx_{2n}, Ix_{2n}) + \delta(Ix_{2n}, z) \leq \delta(Z_{2n}, Z_{2n-1}) + d(Ix_{2n}, z).$$

Therefore, $\delta(Fx_{2n}, z) \longrightarrow 0$. Similarly, it follows that $\delta(Gx_{2n-1}, z) \longrightarrow 0$. By (2), we have for $n = 1, 2, 3, \ldots$.

$$\begin{aligned} \delta(Fx_{2n}, Gv) &\leq \alpha \max\{d(Ix_{2n}, Jv), \delta(Ix_{2n}, Fx_{2n}), \delta(Jv, Gv)\} \\ &\quad + (1-\alpha)\left[aD(Ix_{2n}, Gv) + bD(Jv, Fx_{2n})\right] \\ &\leq \alpha \max\{d(Ix_{2n}, Jv), \delta(Ix_{2n}, Fx_{2n}), \delta(Jv, Gv)\} \\ &\quad + (1-\alpha)\left[a\delta(Ix_{2n}, Gv) + b\delta(Jv, Fx_{2n})\right]. \end{aligned}$$

Since $\delta(Ix_{2n}, Gv) \longrightarrow \delta(z, Gv)$ when $Ix_{2n} \longrightarrow z$, we get as $n \longrightarrow \infty$

$$\delta(z, Gv) \leq \alpha\delta(z, Gv) + (1-\alpha)a\delta(z, Gv) \implies (1-\alpha)(1-a)\delta(z, Gv) \leq 0.$$

Hence $Gv = \{z\} = \{Jv\}$.

Since $\cup G(X) \subseteq I(X)$, there esists $u \in X$ such that $\{Iu\} = Gv = \{Jv\}$. Now if $Fu \neq Gv$, $\delta(Fu, Gv) \neq 0$, so that we have by (2),

$$\begin{aligned} \delta(Fu, Gv) &\leq \alpha \max\{d(Iu, Jv), \delta(Iu, Fu), \delta(Jv, Gv)\} \\ &\quad + (1-\alpha)\left[aD(Iu, Gv) + bD(Jv, Fu)\right] \\ &\leq \alpha \max\{d(Iu, Jv), \delta(Iu, Fu), \delta(Jv, Gv)\} \\ &\quad + (1-\alpha)\left[a\delta(Iu, Gv) + b\delta(Jv, Fu)\right]. \end{aligned}$$

So, we have as $n \longrightarrow \infty$

$$\delta(Fu,z) \le \alpha\delta(Fu,z) + (1-\alpha)a\delta(Fu,z) \implies (1-\alpha)(1-b)\delta(Fu,z) \le 0.$$

It follows that $Fu = \{z\} = Gv = \{Iu\} = \{Jv\}$.

Since $Fu = \{Iu\}$ and the pair $\{F, I\}$ is weakly commuting of type (KB) at coincidence points in X, we obtain

$$\delta(IIu, FIu) \le R\delta(Iu, Fu),$$

which gives $Fz = FIu = IFu = \{Iz\}$. Using (2),

$$\begin{aligned}
\delta(Fz,z) &\le \delta(Fz,Gv)\\
&\le \alpha\max\{d(Iz,Jv), \delta(Iz,Fz), \delta(Jv,Gv)\}\\
&\quad + (1-\alpha)\,[aD(Iz,Gv) + bD(Jv,Fz)]\\
&\le \alpha\delta(Fz,z) + (1-\alpha)(a+b)\delta(Fz,z)\\
\implies & (1-\alpha)[1-(a+b)]\delta(Fz,z) \le 0.
\end{aligned}$$

Since $a + b < 1$, it follows that $\{z\} = \{Fz\}$.

Similarly $\{z\} = Gz = \{Jz\}$, if the pair $\{G, J\}$ is weakly commuting of type (KB) at coincidence points in X. Therefore we obtain $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.

To prove that this $z \in X$ is unique, suppose $w \in X$ is another common fixed point such that $w \ne z$ and $\{w\} = \{Iw\} = \{Jw\} = Fw = Gw$. By (2), we obtain

$$\begin{aligned}
d(z,w) &\le \delta(Fz,Gw)\\
&\le \alpha\max\{d(Iz,Jw), \delta(Iz,Fz), \delta(Jw,Gw)\}\\
&\quad + (1-\alpha)\,[aD(Iz,Gw) + bD(Jw,Fz)]\\
&\le \alpha d(z,w) + (1-\alpha)[ad(z,w) + bd(z,w)]\\
\implies & (1-\alpha)[1-(a+b)]d(z,w) \le 0.
\end{aligned}$$

Since $a + b < 1$, it follows that $z = w$.

This completes the proof. ■

Remark 2. Theorem 1 improves and generalizes the result of Fisher [5]. Also it improves, extends and generalizes the results of Sastry and Naidu [18].

If we put $F = G$ and $I = J$ in Theorem 1, we get the following:

Corollary 1. *Let (X, d) be a metric space and $I : X \longrightarrow X$, $F : X \longrightarrow B(X)$ be mappings satisfying the following conditions:*

(7) $$\cup F(X) \subseteq I(X),$$

$$\begin{aligned}(8)\qquad \delta(Fx,Gy) \ \le\ & \alpha \max\{d(Ix,Iy),\delta(Ix,Fx),\delta(Iy,Fy)\}\\ & + (1-\alpha)\left[aD(Ix,Fy)+bD(Iy,Fx)\right]\end{aligned}$$

for all $x, y \in X$, *where*

$$(9)\qquad 0 \le \alpha < 1,\quad a+b<1,\quad a \ge 0,\quad b \ge 0,\quad \alpha\,|a-b| < 1-(a+b).$$

Suppose $I(X)$ *is complete. If the pair* $\{F, I\}$ *is weakly commuting of type (KB) at coincidence points in* X*, then there exists a unique* $z \in X$ *such that* $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.

Now, we give an example to validate our Theorem 1.

Example 3. Let $X = [0, 5)$ and d be the Eucledian metric.

Let $I, J : X \longrightarrow X$ and $F, G : X \longrightarrow B(X)$ be defined by

$$Ix = \begin{cases} \dfrac{2x^2+x}{8}, & 0 \le x \le 2,\\ \dfrac{x^2+2x-8}{8}, & 2 < x < 5,\end{cases} \qquad Jx = \begin{cases} \dfrac{4x^2+x}{8}, & 0 \le x \le 2,\\ \dfrac{x^2+4x-12}{8}, & 2 < x < 5,\end{cases}$$

$$Fx = \begin{cases} \left[0, \dfrac{x^2}{8}\right], & 0 \le x \le 2,\\ \left[0, \dfrac{x-2}{8}\right], & 2 < x < 5,\end{cases} \qquad Gx = \begin{cases} \left[0, \dfrac{x}{16}\right], & 0 \le x \le 2,\\ \left[0, \dfrac{x^2-4}{16}\right], & 2 < x < 5.\end{cases}$$

Then $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$. If we take $\alpha = \dfrac{1}{2}$, $a = \frac{2}{3}$ and $b = \frac{4}{5}$, then we can see that the condition (2) is satisfied. If we consider the sequence $\{x_n\}$ defined by $x_n = 2 + \frac{1}{n}$, $n = 1, 2, 3...$, then

$$\lim_{n\longrightarrow\infty} Ix_n = 0, \quad \lim_{n\longrightarrow\infty} Fx_n = \{0\}, \quad IFx_n \in (X)$$

and

$$\begin{aligned}\lim_{n\longrightarrow\infty} \delta(FIx_n, IFx_n) = \lim_{n\longrightarrow\infty} \delta\Bigg(&\left[0, \frac{1+12n+48n^2+72n^3+36n^4}{512n^4}\right],\\ &\left[0, \frac{1+4n}{256n^2}\right]\Bigg) \ne 0.\end{aligned}$$

Thus, the pair $\{F, I\}$ is weakly commuting of type (KB) at the coincidence point $x = 0$ but it is not δ-compatible. Similarly, the pair $\{G, J\}$ is weakly commuting of type (KB) at the coincidence point $x = 0$ but it is not $\delta-$compatible. Therefore, all the conditions of Theorem 3.1 are satisfied and $x = 0$ is the unique common fixed point of I, J, F and G. Moreover, all the mappings involved in this example are discontinuous.

Theorem 2. *Let (X, d) be a compact metric space. Let $I, J : X \longrightarrow X$ and $F, G : X \longrightarrow B(X)$ be mappings with $\cup F(X) \subseteq J(X)$, $\cup G(X) \subseteq I(X)$. Suppose that,*

$$\begin{aligned} \delta(Fx, Gy) \quad < \quad & \alpha \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\} \\ & + (1-\alpha)\left[aD(Ix, Gy) + bD(Jy, Fx)\right] \end{aligned} \tag{10}$$

for all $x, y \in X$, where

$$0 \le \alpha < 1, \quad a \ge 0, \quad b \ge 0, \quad a \le 1/2, \quad b < 1/2, \quad \alpha\,|a-b| < 1-(a+b), \tag{11}$$

holds when ever the right hand side of (10) is positive. If the pairs $\{F, I\}$ and $\{G, J\}$ are weakly commuting of type (KB) at coincidence points in X and if F, I are continuous, then there exists a unique $z \in X$ such that $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.

Proof. Let $\eta = \inf\limits_{x \in X}\{\ \delta(Ix, Fx)\}$. Since X is a compact metric space, there exists a convergent sequence $\{x_n\}$ with limit x_0 in X such that,

$$\delta(Ix_n, Fx_n) \longrightarrow \eta \quad \text{as} \quad n \longrightarrow \infty.$$

Since

$$\delta(Ix_0, Fx_0) \le d(Ix_0, Ix_n) + \delta(Ix_n, Fx_n) + \delta(Fx_n, Fx_0),$$

$\lim\limits_{n \longrightarrow \infty} x_n = x_0$ and F, I are continuous, we get $\delta(Ix_0, Fx_0) \le \eta$. Thus, $\delta(Ix_0, Fx_0) = \eta$.

Since $\cup F(X) \subseteq J(X)$, there exists a point $y_0 \in X$ such that $Jy_0 \in Fx_0$ and $d(Ix_0, Jy_0) \le \eta$. If $\eta > 0$, then by (10),

$$\begin{aligned} \delta(Jy_0, Gy_0) \quad & \le \quad \delta(Fx_0, Gy_0) \\ & < \quad \alpha \max\{d(Ix_0, Jy_0), \delta(Ix_0, Fx_0), \delta(Jy_0, Gy_0)\} \\ & \qquad + (1-\alpha)\left[aD(Ix_0, Gy_0) + bD(Jy_0, Fx_0)\right] \\ & \le \quad \alpha \max\{\eta, \delta(Jy_0, Gy_0)\} + (1-\alpha)a\left[d(Ix_0, Gy_0) + \delta(Jy_0, Gy_0)\right] \\ & \le \quad \alpha \max\{\eta, \delta(Jy_0, Gy_0)\} + (1-\alpha)a\left[\eta + \delta(Jy_0, Gy_0)\right]. \end{aligned}$$

If $\delta(Jy_0, Gy_0) > \eta$ in the last inequality, we obtain from $0 \le \alpha < 1$ and $a \le 1/2$ that,

$$\delta(Jy_0, Gy_0) < [\alpha + 2(1-\alpha)a]\delta(Jy_0, Gy_0) \le \delta(Jy_0, Gy_0).$$

This contradiction implies that $\delta(Jy_0, Gy_0) \le \eta$.

Since $\cup G(X) \subseteq I(X)$, there is a point $z_0 \in X$ such that $Iz_0 \in Gy_0$ and $d(Iz_0, Jy_0) < \eta$. Hence we have from $0 \leq \alpha < 1$ and $b < 1/2$ that

$$\begin{aligned} \eta &\leq \delta(Iz_0, Fz_0) \leq \delta(Fz_0, Gy_0) \\ &< \alpha \max\{d(Iz_0, Jy_0), \delta(Iz_0, Fz_0), \delta(Jy_0, Gy_0)\} \\ &\quad + (1-\alpha)\,[aD(Iz_0, Gy_0) + bD(Jy_0, Fz_0)] \\ &\leq \alpha\delta(Iz_0, Fz_0) + (1-\alpha)b\delta(Jy_0, Fz_0) \\ &\leq \alpha\delta(Iz_0, Fz_0) + (1-\alpha)b[d(Jy_0, Iz_0) + \delta(Iz_0, Fz_0)] \\ &< \alpha\delta(Iz_0, Fz_0) + (1-\alpha)b[\eta + \delta(Iz_0, Fz_0)] \\ &\leq \alpha\delta(Iz_0, Fz_0) + 2(1-\alpha)b\delta(Iz_0, Fz_0) < \delta(Iz_0, Fz_0). \end{aligned}$$

This contradiction implies that $\eta = 0$.
Therefore, we have $Gy_0 = \{Jy_0\} = Fx_0 = \{Ix_0\} = \{Iz_0\}$.

Since the pair $\{F, I\}$ is weakly commuting of type (KB) at coincidence points in X we have

$$\delta(IIx_0, FIx_0) \leq R\delta(Ix_0, Fx_0).$$

Since $Fx_0 = \{Ix_0\}$, we get $\{I^2x_0\} = FIx_0 = F^2x_0 = IFx_0$. If $I^2x_0 \neq Ix_0$, then using (10),

$$\begin{aligned} d(I^2x_0, Ix_0) &= \delta(F^2x_0, Gy_0) \\ &< \alpha \max\{d(IFx_0, Jy_0), \delta(IFx_0, F^2x_0), \delta(Jy_0, Gy_0)\} \\ &\quad + (1-\alpha)\left[aD(IFx_0, Gy_0) + bD(Jy_0, F^2x_0)\right] \\ &= \alpha d(I^2x_0, Ix_0) + (1-\alpha)(a+b)d(I^2x_0, Ix_0) \\ &= [\alpha + (1-\alpha)(a+b)]d(I^2x_0, Ix_0). \end{aligned}$$

Since $[\alpha+(1-\alpha)(a+b)] < 1$, we have $I^2x_0 = Ix_0$. Hence $FIx_0 = \{Ix_0\} = \{I^2x_0\}$. Similarly, we have $GJy_0 = \{Jy_0\} = \{J^2y_0\}$. Let $z = Ix_0 = Jy_0$, then $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.

Suppose that the point $w \in X$ is another common fixed point of F, G, I and J with $w \neq z$. If either $\delta(w, Fw) \neq 0$ or $\delta(w, Gw) \neq 0$, then using (10),

$$\begin{aligned} \delta(w, Fw) &\leq \delta(Fw, Gw) \\ &< \alpha \max\{d(w, w), \delta(w, Fw), \delta(w, Gw)\} \\ &\quad + (1-\alpha)\,[aD(w, Gw) + bD(w, Fw)] \\ &\leq \lambda\delta(w, Gw), \quad \text{where} \quad \lambda = \max\left\{\frac{\alpha + (1-\alpha)a}{1-(1-\alpha)b}, \frac{a}{1-b}\right\} < 1. \end{aligned}$$

Hence $\delta(w, Fw) < \delta(w, Gw)$. By symmetry, we have $\delta(w, Gw) < \delta(w, Fw)$. Therefore, $\delta(w, Fw) = \delta(w, Gw) = 0$, so $Fw = Gw = \{w\}$. Now,

$$\begin{aligned} d(w, z) &= \delta(Fw, Gz) \\ &< \alpha \max\{d(w, z), \delta(w, Fw), \delta(z, Gz)\} \\ &\quad + (1 - \alpha)\left[aD(w, Gz) + bD(z, Fw)\right] \\ &= \alpha d(w, z) + (1 - \alpha)(a + b)d(w, z) \\ &= [\alpha + (1 - \alpha)(a + b)]d(w, z). \end{aligned}$$

Since $[\alpha + (1 - \alpha)(a + b)] < 1$, it follows that $w = z$. Therefore z is the unique common fixed point of F, G, I and J.

This completes the proof. ■

Remark 3. Theorem 2 generalizes the result of Fisher [5].

Theorem 3. *Let (X, d) be a metric space. Let $I, J : X \longrightarrow X$ and $F, G : X \longrightarrow B(X)$ be mappings with $\cup F(X) \subseteq J(X)$, $\cup G(X) \subseteq I(X)$. Suppose that the inequality (10) with (11) holds whenever the right hand side of (10) is positive. If the pairs $\{F, I\}$ and $\{G, J\}$ are $D-$mappings and weakly commuting of type (KB) at coincidence points in X and if $\cup F(X)$ (resp. $J(X)$) and $\cup G(X)$ (resp.$I(X)$) are closed, then there exists a unique $z \in X$ such that $\{z\} = \{Iz\} = \{Jz\} = Fz = Gz$.*

Proof. Since the pair $\{F, I\}$ is $D-$mapping, there is a sequence $\{x_n\}$ in X such that,

$$\lim_{n\longrightarrow\infty} Ix_n = t, \quad \lim_{n\longrightarrow\infty} Fx_n = \{t\} \quad \text{for some} \quad t \in X.$$

Since $\cup F(X)$ is closed, there exists a $u \in X$ such that $t = Ju$. By (10),

$$\begin{aligned} \delta(Fx_n, Gu) &< \alpha \max\{d(Ix_n, Ju), \delta(Ix_n, Fx_n), \delta(Ju, Gu)\} \\ &\quad + (1 - \alpha)\left[aD(Ix_n, Gu) + bD(Ju, Fx_n)\right]. \end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned} \delta(Ju, Gu) &< \alpha\delta(Ju, Gu) + (1 - \alpha)aD(Ju, Gu) \\ &< \alpha\delta(Ju, Gu) + (1 - \alpha)a\delta(Ju, Gu) \\ \Longrightarrow\ & \delta(Ju, Gu) < a\delta(Ju, Gu), \end{aligned}$$

which is a contradiction. Thus, $\{Ju\} = Gu$.

Since the pair $\{G, J\}$ is weakly commuting of type (KB) at coincidence points in X, we obtain

$$\delta(JJu, GJu) \le R\delta(Ju, Gu),$$

which gives $\{JJu\} = GJu$. Hence $\{JJu\} = GJu = JGu = GGu$. Again by (10),

$$\begin{aligned} \delta(Fx_n, GGu) &< \alpha \max\{d(Ix_n, JGu), \delta(Ix_n, Fx_n), \delta(JGu, GGu)\} \\ &\quad + (1-\alpha)\left[aD(Ix_n, GGu) + bD(JGu, Fx_n)\right]. \end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned} \delta(Ju, GGu) &< \alpha\delta(Ju, GGu) + (1-\alpha)(a+b)\delta(Ju, GGu) \\ \Longrightarrow\ & \delta(Ju, GGu) < (a+b)\delta(Ju, GGu). \end{aligned}$$

Since $(a+b) < 1$, this gives $\{Ju\} = GGu = JGu$ i.e. $Gu = GGu = JGu$ and Gu is fixed point for G and J.

Similarly, Since the pair $\{G, J\}$ is D-mapping, there is a sequence $\{y_n\}$ in X such that,

$$\lim_{n\longrightarrow\infty} Jy_n = s, \qquad \lim_{n\longrightarrow\infty} Gy_n = \{s\} \quad \text{for some} \quad s \in X.$$

Since $\cup G(X)$ is closed, there exists a $v \in X$ such that $s = Iv$. By (10),

$$\begin{aligned} \delta(Fv, Gy_n) &< \alpha \max\{d(Iv, Jy_n), \delta(Iv, Fv), \delta(Jy_n, Gy_n)\} \\ &\quad + (1-\alpha)\left[aD(Iv, Gy_n) + bD(Jy_n, Fv)\right]. \end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned} \delta(Fv, Iv) &< \alpha\delta(Fv, Iv) + (1-\alpha)bD(Iv, Fv) \\ &< \alpha\delta(Fv, Iv) + (1-\alpha)b\delta(Iv, Fv) \\ \Longrightarrow\ & \delta(Fv, Iv) < b\delta(Fv, Iv), \end{aligned}$$

which is a contradiction. Thus, $Fv = \{Iv\}$.

Since the pair $\{F, I\}$ is weakly commuting of type (KB) at coincidence points in X, we obtain

$$\delta(IIv, FIv) \ \leq\ R\delta(Iv, Fv),$$

which gives $\{IIv\} = FIv$. Hence $\{IIv\} = FIv = FFv = IFv$. Again by (10),

$$\begin{aligned} \delta(FFv, Gy_n) &< \alpha \max\{d(IFv, Jy_n), \delta(IFv, FFv), \delta(Jy_n, Gy_n)\} \\ &\quad + (1-\alpha)\left[aD(IFv, Gy_n) + bD(Jy_n, FFv)\right]. \end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned} \delta(FFv, Iv) &< \alpha\delta(FFv, Iv) + (1-\alpha)\left[aD(IFv, Iv) + bD(Iv, FFv)\right] \\ &< \alpha\delta(FFv, Iv) + (1-\alpha)(a+b)\delta(IFv, Iv) \\ \Longrightarrow\ & \delta(FFv, Iv) < (a+b)\delta(FFv, Iv), \end{aligned}$$

which is a contradiction as $a+b<1$. Thus, it follows that $FFv = Fv = IFv$, that is, Fv is a fixed point for F. Again by (10),

$$\begin{aligned}\delta(Fx_n, Gy_n) &< \alpha \max\{d(Ix_n, Jy_n), \delta(Ix_n, Fx_n), \delta(Jy_n, Gy_n)\} \\ &\quad + (1-\alpha)\left[aD(Ix_n, Gy_n) + bD(Jy_n, Fx_n)\right].\end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned}\delta(Ju, Iv) &< \alpha\delta(Ju, Iv) + (1-\alpha)\left[aD(Ju, Iv) + bD(Iv, Ju)\right] \\ &< \alpha\delta(Ju, Iv) + (1-\alpha)(a+b)\delta(Ju, Iv) \\ \Longrightarrow\ & \delta(Ju, Iv) < (a+b)\delta(Ju, Iv),\end{aligned}$$

which gives $Ju = Iv$ as $a+b<1$. Consequently, $Fv = \{Iv\} = \{Ju\} = Gu$.

Let $Ju = Iv = z$. Then,

$$\{z\} = \{Iz\} = \{Jz\} = Fz = Gz.$$

To prove that this z is unique, suppose $w \in X$ is another common fixed point such that $w \neq z$ and $\{w\} = \{Iw\} = \{Jw\} = Fw = Gw$. By (10),

$$\begin{aligned}d(w,z) &= \delta(Fz, Gw) \\ &< \alpha \max\{d(Iz, Jw), \delta(Iz, Fz), \delta(Jw, Gw)\} \\ &\quad + (1-\alpha)\left[aD(Iz, Gw) + bD(Jw, Fz)\right] \\ &\leq \alpha d(w,z) + (1-\alpha)\left[ad(w,z) + bd(w,z)\right] \\ \Longrightarrow\ & d(w,z) < (a+b)d(w,z).\end{aligned}$$

Since $a+b<1$, it follows that $w = z$.

This completes the proof. ■

Remark 4. (i) Theorem 3 improves the results of Ahmed [2] and Fisher [5].

(ii) Theorem 3 improves our Theorem 2 in the sense that compactness of metric spaces is dropped. Also continuity of any mapping is not assumed to prove common fixed point theorem.

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