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THE SEQUENCE SPACE $m(\phi, \Delta_m, p)^F$

Abstract. The sequence space $m(\phi, \Delta_m, p)^F$ of fuzzy real numbers for $0 < p < 1$ and $1 \leq p < \infty$, are introduced. Some properties of the sequence space like solidness, symmetricity, convergence-free etc. are studied.

Key words: symmetric space; solid space; convergence-free; metric space; completeness.

AMS Mathematics Subject Classification: 40A05, 40A25, 40A30, 40C05.

1. Introduction

The concept of fuzzy set theory was introduced by Zadeh [16]. Later on sequences of fuzzy numbers have been discussed by Matloka [6], Tripathy and Nanda [15], Nuray and Savas [7], Kwon [5] and many others.

Kizmaz [4] defined the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ for crisp sets as follows

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

The above spaces are Banach spaces, normed by,

$$\| x \|_\Delta = |x_1| + \sup_k |\Delta x_k|.$$

The idea of Kizmaz [4] was applied to introduce different type of difference sequence spaces and study their different properties by Tripathy ([11],[12]), Tripathy and Esi [13] and many others.

Tripathy and Esi [13] introduced the new type of difference sequence spaces, for fixed $m \in N$ by

$$Z(\Delta_m) = \{x = (x_k) : (\Delta_m X_k) \in Z\},$$

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for $Z = \ell_\infty, c$ and c_0 where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$.

This generalizes the notion of difference sequence spaces studied by Kizmaz [4].

The above spaces are Banach spaces, normed by

$$\| x \|_{\Delta_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k|.$$

Sargent [9] introduced the crisp set sequence space $m(\phi)$ and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as $m(\phi)$ by Rath and Tripathy [8], Tripathy [11], Tripathy and Sen [14] and others. In this article we introduce the space $m(\phi, \Delta_m, p)^F$ of fuzzy real numbers for $0 < p < \infty$ and $m \geq 0$, an integer.

Throughout the article w^F, ℓ^F, ℓ_∞^F represent the classes of *all*, *absolutely summable* and *bounded* sequences of fuzzy real numbers respectively.

2. Definitions and background

A fuzzy real number X is a fuzzy set on R i.e. a mapping $X : R \rightarrow I$ $(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal.*

A fuzzy real number X is said to be *upper semi continuous* if for each $\varepsilon > 0, X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The class of all *upper semi continuous, normal, convex* fuzzy real numbers is denoted by $R(I)$. For $X \in R(I)$, the α-level set X^α for $0 < \alpha \leq 1$ is defined by, $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level i.e. X^0 is the closure of strong 0-cut, i.e. $X^0 = cl\{t \in R : X(t) > 0\}$.

The absolute value of $X \in R(I)$ i.e. $|X|$ is defined as(see Kaleva and Seikkala [4])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\} & \text{for} \quad t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $r \in R, \overline{r} \in R(I)$ is defined as,

$$\overline{r}(t) = \begin{cases} 1 & \text{for} \quad t = r, \\ 0 & \text{otherwise.} \end{cases}$$

The additive identity and multiplicative identity of $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$ respectively. The zero sequence of fuzzy real numbers is denoted by $\overline{\theta}$.

Let D be the set of all closed bounded intervals $X = [X^L, X^R]$.

Define $d : D \times D \to R$ by $d(X, Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. Then clearly (D, d) is a complete metric space.

Define $\overline{d} : R(I) \times R(I) \to R$ by $\overline{d}(X, Y) = \sup_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$, for $X, Y \in R(I)$. Then it is well known that $(R(I), \overline{d})$ is a complete metric space.

A sequence $X = (X_k)$ of fuzzy real numbers is said to converge to the fuzzy number X_0, if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $\overline{d}(X_k, X_0) < \varepsilon$, for all $k \geq k_0$.

A sequence space E is said to be *solid* if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \leq |X_n|$, for all $n \in N$.

A sequence space E is said to be *monotone* if E contains the canonical pre-images of all its step spaces.

Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of (X_n) i.e. $S(X) = \{(X_{\pi(n)}) : \pi$ is a permutation of $N\}$. A sequence space E is said to be *symmetric* if $S(X) \subset E$ for all $X \in E$.

A sequence space E is said to be *convergence-free* if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = \overline{0}$ implies $Y_n = \overline{0}$.

Remark. A sequence space E is solid implies that E is monotone.

Let $\wp_s$ be the class of all subsets of N those do not contain more than S number of elements.

Throughout (ϕ_n) is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in N$.

The space $m(\phi)$ introduced by Sargent [9] is defined as,

$$m(\phi) = \{(x_k) \in w : \| x \|_{m(\phi)} = \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty\}.$$

Tripathy and Sen [14] generalized this sequence space and introduced the sequence space $m(\phi, p)$ defined as follows

$$\begin{aligned} m(\phi, p) &= \{(x_k) \in w : \| x \|_{m(\phi,p)} \\ &= \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k|^p < \infty\} \qquad \text{for } 0 < p < \infty. \end{aligned}$$

We introduce the sequence space $m(\phi, \Delta_m, p)^F$ of fuzzy real numbers as follows

$$\begin{aligned} m(\phi, \Delta_m, p)^F &= \{X = (X_k) : \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} (\overline{d}(\Delta_m X_k, \overline{0}))^p < \infty\} \\ &\qquad \text{for } 0 < p < \infty. \end{aligned}$$

3. Main results

In this section, we prove some results involving the sequence space $m(\phi, \Delta_m, p)^F$ with two values of p such that $0 < p < \infty$.

Theorem 1. (*a*) *The sequence space* $m(\phi, \Delta_m, p)^F$ *for* $0 < p < 1$ *is a complete metric space by the metric,*

$$\rho(X,Y) = \sum_{r=1}^{m} \overline{d}(X_r, Y_r) + \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[\overline{d}(\Delta_m X_k, \Delta_m Y_k)\right]^p \quad \textit{for} \quad X, Y \in m(\phi, \Delta_m, p)^F.$$

(*b*) *The sequence space* $m(\phi, \Delta_m, p)^F$ *for* $0 < p < 1$ *is a complete metric space by the metric,*

$$\eta(X,Y) = \sum_{r=1}^{m} \overline{d}(X_r, Y_r) + \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \left[\sum_{k\in\sigma} \left[\overline{d}(\Delta_m X_k, \Delta_m Y_k)\right]^p\right]^{\frac{1}{p}} \quad \textit{for} \quad X, Y \in m(\phi, \Delta_m, p)^F.$$

Proof. (*a*) Clearly, $m(\phi, \Delta_m, p)^F$ is a metric space with the above defined metric ρ.

We have to prove that it is a complete metric space.

Let $(X^{(i)})$ be a Cauchy sequence in $m(\phi, \Delta_m, p)^F$ such that $X^{(i)} = (X^{(i)})_{n=1}^{\infty}$. Then we have for any $\epsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that,

$$\rho(X^{(i)}, X^{(j)}) < \varepsilon \qquad \text{for} \quad i, j \geq n_0$$

$$(1) \quad \Rightarrow \sum_{r=1}^{m} \overline{d}(X_r^{(i)}, X_r^{(j)}) + \sup_{s\geq 1,\, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \left[\overline{d}(\Delta_m X_k^{(i)}, \Delta_m X_k^{(j)})\right]^p < \varepsilon \quad \text{for all} \quad i, j \geq n_0.$$

Which implies that,

$$\sum_{r=1}^{m} \overline{d}(X_r^{(i)}, X_r^{(j)}) < \varepsilon \qquad \text{for all} \quad i, j \geq n_0$$

$$\Rightarrow \overline{d}(X_r^{(i)}, X_r^{(j)}) < \varepsilon \qquad \text{for all} \quad i, j \geq n_0, \quad r = 1, 2, 3, ...m.$$

Hence, $(X_r^{(i)})$ is a Cauchy sequence in $R(I)$, so it is convergent in $R(I)$, by the completeness property of $R(I)$, for $r = 1, 2, 3, ...m$.

Let
$$\lim_{i\to\infty} X_r^{(i)} \;=\; X_r, \qquad \text{for} \quad r=1,2,3,....m.$$

Also, $\sup\limits_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\sum\limits_{k\in\sigma}[\overline{d}(\Delta_m X_k^{(i)},\Delta_m X_k^{(j)})]^p<\varepsilon$, for all $i,j\geq n_0$.

On taking $s=1$, we have,
$$\overline{d}(\Delta_m X_k^{(i)},\Delta_m X_k^{(j)})<(\varepsilon\phi_1)^{\frac{1}{p}}, \qquad \text{for all} \quad i,j\geq n_0 \ \text{ and } \ k\in N.$$

Which implies that for each fixed $k(1\leq k<\infty)$, the sequence $(\Delta_m X_k^{(i)})$ is a Cauchy sequence in $R(I)$, hence converges in $R(I)$.

Let, $\lim\limits_{i\to\infty}\Delta_m X_k^{(i)}=Y_k$ (say), in $R(I)$, for each $k\in N$.

For $k=1$, we get, $(X_1^{(i)})$ and $(X_1^{(i)}-X_{m+1}^{(i)})$ are convergent. Hence $(X_{m+1}^{(i)})$ is convergent.

On applying the principle of induction, we get, $\lim\limits_{i\to\infty}X_k^{(i)}=X_k$ exists for each $k\in N$.

Taking limit as $j\to\infty$ in (1), we have,
$$(2)\quad \sum_{r=1}^{m}\overline{d}(X_r^{(i)},X_r) \;+\; \sup_{s\geq 1,\,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}[\overline{d}(\Delta_m X_k^{(i)},\Delta_m X_k)]^p<\;\varepsilon,$$
$$\text{for all} \quad i\geq n_0,\quad m\geq 0.$$

$$\Rightarrow \rho(X^{(n)},X)<\;\varepsilon, \qquad \text{for all} \quad n\geq n_0.$$

Since $(X^{(i)})\in m(\phi,\Delta_m,p)^F$ and by (2), for all $i\geq n_0$, we have, $\rho(X,\theta)\leq \rho(X^{(i)},X)+\rho(X^{(i)},\theta)<\infty$. Hence, $X\in m(\phi,\Delta_m,p)^F$. Hence, $m(\phi,\Delta_m,p)^F$ is a complete metric space.

This completes the proof of the theorem. ■

(b) This part can be proved by following similar techniques.

Theorem 2. *The sequence space $m(\phi,\Delta_m,p)^F$ is not solid for $0<\;p<\;\infty$.*

Proof. The proof follows from the following example. ■

Example 1. Let $m=3,p=2$ and $\phi_s=1$, for all $s\in\; N$.

Let $X_k=\overline{1}$ for all $k\in N$. Then, we have, $\overline{d}(\Delta_3 X_k,\overline{0})=0$ for all $k\in N$. Hence, $\sup\limits_{s\geq 1,\,\sigma\in\wp_s}\frac{1}{\phi_s}\sum\limits_{k\in\sigma}[\overline{d}(\Delta_m X_k,\overline{0})]^p=0$. Which implies that, $(X_k)\in m(\phi,\Delta_3,2)^F$. Consider the sequence (α_k) of scalars defined by
$$\alpha_k=\begin{cases}1 & \text{for } k \text{ is even,}\\ 0 & \text{otherwise.}\end{cases}$$

So, $\overline{d}(\Delta_3\alpha_k X_k, \overline{0}) = 1$ for all $k \in N$. Which implies that,

$$\sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}[\overline{d}(\Delta_m X_k, \overline{0})]^p = \sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{1}\sum_{k\in\sigma}1 = \sup_{s\geq 1,\sigma\in\wp_s} s = \infty.$$

Which shows that,$(\alpha_k X_k) \notin m(\phi, \Delta_3, 2)^F$. Hence, $m(\phi, \Delta_m, p)^F$ is not solid.

Theorem 3. *The sequence space $m(\phi, \Delta_m, p)^F$ is not symmetric for $0 < p < \infty$.*

Proof. The result follows from the following example. ■

Example 2. Let $m = 1$, $\phi_s = s$, for all $s \in N$. Let, $X_k = \overline{k}$, for all $k \in N$. Then, $\overline{d}(\Delta X_k, \overline{0}) = 1$, for all $k \in N$. Let (Y_k) be a rearrangement of (X_k) such that,

$$(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, ...).$$

Which shows that, $\sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}[\overline{d}(\Delta_m Y_k, \overline{0})]^p = \infty$. Hence, $(Y_k) \notin m(\phi, \Delta_m, p)^F$. Thus, $m(\phi, \Delta_m, p)^F$ is not symmetric.

Proposition 1. *The sequence space $m(\phi, \Delta_m, p)^F$ is not convergence-free, for $0 < p < 1$ and $1 \leq p < \infty$.*

Proof. The result follows from the following example. ■

Example 3. Let $p = \frac{1}{2}$ and $\phi_s = s$ for all $s \in N$. Consider the sequence (X_k) defined as follows:

$$X_k(t) = \begin{cases} 1 + kt & \text{for} \quad t \in [-\frac{1}{k}, 0]\,, \\ 1 - kt & \text{for} \quad t \in [0, \frac{1}{k}]\,, \\ 0 & \quad\text{otherwise}\,. \end{cases}$$

Then,

$$\Delta_m X_k(t) = \begin{cases} 1 + \frac{k(k+m)}{2k+m}t & \text{for} \quad t \in [-\frac{2k+m}{k(k+m)}, 0]\,, \\ 1 - \frac{k(k+m)}{2k+m}t & \text{for} \quad t \in [0, \frac{2k+m}{k(k+m)}]\,, \\ 0 & \quad\text{otherwise}\,. \end{cases}$$

Such that, $\overline{d}(\Delta_m X_k, \overline{0}) = \frac{2k+m}{k(k+m)} = \frac{2}{(k+m)} + \frac{m}{k(k+m)} < \infty$, $m \geq 1$.

Then,

$$\sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\left[\overline{d}(\Delta_m X_k,\overline{0})\right]^p$$
$$= \sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{s}\sum_{k\in\sigma}\left[\frac{2}{(k+m)}+\frac{m}{k(k+m)}\right]^{\frac{1}{2}} < \infty.$$

Thus, $(X_k) \in m(s, \Delta_m, \frac{1}{2})^F$.

Now, let us take another sequence (Y_k) such that,

$$Y_k(t) = \begin{cases} 1+\frac{t}{k^2} & \text{for} \quad t\in[-k^2,0]\,, \\ 1-\frac{t}{k^2} & \text{for} \quad t\in[0,k^2]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

for all $k \in N$. So that,

$$\Delta_m Y_k(t) = \begin{cases} 1+\frac{t}{2k^2+2km+m^2} & \text{for} \quad t\in[-(2k^2+2km+m^2),0]\,, \\ 1-\frac{t}{2k^2+2km+m^2} & \text{for} \quad t\in[0,(2k^2+2km+m^2)]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

for all $k \in N$. But, $\overline{d}(\Delta_m Y_k, \overline{0}) = (2k^2 + 2km + m^2)$, for all $m \geq 1$. Which implies that, $\sup\limits_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{s}\sum\limits_{k\in\sigma}(2k^2+2km+m^2)^{\frac{1}{2}} = \infty$. Thus, $(Y_k) \notin m(s, \Delta_m, \frac{1}{2})^F$. Hence $m(\phi, \Delta_m, p)^F$ is not convergence-free, for $0 < p < 1$. Similarly, it can be proved that $m(\phi, \Delta_m, p)^F$ is not convergence-free for $1 \leq p < \infty$. The following result is a consequence of Lemma and Proposition 1.

Proposition 2. *$m(\phi, \Delta_m)^F \subseteq m(\phi, \Delta_m, p)^F$.*

Proof. Let $X \in m(\phi, \Delta_m)^F$, then we have

$$\sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\overline{d}(\Delta_m X_k,\overline{0}) = K(<\infty).$$

Hence, for each fixed s, we have

$$\sum_{k\in\sigma}\overline{d}(\Delta_m X_k,\overline{0}) \leq K\phi_s, \qquad \text{for} \quad \sigma\in\wp_s,\;\; m\geq 1.$$

$$\Rightarrow \left[\sum_{k\in\sigma}\left\{\overline{d}\left(\Delta_m X_k,\overline{0}\right)\right\}^p\right]^{\frac{1}{p}} \leq K\phi_s$$

$$\Rightarrow \sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\left[\sum_{k\in\sigma}\left\{\bar{d}\left(\Delta_m X_k,\bar{0}\right)\right\}^p\right]^{\frac{1}{p}} \leq K, \quad m\geq 1.$$

$$\text{i.e.} \quad \sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\left[\sum_{k\in\sigma}\left\{\bar{d}\left(\Delta_m X_k,\bar{0}\right)\right\}^p\right]^{\frac{1}{p}} < \infty.$$

Which implies that, $X\in m(\phi,\Delta_m,p)^F$, for $1\leq p<\infty$. This completes the proof. ■

Proposition 3. *$m(\phi,\Delta_m,p)^F\subseteq m(\psi,\Delta_m,p)^F$, if and only if $\sup\limits_{s\geq 1}(\frac{\phi_s}{\psi_s}) < \infty$.*

Proof. Suppose, $\sup\limits_{s\geq 1}(\frac{\phi_s}{\psi_s})=K(<\infty)$, then we have, $\phi_s\leq K\psi_s$.
Now, if $(X_k)\in m(\phi,\Delta_m,p)^F$, then

$$\sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{\phi_s}\left[\sum_{k\in\sigma}\left\{\bar{d}\left(\Delta_m X_k,\bar{0}\right)\right\}^p\right]^{\frac{1}{p}} < \infty$$

$$\Rightarrow \sup_{s\geq 1,\,\sigma\in\wp_s} \frac{1}{K\psi_s}\left[\sum_{k\in\sigma}\left\{\bar{d}\left(\Delta_m X_k,\bar{0}\right)\right\}^p\right]^{\frac{1}{p}} < \infty,$$

i.e. $(X_k)\in m(\psi,\Delta_m,p)^F$. Hence, $m(\phi,\Delta_m,p)^F\subseteq m(\psi,\Delta_m,p)^F$. Conversely, suppose that $m(\phi,\Delta_m,p)^F\subseteq m(\psi,\Delta_m,p)^F$. To show that, $\sup\limits_{s\geq 1}(\frac{\phi_s}{\psi_s})=\sup\limits_{s\geq 1}(\eta_s)<\infty$. Suppose if possible, $\sup\limits_{s\geq 1}(\eta_s)=\infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that,

$$\lim_{i\to\infty}(\eta_{s_i})=\infty.$$

Then for $(X_k)\in m(\phi,\Delta_m,p)^F$, we have $\sup\limits_{s\geq 1,\,\sigma\in\wp_s}\frac{1}{\psi_s}[\sum\limits_{k\in\sigma}\{\bar{d}(\Delta_m X_k,\bar{0})\}^p]^{\frac{1}{p}}\geq \sup\limits_{s\geq 1,\sigma\in\wp_s}\frac{\eta_{s_i}}{\phi_{s_i}}[\sum\limits_{k\in\sigma}\{\bar{d}(\Delta_m X_k,\bar{0})\}^p]^{\frac{1}{p}}=\infty$. i.e. $\sup\limits_{s\geq 1,\,\sigma\in\wp_s}\frac{1}{\psi_s}[\sum\limits_{k\in\sigma}\{\bar{d}(\Delta_m X_k,\bar{0})\}^p]^{\frac{1}{p}}=\infty$, which implies that $(X_k)\notin m(\psi,\Delta_m,p)^F$, a contradiction.
This completes the proof. ■

Corollary 1. *$m(\phi,\Delta_m,p)^F=m(\psi,\Delta_m,p)^F$, if and only if $\sup\limits_{s\geq 1}(\eta_s)<\infty$ and $\sup\limits_{s\geq 1}(\eta_s^{-1})<\infty$, where $\eta_s=\frac{\phi_s}{\psi_s}$ for $0<p<\infty$.*

Theorem 4. *$\ell_p(\Delta_m)^F\subseteq m(\phi,\Delta_m,p)^F\subseteq \ell_\infty(\Delta_m)^F$ for $1\leq p<\infty$.*

Proof. Since $m(\phi, \Delta_m, p)^F = \ell_p(\Delta_m)^F$ for $\phi_n = 1$ and $0 < p < 1$ and for all $n \in N$.

So, the first inclusion is clear. Next, suppose that, $(X_k) \in m(\phi, \Delta_m, p)^F$, that implies that,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} [\sum_{k \in \sigma} \{\overline{d}(\Delta_m X_k, \overline{0})\}^p]^{\frac{1}{p}} = K(< \infty).$$

For $s = 1$, $\overline{d}(\Delta_m X_k, \overline{0}) \leq K\phi_1$, $k \in \sigma$, which implies that, $\sup_{k \geq 1}\{\overline{d}(\Delta_m X_k, \overline{0})\} < \infty$ which implies that, $X_k \in \ell_\infty(\Delta_m)^F$.

This completes the proof. ■

Putting $\psi_n = 1$, for all $n \in N$, in Corollary 1, we get

Proposition 4. *$m(\phi, \Delta_m, p)^F = \ell_p(\Delta_m)^F$ if and only if $\sup_{s \geq 1}(\phi_s) < \infty$ and $\sup_{s \geq 1}(\phi_s^{-1}) < \infty$.*

Using the properties of ℓ_p spaces, we get the following results.

Proposition 5. *If $p < q$, then $m(\phi, \Delta_m, p)^F \subset m(\phi, \Delta_m, q)^F$.*

Proposition 6. *$m(\phi, \Delta_m, p)^F \subset m(\psi, \Delta_m, q)^F$ if $p < q$ and $\sup_{s \geq 1}(\frac{\phi_s}{\psi_s}) < \infty$.*

Corollary 2. *$m(\phi, \Delta_m, p)^F = \ell_p(\Delta_m)^F$ if $\lim_{s \to \infty}(\frac{\phi_s}{s}) > 0$.*

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