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# STATISTICALLY CONVERGENT DIFFERENCE SEQUENCE SPACES OF FUZZY REAL NUMBERS

Abstract. In this article we introduce the notion of statistical convergence difference sequences of fuzzy real numbers, $\bar{c}^F(\Delta)$. We study some properties of the statistically convergent and statistically null difference sequence spaces of fuzzy real numbers, like completeness, solidness, sequence algebra, symmetricity, convergence free, nowhere denseness and some inclusion results.

Key words: fuzzy real number, statistical convergence, difference sequence, solid space, symmetric space, sequence algebra, convergence free.

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## 1. Introduction

In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [2] and Schoenberg [11] independently. It was also found in Zygmund [15]. Later on it was studied from sequence space point of view and linked with summability by Fridy [3], Šalát [9], Tripathy ([12], [13]), Rath and Tripathy [8] and many others.

The idea depends on certain density of the subsets of the set $N$ of natural numbers.

A subset $E$ of $N$ is said to have *natural density* $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \ \text{ exists},$$

where $\chi_E(k)$ is the characteristic function of $E$. Clearly all finite subsets of $N$ have zero natural density and $\delta(E^c) = \delta(N - E) = 1 - \delta(E)$.

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A sequence $(X_k)$ is said to be *statistically convergent* to $L$ if for every $\varepsilon > 0$, $\delta(\{k \in N : |X_k - L| \geq \varepsilon\}) = 0$. We write $X_k \xrightarrow{stat} L$ or stat-lim $X_k = L$.

Let $(X_k)$ and $(Y_k)$ be two sequences, then we say that $X_k = Y_k$ for *almost all* $k$ (in short *a.a.k.* ) if $\delta(k \in N : X_k \neq Y_k) = 0$.

Kizmaz [5] initiated the works on classical difference sequence spaces $c(\Delta)$, $c_0(\Delta)$ and $\ell_\infty(\Delta)$ defined as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c$, $c_0$ and $\ell_\infty$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in N$.

Let $D$ denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on $R$, the real line. For $X, Y \in D$ we define

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that $(D, d)$ is a complete metric space.

A *fuzzy real number* $X$ is a fuzzy set on $R$, i.e. a mapping $X : R \to I = ([0, 1])$ associating each real number $t$ with its grade of membership $X(t)$.

A fuzzy real number $X$ is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number $X$ is called *normal*.

A fuzzy real number $X$ is said to be *upper-semi continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of $R$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The *$\alpha$-level set* $[X]^\alpha$ of the fuzzy real number $X$, for $0 < \alpha \leq 1$, defined as $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-cut.

The arithmetic operations for $\alpha$-level sets are defined as follows:

Let $X, Y \in R(I)$ and $\alpha$-level sets be $[X]^\alpha = [a_1^\alpha, b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha, b_2^\alpha]$, $\alpha \in [0, 1]$. Then

$$\begin{aligned}
[X \oplus Y]^\alpha &= [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha], \\
[X - Y]^\alpha &= [a_1^\alpha - b_2^\alpha, b_1^\alpha - a_2^\alpha], \\
[X \otimes Y]^\alpha &= \left[\min_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha, \max_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha\right] \\
\text{and} \qquad [Y^{-1}]^\alpha &= \left[\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha}\right], \quad 0 \notin Y.
\end{aligned}$$

The set $R$ of all real numbers can be embedded in $R(I)$. For $r \in R$, $\overline{r} \in R(I)$is defined by

$$\bar{r}(t) = \begin{cases} 1 & \text{for} \quad t = r, \\ 0 & \text{for} \quad t \neq r, \end{cases}$$

For $r \in R$ and $X \in R(I)$ we define

$$rX(t) = \begin{cases} X(r^{-1}t) & \text{for} \quad r \neq 0, \\ \bar{0} & \text{for} \quad r = 0, \end{cases}$$

The *absolute value*, $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and Seikkala [4] )

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\} & \text{for} \quad t \geq 0, \\ 0 & \text{for} \quad t < 0, \end{cases}$$

A fuzzy real number $X$ is called *non-negative* if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d\left([X]^\alpha, [Y]^\alpha\right).$$

Then $\bar{d}$ defines a metric on $R(I)$.

The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

A sequence $(X_k)$ of fuzzy real numbers is said to be *convergent* to the fuzzy real number $X_0$ if, for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \geq n_0$.

A fuzzy real number sequence $(X_k)$ is said to be *bounded* if $|X_k| \leq \mu$, for some $\mu \in R^*(I)$.

## 2. Definitions and preliminaries

Nuray and Savas [7] defined the notion of statistical convergence for sequences of fuzzy real numbers and studied some properties.

A fuzzy real number sequence $(X_k)$ is said to be *statistically convergent* to the fuzzy real number $X_0$ if, for every $\varepsilon > 0$, $\delta(\{k \in N : \bar{d}(X_k, X_0) \geq \varepsilon\}) = 0$.

Savas [10] studied the difference sequence spaces $c^F(\Delta)$ and $\ell_\infty^F(\Delta)$ of fuzzy real numbers.

A fuzzy real number sequence $\Delta X = (\Delta X_k)$ is said to be *convergent* to a fuzzy real number $X$, written as $\lim_{k\to\infty} \Delta X_k = X$ if, for every $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$\bar{d}\,(\Delta X_k, X) < \varepsilon \quad \text{for all} \quad k > n_0.$$

A fuzzy real number sequence $\Delta X = (\Delta X_k)$ is said to be *bounded* if $|\Delta X_k| \leq \mu$, for some $\mu \in R^*(I)$.

In this article we define the statistical convergence difference sequences of fuzzy real numbers as follows:

A fuzzy real number sequence $\Delta X = (\Delta X_k)$ is said to be *statistically convergent* to the fuzzy real number $X_0$ if, for every $\varepsilon > 0$, $\delta(\{k \in N : \bar{d}(\Delta X_k, X_0) \geq \varepsilon\}) = 0$.

Throughout the article $w^F$, $\ell_\infty^F$, $c^F$, $c_0^F$, $\bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$ denote the classes of *all*, *bounded*, *convergent*, *null*, *statistically convergent*, *bounded statistically convergent*, *statistically null* and *bounded statistically null* fuzzy real number sequences respectively. Similarly $c^F(\Delta)$, $c_0^F(\Delta)$, $\ell_\infty^F(\Delta)$, $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ denote the classes of *convergent*, *null*, *bounded*, *statistically convergent*, *bounded statistically convergent*, *statistically null* and *bounded statistically null difference sequences* of fuzzy real numbers.

A sequence space $E^F$ is said to be *normal* (or *solid*) if $(Y_k) \in E^F$, whenever $|Y_k| \leq |X_k|$, for all $k \in N$, for some $(X_k) \in E^F$.

A sequence space $E^F$ is said to be *monotone* if $E^F$ contains the canonical pre-images of all its step spaces.

Let $K = \{k_1 < k_2 < k_3 \ldots\} \subseteq N$ and $E^F$ be a sequence space. A *K-step space* of $E^F$ is a sequence space $\lambda_k^{E^F} = \{(X_{k_n}) \in w^F : (X_n) \in E^F\}$.

A *canonical pre-image of a sequence* $(X_{k_n}) \in \lambda_k^{E^F}$ is a sequence $(Y_n) \in w^F$ defined as follows:

$$Y_n = \begin{cases} X_n & \text{for} \quad n \in K, \\ \bar{0} & \text{otherwise.} \end{cases}$$

A *canonical pre-image of a step space* $\lambda_k^{E^F}$ is a set of canonical pre-images of all elements in $\lambda_k^{E^F}$, i.e. $Y$ is in canonical pre-image $\lambda_k^{E^F}$ if and only if $Y$ is canonical pre-image of some $X \in \lambda_k^{E^F}$.

From the above definitions we have the following remark.

**Remark 1.** A sequence space $E^F$ is solid $\Rightarrow$ $E^F$ is monotone.

A sequence space $E^F$ is is said to be *symmetric* if $(X_{\pi(n)}) \in E^F$, whenever $(X_k) \in E^F$, where $\pi$ is a permutation of $N$.

A sequence space $E^F$ is is said to be *sequence algebra* if $(X_k \otimes Y_k) \in E^F$, whenever $(X_k), (Y_k) \in E^F$.

A sequence space $E^F$ is is said to be *convergence free* if $(Y_k) \in E^F$, whenever $(X_k) \in E^F$ and $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

**Remark 2.** For the crisp set, $x_k \xrightarrow{stat} L$ implies $\Delta x_k \xrightarrow{stat} 0$. This conjecture fails in case of sequences of fuzzy real numbers and $\Delta X_k \xrightarrow{stat} X$, where $X$ is of particular type, defined by $[X]^\alpha = [-a, a]$ for some $a = a(\alpha) \in R_+ \cup \{0\}$, the set of non-negative real numbers. This is clear from the following example.

**Example 1.** Consider the sequence $(X_k)$ as follows:
For $k = n^2$, $n \in N$,

$$X_k(t) = \begin{cases} 1 + 2^{-1}k(t-1) & \text{for} \quad 1 - 2k^{-1} \le t \le 1, \\ 1 < t \le 1 + 2k^{-1} & \text{for} \quad 1 < t \le 1 + 2k^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise,

$$X_k(t) = \begin{cases} t - 3 & \text{for} \quad 3 \le t \le 4, \\ 1 < t \le 1 + 2k^{-1} & \text{for} \quad 4 < t \le 5, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} 1 - 2k^{-1}(1-\alpha), 1 + 2k^{-1}(1-\alpha)] & \text{for} \quad k = n^2, \quad n \in N, \\ [3+\alpha,\ 5-\alpha] & \text{otherwise.} \end{cases}$$

and

$$[\Delta X_K]^\alpha = \begin{cases} [(\alpha - 4 - \frac{2(1-\alpha)}{k}), (\frac{2(1-\alpha)}{k} - 2 - \alpha)] & \text{for} \quad k = n^2, \quad n \in N, \\ [(2 + \alpha - \frac{2(1-\alpha)}{(k+1)}), (4 - \alpha + \frac{2(1-\alpha)}{(k+1)})] & \text{for} \quad k = n^2 - 1, \\ & \qquad n > 1 \text{ with } n \in N, \\ [2\alpha - 2,\ 2 - 2\alpha] & \text{otherwise.} \end{cases}$$

i.e. $X_k \xrightarrow{stat} L$, where $[L]^\alpha = [3+\alpha, 5-\alpha]$ and $\Delta X_k \xrightarrow{stat} X$, where $[X]^\alpha = [2\alpha - 2, 2 - 2\alpha]$.

Thus $(X_k) \in \bar{c}^F$ but $(X_k) \notin \bar{c_0}^F(\Delta)$.

**Lemma (Savas [10], Theorem 1).** *$\ell_\infty^F(\Delta)$ and $c^F(\Delta)$ are complete metric spaces with the metric*

$$\rho(X, Y) = \bar{d}(X_1, Y_1) + \sup_k \bar{d}(\Delta X_k, \Delta Y_k),$$

*where $X = (X_k)$ and $Y = (Y_k)$ are in $\ell_\infty^F(\Delta)$ and $c^F(\Delta)$.*

## 3. Main results

In this section we prove the results of this article.

**Theorem 1.** *The class of sequences $c^F(\Delta)$, ${c_0}^F(\Delta)$ and $\ell^F_\infty(\Delta)$ are closed under the operations of addition and scalar multiplication.*

**Proof.** We prove the result for the class of sequences $c^F(\Delta)$. The proof for the other two classes of sequences can be established following similar method.

Let us consider sequences $(X_k), (Y_k) \in c^F(\Delta)$.

Then

$$\begin{aligned}(\Delta[X_k \oplus Y_k]) &= ([X_k \oplus Y_k] - [X_{k+1} \oplus Y_{k+1}]) \\ &= ([X_k - X_{k+1}] \oplus [Y_k - Y_{k+1}]) = (\Delta X_k \oplus \Delta Y_k) \in c^F, \\ &\quad [\text{since } (\Delta X_k), (\Delta Y_k) \in c^F \text{and } c^F \text{is closed under addition} \\ &\quad \text{and scalar multiplication}]\end{aligned}$$

Again, for $r \in R$, $(X_k) \in c^F(\Delta)$, we have

$$\begin{aligned}(\Delta r X_k) &= ([rX_k] - [rX_{k+1}]) = (r[X_k - X_{k+1}]) = (r\Delta X_k) \in c^F, \\ &\quad [\text{since } r \in R,\ (\Delta X_k) \in c^F \text{and } c^F \text{is a closed under scalar} \\ &\quad \text{multiplication}].\end{aligned}$$

Hence $c^F(\Delta)$ is closed under addition and scalar multiplication. ■

**Theorem 2.** $m^F(\Delta) = \bar{c}^F(\Delta) \cap \ell^F_\infty(\Delta)$ *and* $m^F_0(\Delta) = \bar{c_0}^F(\Delta) \cap \ell^F_\infty(\Delta)$ *are closed subspaces of the complete metric space* $\ell^F_\infty(\Delta)$ *with the metric* $\rho$ *defined by*

$$\rho(X, Y) = \bar{d}(X_1, Y_1) + \sup_k \bar{d}(\Delta X_k, \Delta Y_k),$$

*where* $X = (X_k)$ *and* $Y = (Y_k)$ *are in* $m^F(\Delta)$ *or* $m^F_0(\Delta)$.

**Proof.** We prove the result for the case of $m^F(\Delta)$. Another can be established by similar technique.

Let $(X^{(n)})$ be a Cauchy sequence in $m^F(\Delta)$. Then $(X^{(n)})$ is a Cauchy sequence in $\ell^F_\infty(\Delta)$. Since $\ell^F_\infty(\Delta)$ is complete (see [10]), so $X^{(n)} \to X$ in $\ell^F_\infty(\Delta)$. We shall show that

$$X \in m^F(\Delta).$$

Since $X^{(n)} = (X^{(n)}_k) = (X^{(n)}_1, X^{(n)}_2, X^{(n)}_3, \ldots) \in m^F(\Delta)$, so for each $n \in N$ there exists $A_n \in R(I)$ such that

$$\text{stat} - \lim \Delta X^{(n)}_k = A_n.$$

We prove the followings:

$(i)$ $\lim_{n\to\infty} A_n = A$.

$(ii)$ stat-lim $\Delta X_k = A$.

$(i)$. Since $(X^{(n)})$ is a convergent sequence, so for a given $\varepsilon > 0$, there exists such a $n_0 \in N$ that for each $m, n > n_0$ we have

$$\begin{aligned} (1) \quad \rho(X^{(m)}, X^{(n)}) &= \bar{d}(X_1^{(m)}, X_1^{(n)}) + \sup_k \bar{d}(\Delta X_k^{(m)}, \Delta X_k^{(n)}) < \frac{\varepsilon}{3} \\ &\Rightarrow \bar{d}(\Delta X_k^{(m)}, \Delta X_k^{(n)}) < \frac{\varepsilon}{3} \quad \text{for each} \quad k \in N \end{aligned}$$

Again, since $X^{(n)} = (X_k^{(n)}) \in m^F(\Delta)$, so for a given $\varepsilon > 0$, we have

$$(2) \qquad \bar{d}(\Delta X_k^{(m)}, A_m) < \frac{\varepsilon}{3} \quad \text{for} \quad a.a.k.$$

and

$$(3) \qquad \bar{d}(\Delta X_k^{(n)}, A_n) < \frac{\varepsilon}{3} \quad \text{for} \quad a.a.k.$$

Now for each $m, n > n_0 \in N$ and from the inequalities (1), (2) and (3), we get

$$\begin{aligned} \bar{d}(A_m, A_n) &\leq \bar{d}(A_m, \Delta X_k^{(m)}) + \bar{d}(\Delta X_k^{(m)}, \Delta X_k^{(n)}) \\ &\quad + \bar{d}(\Delta X_k^{(n)}, A_n), \text{ for } a.a.k. \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus $(A_n)$ is a Cauchy sequence in $R(I)$. Since $R(I)$ complete, so there exists such a number $A \in R(I)$ such that

$$\lim_{n\to\infty} A_n = A.$$

$(ii)$. We have $X^{(n)} \to X$. For a given $\lambda > 0$, there exists such a $q \in N$ that

$$\begin{aligned} (4) \qquad & \bar{d}(X_1^{(q)}, X_1) + \sup_k \bar{d}(\Delta X_k^{(q)}, \Delta X_k) < \frac{\lambda}{3}. \\ & \Rightarrow \bar{d}(\Delta X_k^{(q)}, \Delta X_k) < \frac{\lambda}{3}, \quad \text{for each } k \in N. \end{aligned}$$

The number $q$ can be chosen in such a way that together with (4), we get

$$(5) \qquad \bar{d}(A_q, A) < \frac{\lambda}{3}.$$

Since, stat-lim $\Delta X_k^{(q)} = A_q$. For a given $\lambda > 0$,

$$\bar{d}(\Delta X_k^{(q)}, A_q) < \frac{\lambda}{3}, \quad \text{for} \quad a.a.k. \tag{6}$$

Now,

$$\begin{aligned}\bar{d}(\Delta X_k, A) &\leq \bar{d}(\Delta X_k, \Delta X_k^{(q)}) + \bar{d}(\Delta X_k^{(q)}, A_q) + \bar{d}(A_q, A), \text{ for } a.a.k. \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \lambda, \quad \text{by (4), (5) and (6).}\end{aligned}$$

Hence stat-lim $\Delta X_k = A$. This proves the result. ∎

**Theorem 3.** *The sequence spaces $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are neither monotone nor solid.*

**Proof.** The result follows from the following example. ∎

**Example 2.** Consider the sequence $(X_k) \in Z(\Delta)$, for $Z = \bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$ defined as follows:

For $k = n^2$, $n \in N$,

$$X_k(t) = \begin{cases} 1 + k(t-4) & \text{for} \quad 4 - k^{-1} \leq t \leq 4, \\ 1 - k(t-4) & \text{for} \quad 4 < t \leq 4 + k^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise,

$$X_k(t) = \begin{cases} 1 - 2^{-1}k(t-1) & \text{for} \quad 1 \leq t \leq 1 + 2k^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [4 - k^{-1}(1-\alpha), 4 + k^{-1}(1-\alpha)] & \text{for} \quad k = n^2, \ n \in N, \\ [1, 1 + 2k^{-1}(1-\alpha)] & \text{otherwise.} \end{cases}$$

and

$$[\Delta X_K]^\alpha = \begin{cases} [3 + (\alpha - 1)(\frac{1}{k} + \frac{2}{(k+1)}), \frac{(1-\alpha)}{k} + 3] & \text{for} \quad k = n^2, \ n \in N, \\ [\frac{(\alpha-1)}{(k+1)} - 3, (1-\alpha)(\frac{2}{k} + \frac{1}{(k+1)}) - 3] & \text{for} \quad k = n^2 - 1, \\ & \qquad n > 1 \text{ with } n \in N, \\ [\frac{2(\alpha-1)}{(k+1)}, \frac{2(1-\alpha)}{k}] & \text{otherwise.} \end{cases}$$

Thus $(X_k) \in Z(\Delta)$, for $Z = \bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$.

Let $J = \{k \in N : k = 2i, i \in N\}$ be a subset of $N$ and let $\overbrace{m^F(\Delta)_J}$ be the canonical pre-image of the $J$-step space $m^F(\Delta)_J$ of $m^F(\Delta)$, defined as follows:

$(Y_k) \in \overbrace{m^F(\Delta)_J}$ is the canonical pre-image of $(X_k) \in m^F(\Delta)$ implies

$$Y_k = \begin{cases} X_k & \text{for} \quad k \in J, \\ \bar{0} & \text{for} \quad k \notin J. \end{cases}$$

Now,

$$[Y_k]^\alpha = \begin{cases} [4 - k^{-1}(1-\alpha), 4 + k^{-1}(1-\alpha)] & \text{for } k \in J \text{ and } k = n^2,\ n \in N, \\ [1, 1 + 2k^{-1}(1-\alpha)] & \text{for } k \in J \text{ and } k \neq n^2 \text{ for any } n \in N, \\ [0, 0] & k \notin J. \end{cases}$$

and

$$[\Delta Y_K]^\alpha = \begin{cases} [4 - k^{-1}(1-\alpha), 4 + k^{-1}(1-\alpha)] & \text{for } k \in J \text{ and } k = n^2,\ n \in N, \\ [1, 1 + 2k^{-1}(1-\alpha)] & \text{for } k \in J \text{ and } k \neq n^2, \text{ for any } n \in N \\ & \text{for} \quad k = n^2 - 1,\quad n > 1 \text{ with } n \in N, \\ [(k+1)^{-1}(\alpha - 1) - 4, (k+1)^{-1}(1-\alpha) - 4] & \text{for } k \notin J \\ & \text{and } k = n^2 - 1,\ n \in N, \\ [2(k+1)^{-1}(\alpha - 1) - 1, -1] & \text{otherwise.} \end{cases}$$

Thus $(Y_k) \notin Z(\Delta)$ for $Z = \bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$.

Therefore, the spaces $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are not monotone.

The spaces $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are not solid follows from Remark 1.

**Theorem 4.** *The spaces $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are not symmetric.*

**Proof.** The result follows from the following example. ■

**Example 3.** Consider the sequence $(X_k)$, defined in Example 2.
Here we have,

$$(X_k) \in Z(\Delta) \quad \text{for} \quad Z = \bar{c}^F,\ m^F,\ \bar{c_0}^F \text{ and } m_0^F$$

Let $(Y_k)$ be a rearrangement of the sequence $(X_k)$, defined as follows:

$$(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, X_7\ \ldots\ ).$$

Then

$$[Y_K]^\alpha = \begin{cases} [4 - k^{-1}(1-\alpha), 4 + k^{-1}(1-\alpha)] & \text{for } k \text{ odd,} \\ [1, 1 + 2k^{-1}(1-\alpha)] & \text{for } k \text{ even} \end{cases}$$

and

$$[\Delta Y_K]^\alpha = \begin{cases} [3 - (1-\alpha)(k^{-1} + 2(k+1)^{-1}), 3 + k^{-1}(1-\alpha)] & \text{for } k \text{ odd,} \\ [(k+1)^{-1}(\alpha - 1) - 3, (1-\alpha)(2k^{-1} + (k+1)^{-1}) - 3] & \text{for } k \text{ even.} \end{cases}$$

Thus $(Y_k) \notin Z(\Delta)$, for $Z = \bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$.

Hence, the spaces $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are not symmetric.

**Theorem 5.** *The spaces $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are not convergence free.*

**Proof.** The result follows from the following example. ∎

**Example 4.** Consider the sequence $(X_k) \in Z(\Delta)$, for $Z = \bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$, defined as follows:

For $k = n^2$, $n \in N$, $X_k = \bar{0}$.

Otherwise,

$$X_k(t) = \begin{cases} 1 + k(t-1) & \text{for } 1 - k^{-1} \le t \le 1, \\ 1 - 2^{-1}k(t-1) & \text{for } 1 < t \le 1 + 2k^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [0, 0] & \text{for } k = n^2, \ n \in N, \\ \left[1 + k^{-1}(\alpha - 1), 1 + 2k^{-1}(1-\alpha)\right] & \text{otherwise} \end{cases}$$

and

$$[\Delta X_K]^\alpha = \begin{cases} \left[\frac{2(\alpha-1)}{(k+1)} - 1, \frac{(1-\alpha)}{(k+1)} - 1\right] & \text{for } k = n^2, \ n \in N, \\ \left[1 + \frac{(\alpha-1)}{k}, 1 + \frac{2(1-\alpha)}{k}\right] & \text{for } k = n^2 - 1, \ n > 1 \text{ with } n \in N, \\ \left[(\alpha - 1)(\frac{1}{k} + \frac{2}{(k+1)}), (1-\alpha)(\frac{2}{k} + \frac{1}{(k+1)})\right] & \text{otherwise.} \end{cases}$$

Thus $(X_k) \in Z(\Delta)$, for $Z = \bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$.

Let the sequence $(Y_k)$ be defined as follows:

For $k = n^2$, $n \in N$, $Y_k = \bar{0}$.

Otherwise,

$$Y_k(t) = \begin{cases} 1 & \text{for } 1 \le t \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[Y_k]^\alpha = \begin{cases} [0,0] & \text{for } k = n^2, \ \ n \in N, \\ [1,k] & \text{otherwise.} \end{cases}$$

and

$$[\Delta Y_K]^\alpha = \begin{cases} [-(k+1),-1] & \text{for } k = n^2, \ \ n \in N, \\ [1,k] & \text{for } k = n^2-1, \ \ n > 1 \text{ with } n \in N, \\ [-1,k-1] & \text{otherwise.} \end{cases}$$

Thus $(Y_k) \notin Z(\Delta)$ for $Z = \bar{c}^F$, $m^F$, $\bar{c_0}^F$ and $m_0^F$.

Hence the spaces $\bar{c}^F(\Delta)$, $m^F(\Delta)$, $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are not convergence free.

**Theorem 6.** *The spaces $\bar{c_0}^F(\Delta)$ and $m_0^F(\Delta)$ are sequence algebra.*

**Proof.** Let $0 < \varepsilon < 1$ be given. Suppose $(X_k), (Y_k) \in \bar{c_0}^F(\Delta)$. Then we have

$$\begin{aligned} (7) \qquad & \left\{k \in N : \bar{d}\,(\Delta X_K \otimes \Delta Y_K, \ \bar{0}) < \varepsilon\right\} \\ & \supseteq \left\{k \in N : \bar{d}\,(\Delta X_K, \bar{0}) < \sqrt{\varepsilon}\right\} \cap \left\{k \in N : \bar{d}\,(\Delta Y_K, \bar{0}) < \sqrt{\varepsilon}\right\} \end{aligned}$$

Again, $\Delta\{X_k \otimes Y_k\} = X_k \otimes \Delta Y_k + Y_{k+1} \otimes \Delta X_k$ and

$$\begin{aligned} \Delta X_k \otimes \Delta Y_k &= X_k \otimes \Delta Y_k - X_{k+1} \otimes \Delta Y_k \\ &= X_k \otimes \Delta Y_k + Y_{k+1} \otimes \Delta X_k + 2X_{k+1} \otimes Y_{k+1} \\ &\quad - \{Y_{k+1} \otimes X_k + X_{k+1} \otimes Y_k\}. \end{aligned}$$

Since $(X_k), (Y_k) \in \bar{c_0}^F(\Delta)$, so

$$\text{stat} - \lim 2\, X_{k+1} \otimes Y_{k+1} = \ \text{stat} - \lim Y_{k+1} \otimes X_k + \text{stat} - \lim X_{k+1} \otimes Y_k.$$

Hence

$$(8) \qquad \text{stat} - \lim \Delta(X_k \otimes Y_k) = \text{stat} - \lim \Delta X_k \otimes \Delta Y_k$$

Since $\delta(\{k \in N : \bar{d}(\Delta X_K, \bar{0}) < \sqrt{\varepsilon}\}) = 1$ and $\delta(\{k \in N : \bar{d}(\Delta Y_K, \bar{0}) < \sqrt{\varepsilon}\}) = 1$.

Hence by (7) and (8), we have

$$\begin{aligned} & \delta\left(\left\{k \in N : \bar{d}(\Delta X_K \otimes \Delta Y_k, \bar{0}) < \varepsilon\right\}\right) \\ & \qquad = \ \delta\left(\left\{k \in N : \bar{d}\,(\Delta(X_K \otimes Y_k), \bar{0}) < \varepsilon\right\}\right) = 1. \end{aligned}$$

Thus $(X_k \otimes Y_k) \in \bar{c_0}^F(\Delta)$. Hence $\bar{c_0}^F(\Delta)$ is sequence algebra. The rest of the proof follows similarly. ■

**Theorem 7.** *The sequence spaces $\bar{c}^F(\Delta)$ and $m^F(\Delta) = \bar{c}^F(\Delta) \cap \ell_\infty^F(\Delta)$ are not sequence algebra.*

**Proof.** The result follows from the following example. ∎

**Example 5.** Consider the two sequences $(X_k), (Y_k) \in m^F(\Delta) \subset \bar{c}^F(\Delta)$, defined as follows:

For $k = n^2$, $n \in N$

$$X_k(t) = \begin{cases} t-k & \text{for } k \le t \le k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise,

$$X_k(t) = \begin{cases} t-k+1 & \text{for } k-1 \le t \le k, \\ k+1-t & \text{for } k < t \le k+1, \\ 0 & \text{otherwise.} \end{cases}$$

And for $k = n^2$, $n \in N$

$$Y_k(t) = \begin{cases} k+1-t & \text{for } k \le t \le k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise,

$$Y_k(t) = \begin{cases} t-k & \text{for } k \le t \le k+1, \\ k+2-t & \text{for } k+1 < t \le k+2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [k+\alpha, k+1] & \text{for } k = n^2,\ n \in N, \\ [k-1+\alpha, k+1-\alpha] & \text{otherwise.} \end{cases}$$

and

$$[Y_k]^\alpha = \begin{cases} [k, k+1-\alpha] & \text{for } k = n^2,\ n \in N, \\ [k+\alpha, k+2-\alpha] & \text{otherwise.} \end{cases}$$

Now

$$[\Delta X_k]^\alpha = \begin{cases} [2\alpha-2, 1-\alpha] & \text{for } k = n^2,\ n \in N, \\ [\alpha-3, 2-2\alpha] & \text{for } k = n^2-1, n > 1 \text{ with } n \in N, \\ [2\alpha-3, 1-2\alpha] & \text{otherwise.} \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} [\alpha-3,-2\alpha] & \text{for } k=n^2,\ n\in N, \\ [2\alpha-2,1-\alpha] & \text{for } k=n^2-1, n>1 \text{ with } n\in N, \\ [2\alpha-3,1-2\alpha] & \text{otherwise.} \end{cases}$$

Thus $(X_k),(Y_k)\in m^F(\Delta)\subset \bar{c}^F(\Delta)$.
Again,

$$\begin{aligned} \Delta(X_k\otimes Y_k)]^\alpha &= [(X_k\otimes Y_k)-(X_{k+1}\otimes Y_k+1)]^\alpha \\ &= \begin{cases} [-(\alpha^2-3k\alpha+5(k-\alpha)+6),-(\alpha^2+3k\alpha-k+2\alpha-1)] \\ \hfill \text{for } k=n^2,\ n\in N, \\ [(\alpha^2+3k\alpha-5k+\alpha-4),(\alpha^2-3k\alpha+k-4\alpha+1)] \\ \hfill \text{for } k=n^2-1, n>1 \text{ with } n\in N, \\ [(4k\alpha-6k+4\alpha-6),-(4k\alpha-2k+4\alpha-2] \qquad \text{otherwise.} \end{cases} \end{aligned}$$

Thus $(X_k\otimes Y_k)\notin \bar{c}^F(\Delta)\big(\supset m^F(\Delta)\big)$.
Hence the spaces $m^F(\Delta)$ and $\bar{c}^F(\Delta)$ are not sequence algebra.

**Theorem 8.** *(a) $m_0^F\subset m_0^F(\Delta)$ and the inclusion is strict.*
*(b) $m^F\subset m^F(\Delta)$ and the inclusion is strict.*

**Proof.** (a) Let us consider a sequence $(X_k)\in m_0^F$. Clearly (from Remark 2), we have

$$\Delta X_k \xrightarrow{stat} \bar{0} \text{ and hence } m_0^F\subset m_0^F(\Delta).$$

(b) Consider a sequence $(X_k)\in m^F$. Then, we have

$$\Delta X_k \xrightarrow{stat} X,$$

where $X$ is of particular type, defined by $[X]^\alpha=[-a,a]$ for some $a=a(\alpha)\in R_+\cup\{0\}$, the set of non-negative real numbers (see Remark 2 ).
Hence $m^F\subset m^F(\Delta)$. ■

The strictness of the inclusions of $(a)$ and $(b)$ follow from the following example.

**Example 6.** Consider the sequence $(X_k)$, defined as follows:

For $k=n^2$, $n\in N$

$$X_k(t) = \begin{cases} k+1-t & \text{for } k\le t\le k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise

$$X_k(t)=\begin{cases}1+2^{-1}k(t-3) & \text{for } 3-2k^{-1}\le t\le 3,\\ 1-2^{-1}k(t-3) & \text{for } 3\le t\le 3+2k^{-1},\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$[X_k]^\alpha=\begin{cases}[k,k+1-\alpha] & \text{for } k=n^2,\ n\in N,\\ [3+2k^{-1}(\alpha-1),3+2k^{-1}(1-\alpha)] & \text{otherwise.}\end{cases}$$

and

$$[\Delta X_k]^\alpha=\begin{cases}\left[k-3+\frac{2(\alpha-1)}{(k+1)},k-2-\alpha+\frac{2(1-\alpha)}{(k+1)}\right] & \text{for } k=n^2,\ n\in N,\\ \left[1-k+\alpha+\frac{2(\alpha-1)}{k},2-k+\frac{2(1-\alpha)}{k}\right] & \text{for } k=n^2-1,\\ & n>1 \text{ with } n\in N,\\ \left[2(\alpha-1)(\frac{1}{k}+\frac{1}{(k+1)}),2(1-\alpha)(\frac{1}{k}+\frac{1}{(k+1)})\right] & \text{otherwise.}\end{cases}$$

Thus $(X_k)\notin m^F(\supset m_0^F)$ and $(X_k)\in m_0^F(\Delta)\subset m^F(\Delta)$.
Hence the strictness of inclusions for both $(a)$ and $(b)$ are satisfied.

**Theorem 9.** *The spaces $m_0^F(\Delta)$ and $m^F(\Delta)$ are nowhere dense subsets of $\ell_\infty^F(\Delta)$.*

**Proof.** Clearly, we have $m_0^F(\Delta)$ and $m^F(\Delta)$ are closed subsets of the complete metric space $\ell_\infty^F(\Delta)$. Also $m_0^F(\Delta)$ and $m^F(\Delta)$ are proper subspaces of $\ell_\infty^F(\Delta)$ which follow from the following example. ∎

**Example 7.** Consider the sequence $(X_k)$, defined as follows:
For $k$ even

$$X_k(t)=\begin{cases}1-k(t-1)(k+2)^{-1} & \text{for } 1\le t\le 2+2k^{-1},\\ 0 & \text{otherwise}\end{cases}$$

and for $k$ odd,

$$X_k(t)=\begin{cases}1+kt(k+1)^{-1} & \text{for } -(1+k)^{-1}\le t\le 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$[X_k]^\alpha=\begin{cases}[1,1+(1+2k^{-1})(1-\alpha)] & \text{for } k \text{ even},\\ [(1+k^{-1})(\alpha-1),0] & \text{for } k \text{ odd.}\end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} 1, 1 + (1-\alpha)(2 + 2k^{-1} + (k+1)^{-1})] & \text{for } k \text{ even,} \\ [(\alpha - 1)(2 + k^{-1} + 2(k+1)^{-1}) - 1, -1] & \text{for } k \text{ odd.} \end{cases}$$

Thus $(X_k) \notin m^F(\Delta) \left(\supset m_0^F(\Delta)\right)$, but $(X_k) \in \ell_\infty^F(\Delta)$.
Hence the result.

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