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SOME PARANORMED DIFFERENCE DOUBLE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

Abstract. In this article we introduce some vector valued difference paranormed double sequence spaces defined by Orlicz function. We study some of their properties like solidness, symmetricity, completeness etc. and prove some inclusion results.

Key words: Orlicz function, difference sequence, completeness, seminorm, regular convergence, solid space, symmetric space.

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1. Introduction

Throughout the article w, ℓ_∞, c and c_0 denote the classes of all, bounded, convergent and null single sequence spaces of complex numbers respectively.

Throughout the article ${}_2w(q)$, ${}_2\ell_\infty(q)$, ${}_2c(q)$, ${}_2c_0(q)$, ${}_2c^R(q)$, ${}_2c_0^R(q)$, ${}_2c^B(q)$, ${}_2c_0^B(q)$ denote the spaces of all, bounded, convergent in Pringsheim's sense, null in Pringsheim's sense, regularly convergent, regularly null, convergent in Pringsheim's sense and bounded and null in Pringsheim's sense and bounded double sequences, defined over a seminormed space (X, q), seminormed by q. For $X = C$, the field of complex numbers, these represent the corresponding scalar sequence spaces. The zero element of X is denoted by θ.

An *Orlicz function M* is mapping $M : [0,\infty) \to [0,\infty)$ such that it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct the sequence space,

$$\ell^M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space normed by

$$||(x_k)|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

The space ℓ^M is closely related to the space ℓ^p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq K(Mu)$, $u \geq 0$.

Remark 1. Let $0 < \lambda < 1$, then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$.

2. Definitions and preliminaries

Throughout, a double sequence is denoted by $A = < a_{nk} >$, a double infinite array of elements $a_{nk} \in X$, for all $n, k \in N$ and $p = < p_{nk} >$ is a sequence of positive real numbers.

The initial works on double sequences are found in Bromwich [2]. Later on it is studied by Hardy [3], Moricz [8], Moricz and Rhoades [9], Tripathy ([12], [13]), Basarir and Sonalcan [1] and many others. Hardy [3] introduced the notion of regular convergence for double sequences.

The concept of paranormed sequences was studied by Nakano [10] and Simmons [11] at the initial stage. Later on it was studied by many others.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [5] as follows:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

The above spaces are Banach spaces normed by

$$||(x_k)|| = |x_1| + \sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by Tripathy [13] and many others.

The notion of difference double sequence spaces was introduced by Tripathy and Sarma [16]. These notions are further studied by Tripathy, Choudhary and Sarma [17].

Let $< a_{nk} >$ be a double sequence. Then the operator Δ is defined as:

$$\Delta a_{nk} = a_{nk} - a_{n+1,k} - a_{n,k+1} + a_{n+1,k+1}, \text{ for all } n, k \in N.$$

Definition 1. *A double sequence space E is said to be solid if $< \alpha_{nk} a_{nk} > \in E$ whenever $< a_{nk} > \in E$ for all double sequences $< \alpha_{nk} >$ of scalars with $|\alpha_{nk}| \leq 1$, for all $n, k \in N$.*

Definition 2. *Let $K = \{(n_i, k_j) : i, j \in N; n_1 < n_2 < \ldots$ and $k_1 < k_2 < \ldots\} \subseteq N \times N$ and E be a double sequence space. A K-step space of E is a sequence space*

$$\lambda_K^E = \{< a_{n_i k_j} > \in {}_2w : < a_{nk} > \in E\}.$$

A canonical pre-image of a sequence $< a_{nk} > \in E$ *is a sequence* $< b_{nk} > \in E$ *defined as follows:*

$$b_{nk} = \begin{cases} a_{nk} & \text{if} \quad (n,k) \in K, \\ 0 & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_K^E *is a set of canonical pre-images of all elements in* λ_K^E.

Definition 3. *A double sequence space* E *is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Remark 2. From the above notions, it follows that "If a sequence space E is solid, then E is monotone".

Definition 4. *A double sequence space* E *is said to be symmetric if* $< a_{nk} > \in E$ *implies* $< a_{\pi(n)\pi(k)} > \in E$, *where* π *is a permutation of* N.

Definition 5. *A normed (paranormed) space with norm (paranorm)* g *is said to be a* K*-space if the co-ordinatewise maps are continuous, i.e.*

$$\left| x_k^{(n)} - x_k \right| \to 0, \text{ whenever } g\left(x^{(n)} - x \right) \to \theta, \text{ as } n \to \infty,$$
$$\text{where } x^{(n)} = \left(x_k^{(n)} \right) \text{ and } x = x\left(x_k \right).$$

Remark 3. Let $p = (p_k)$ be a sequence of positive real numbers. If $0 < p_k \le \sup p_k = H$ and $D = \max(1, 2^{H-1})$, then for $a_k, b_k \in C$, for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \le D \left\{ |a_k|^{p_k} + |b_k|^{p_k} \right\}.$$

Let M be an Orlicz function. We have the following double sequence spaces:

$${}_2\ell_\infty(M,q) = \left\{ < a_{nk} > \in {}_2w(q) : \sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\},$$

$${}_2c(M,q) = \left\{ < a_{nk} > \in {}_2w(q) : M\left(q\left(\frac{a_{nk} - L}{\rho} \right) \right) \to 0, \text{ as } n,k \to \infty, \right.$$
$$\left. \text{for some } \rho > 0 \text{ and some } L \in X \right\}.$$

Also $< a_{nk} > \in {}_2c^R(M,q)$ *i.e.* regularly convergent if $< a_{nk} > \in {}_2c(M,q)$ and the following limits hold.

There exists $L_k \in X$, such that $M\left(q\left(\frac{a_{nk} - L_k}{\rho} \right) \right) \to 0$, as $n \to \infty$, for some $\rho > 0$ and all $k \in N$.

There exists $J_n \in X$, such that $M\left(q\left(\frac{a_{nk}-J_n}{\rho}\right)\right) \to 0$, as $k \to \infty$, for some $\rho > 0$ and all $n \in N$.

The definition of ${}_2c_0(M,q)$ and ${}_2c_0^R(M,q)$ follows from the above definition on taking $L = L_k = J_n = \theta$, for all $n, k \in N$.

We introduce the following difference double sequence spaces. Let $< p_{nk} >$ be a double sequence of positive real numbers.

$$
\begin{aligned}
&{}_2\ell_\infty(M,\Delta,p,q)\\
&= \left\{ < a_{nk} > \in {}_2w(q) : \sup_{n,k}\left[M\left(q\left(\frac{\Delta a_{nk}}{\rho}\right)\right)\right]^{p_{nk}} < \infty, \text{ for some } \rho > 0 \right\}.
\end{aligned}
$$

$$
\begin{aligned}
&{}_2c(M,\Delta,p,q)\\
&= \left\{ < a_{nk} > \in {}_2w(q) : \left[M\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{p_{nk}} \to 0, \ as\ n,k\to\infty \right.\\
&\qquad\qquad \left. \text{for some } \rho>0, \text{ for some } L\in X \right\}.
\end{aligned}
$$

Also $< a_{nk} > \in {}_2c^R(M,\Delta,p,q)$ *i.e.* Δ*-regularly convergent* if $< a_{nk} > \in {}_2c(M,\Delta,p,q)$ and the following limits hold.

There exists $L_k \in X$, such that $\left[M\left(q\left(\frac{\Delta a_{nk}-L_k}{\rho}\right)\right)\right]^{p_{nk}} \to 0$, as $n \to \infty$, for some $\rho > 0$ and for all $k \in N$.

There exists $J_n \in X$, such that $\left[M\left(q\left(\frac{\Delta a_{nk}-J_n}{\rho}\right)\right)\right]^{p_{nk}} \to 0$, as $k \to \infty$, for some $\rho > 0$ and for all $n \in N$.

The definitions of ${}_2c_0(M,\Delta,p,q)$ and ${}_2c_0^R(M,\Delta,p,q)$ follow from the above definition on taking $L = L_k = J_n = \theta$, for all $n, k \in N$.

3. Main results

The proof of following two results is easy, so omitted.

Theorem 1. *The classes of sequences $Z(M,\Delta,p,q)$, where $Z = {}_2c$, ${}_2c_0$, ${}_2c^B$, ${}_2c_0^B$, ${}_2c^R$, ${}_2c_0^R$ and ${}_2\ell_\infty$ are linear spaces.*

Theorem 2. *The sequence spaces $Z(M,\Delta,p,q)$, where $Z = {}_2c^B$, ${}_2c_0^B$, ${}_2c^R$, ${}_2c_0^R$ and ${}_2\ell_\infty$ are paranormed spaces paranormed by*

$$
\begin{aligned}
(1)\qquad f(< a_{nk} >) = \inf\Bigg\{ &\rho^{\frac{p_{nk}}{J}} > 0 : \sup_n M\left(q\left(\frac{a_{n1}}{\rho}\right)\right)\\
&+ \sup_k M\left(q\left(\frac{a_{1k}}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a_{nk}}{\rho}\right)\right) \le 1, \Bigg\}
\end{aligned}
$$

where $J = \max(1, 2^{H-1})$

Remark 4. Theorem 2 holds good if the function f is replaced by the function g, where

$$g(< a_{nk} >) = \inf \left\{ \rho^{\frac{p_{nk}}{J}} > 0 : \sup_{n,k} M \left(q \left(\frac{\Delta a_{nk}}{\rho} \right) \right) \leq 1 \right\}.$$

Theorem 3. *Let (X, q) be a complete seminormed space. Then the spaces $Z(M, \Delta, p, q)$, where $Z = {}_2c^B$, ${}_2c_0^B$, ${}_2c^R$, ${}_2c_0^R$ and ${}_2\ell_\infty$ are complete paranormed spaces paranormed by f.*

Proof. Let us consider the space ${}_2\ell_\infty(M, \Delta, p, q)$. Let $< a^i_{nk} >$ be a Cauchy sequence in ${}_2\ell_\infty(M, \Delta, p, q)$. For fixed $x_0 > 0$, $r > 0$, choose

$$M\left(\frac{rx_0}{2}\right) \geq 1. \tag{2}$$

Then for a given $\varepsilon > 0$, there exists $m_0 \in N$ such that

$$f(< a^i_{nk} - a^j_{nk} >) < \frac{\varepsilon}{rx_0} \quad \text{for all} \quad i, j \geq m_0. \tag{3}$$

By (1), the definition of the paranorm f, we have

$$\begin{aligned}
(4) \qquad & \sup_n M\left(q\left(\frac{a^i_{n1} - a^j_{n1}}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a^i_{1k} - a^j_{1k}}{\rho}\right)\right) \\
& + \sup_{n,k} M\left(q\left(\frac{\Delta a^i_{nk} - \Delta a^j_{nk}}{\rho}\right)\right) \leq 1 \leq M\left(\frac{rx_0}{2}\right) \\
& \Rightarrow M\left(q\left(\frac{a^i_{n1} - a^j_{n1}}{f(< a^i_{nk} - a^j_{nk} >)}\right)\right) \leq M\left(\frac{rx_0}{2}\right), \\
& M\left(\left(\frac{a^i_{1k} - a^j_{1k}}{f(< a^i_{nk} - a^j_{nk} >)}\right)\right) \leq M\left(\frac{rx_0}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
& M\left(q\left(\frac{\Delta a^i_{nk} - \Delta a^j_{nk}}{f(< a^i_{nk} - a^j_{nk} >)}\right)\right) \leq M\left(\frac{rx_0}{2}\right) \\
& \Rightarrow q\left(a^i_{n1} - a^j_{n1}\right) < \frac{rx_0}{2} \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2} \text{ for all } i, j \geq m_0, \\
& q\left(a^i_{1k} - a^j_{1k}\right) < \frac{rx_0}{2} \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2} \text{ for all } i, j \geq m_0, \\
& q\left(\Delta a^i_{nk} - \Delta a^j_{nk}\right) < \frac{rx_0}{2} \frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2} \text{ for all } i, j \geq m_0.
\end{aligned}$$

Thus $< a^i_{n1} >$, $< a^i_{1k} >$ and $< \Delta a^i_{nk} >$ are Cauchy sequences in X. Since X is complete, so there exist a_{n1}, a_{1k}, $y_{nk} \in X$ such that

$$\lim_{i\to\infty} a^i_{n1} = a_{n1}, \quad \lim_{i\to\infty} a^i_{1k} = a_{1k} \text{ and } \lim_{i\to\infty} \Delta a^i_{nk} = y_{nk}.$$

From this it is clear that $\lim\limits_{i\to\infty} \Delta a^i_{nk} \in X$, for each $n, k \in N$.

Since M is continuous, so taking $j \to \infty$ in (4) we get

$$\sup_n M\left(q\left(\frac{a^i_{n1} - a_{n1}}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a^i_{1k} - a_{1k}}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a^i_{nk} - \Delta a_{nk}}{\rho}\right)\right) \le 1.$$

Taking infimum of such ρ's, we get

$$\inf\left\{\rho^{\frac{p_{nk}}{J}} : \sup_n M\left(q\left(\frac{a^i_{n1} - a_{n1}}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a^i_{1k} - a_{1k}}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a^i_{nk} - \Delta a_{nk}}{\rho}\right)\right) \le 1\right\} < \varepsilon, \quad \text{for all } i \ge m_0.$$

Hence $< a^i_{nk} - a_{nk} > \in {}_2\ell_\infty(M, \Delta, p, q)$. Since ${}_2\ell_\infty(M, \Delta, p, q)$ is linear, so $< a_{nk} > = < a^i_{nk} > - < a^i_{nk} - a_{nk} > \in {}_2\ell_\infty(M, \Delta, p, q)$.

Thus ${}_2\ell_\infty(M, \Delta, p, q)$ is complete. The other cases can be proved similarly. ■

Proposition 1. *The spaces $Z(M, \Delta, p, q)$, for $Z = {}_2c^R$, ${}_2c^R_0$ and ${}_2\ell_\infty$ are K-spaces.*

Proof. Let us consider the sequence space ${}_2\ell_\infty(M, \Delta, p, q)$. Let $< a^i_{nk} >$ be a sequence in ${}_2\ell_\infty(M, \Delta, p, q)$ such that $f(< a^i_{nk} - a_{nk} >) \to \infty$, as $i \to \infty$.

For fixed x_0, $r > 0$, choose $M(rx_0) \ge 1$. Then for a given $\varepsilon > 0$ there exists $m_0 \in N$ such that

$$f(< a^i_{nk} - a_{nk} >) < \frac{\varepsilon}{rx_0} \quad \text{for all } i \ge m_0.$$

By the definition of the paranorm f, we have

$$\sup_n M\left(q\left(\frac{a^i_{n1} - a_{n1}}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a^i_{1k} - a_{1k}}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a^i_{nk} - \Delta a_{nk}}{\rho}\right)\right) \le 1 \le M\left(\frac{rx_0}{2}\right)$$

$$\Rightarrow M\left(q\left(\frac{a^i_{n1} - a_{n1}}{f(< a^i_{nk} - a_{nk} >)}\right)\right) \le M\left(rx_0\right),$$

$$M\left(\left(\frac{a_{1k}^i - a_{1k}}{f(< a_{nk}^i - a_{nk} >)}\right)\right) \le M(rx_0)$$
$$\text{and}\quad M\left(q\left(\frac{\Delta a_{nk}^i - \Delta a_{nk}}{f(< a_{nk}^i - a_{nk} >)}\right)\right) \le M(rx_0)$$
$$\Rightarrow q\left(a_{n1}^i - a_{n1}\right) < rx_0 2\frac{\varepsilon}{rx_0} = \varepsilon \text{ for all } i \ge m_0, \text{ and for all } n \in N.$$
$$q\left(a_{1k}^i - a_{1k}\right) < rx_0\frac{\varepsilon}{rx_0} = \varepsilon \text{ for all } i \ge m_0 \text{ and for all } k \in N$$
$$\text{and } q\left(\Delta a_{nk}^i - \Delta a_{nk}\right) < rx_0\frac{\varepsilon}{rx_0} = \varepsilon \text{ for all } i \ge m_0 \text{ and for all } n, k \in N.$$

Thus

$$q\left(a_{n1}^i - a_{n1}\right) \to 0, \quad q\left(a_{1k}^i - a_{1k}\right) \to 0 \quad \text{and} \quad q\left(\Delta a_{nk}^i - \Delta a_{nk}\right) \to 0, \text{ as } i \to \infty. \tag{5}$$

Using the expressions $\Delta a_{nk} = a_{nk} - a_{n+1,k} - a_{n,k+1} + a_{n+1,k+1}$ and $\Delta a_{nk}^i = a_{nk}^i - a_{n+1,k}^i - a_{n,k+1}^i + a_{n+1,k+1}^i$ for all $n, k \in N$, from (5) we get from the $q\left(a_{nk}^i - a_{nk}\right) \to 0$. ■

Hence ${}_2\ell_\infty(M, \Delta, p, q)$ is K-space. Similarly the other spaces are also K-spaces.

Result 1. *The spaces $Z(M, \Delta, p, q)$, for $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ and ${}_2\ell_\infty$ are not symmetric.*

The result follows from the following example.

Example 1. Consider the sequence space ${}_2c_0(M, \Delta, p, q)$. Let $X = \ell_2$, $M(x) = x$ and $q(x) = \left(\sum\limits_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$.

Let the sequence $< a_{nk} >=< (a_{nk}^i) >$ be defined by

$$a_{1k} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \quad \text{for all} \quad k \in N,$$
$$a_{nk} = (0, 0, 0, 0, \ldots) \quad \text{for all} \quad k \in N \text{ and all } n \ge 2.$$

$$\begin{aligned}\Delta a_{1k} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) - \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) - (0,0,0,\ldots) + (0,0,0,\ldots) \\ = (0,0,0,\ldots) \text{ for all } k \in N.\end{aligned}$$

$$\begin{aligned}\Delta a_{nk} = (0,0,0,\ldots) - (0,0,0,\ldots) - (0,0,0,\ldots) + (0,0,0,\ldots) \\ = (0,0,0,\ldots) \text{ for all } n \ge 1 \text{ and for all } k \in N.\end{aligned}$$

Thus $\Delta a_{nk} = (0, 0, 0, 0, \ldots)$, for all $n, k \in N$.

Let $< b_{nk} >$ be a rearrangement of $< a_{nk} >$ defined by

$$b_{nk} = \begin{cases} \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) & \text{for} \quad n = k, \\ (0,0,0,0,\ldots) & \text{otherwise,} \end{cases}$$

Then

$$\begin{aligned} \Delta b_{nn} &= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) - (0,0,0,\ldots) - (0,0,0,\ldots) + \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \\ &= 2\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \quad \text{for all } n \in N \end{aligned}$$

and

$$\Delta b_{nk} = \begin{cases} -\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) & \text{for} \quad n = k \pm 1, \\ (0,0,0,0,\ldots) & \text{otherwise.} \end{cases}$$

The sequence $< a_{nk} > \in {}_2c_0(M, \Delta, p, q) \subset {}_2c(M, \Delta, p, q)$ but $< b_{nk} > \notin {}_2c(M, \Delta, p, q)$. Similarly it can be shown that the other spaces are not symmetric.

Result 2. *The spaces $Z(M, \Delta, p, q)$, for $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ and ${}_2\ell_\infty$ are neither solid nor monotone.*

The result follows from the following example.

Example 2. Consider the sequence space ${}_2c(M, \Delta, p, q)$. Let $X = c$, $q(x) = \sup_i |x^i|$ and

$$(M(x))^{p_{nk}} = \begin{cases} x^3 & \text{if} \quad n = k, \\ x^2 & \text{otherwise,} \end{cases} \quad \text{for } x \in [0, \infty) \text{ and all } n, k \in N.$$

Let the sequence $< a_{nk} >$ be defined by

$$a_{nk} = (n+k, n+k, n+k, \ldots), \quad \text{for all } n, k \in N.$$

Then $\Delta a_{nk} = (0,0,0,\ldots)$, for all $n, k \in N$.

Let $J = \{(n,k) : n = k\} \subset N \times N$. Let ${}_2c(M, \Delta, p, q)_J^*$ be the canonical pre-image of the J step space ${}_2c(M, \Delta, p, q)_J$ of ${}_2c(M, \Delta, p, q)$. Let $< a_{nk} > \in {}_2c(M, \Delta, p, q)_J^*$. Then

$$b_{nk} = \begin{cases} a_{nk} & \text{for all} \quad (n,k) \in J, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = k$, $\Delta b_{nk} = (4n+2, 4n+2, 4n+2, \ldots)$ which tends to infinity as n tends to infinity.

The sequence $< a_{nk} > \in {}_2c(M, \Delta, p, q)$ but $< b_{nk} > \notin {}_2c(M, \Delta, p, q)$. Hence the sequence space ${}_2c(M, \Delta, p, q)$ is not monotone. Similarly it can be shown for the other spaces too.

Proposition 2. *(i) $Z(M,\Delta,p,q) \subset {}_2\ell_\infty(M,\Delta,p,q)$, for $Z = {}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$. The inclusions are strict.*

(ii) If $\sup\limits_{n,k} \frac{p_{nk}}{p_{n+1,k}} < \infty$, $\sup\limits_{n,k} \frac{p_{nk}}{p_{n,k+1}} < \infty$, for all $n,k \in N$, then $Z(M,p,q) \subset Y(M,\Delta,p,q)$, for $Z = {}_2c$, ${}_2c^R$, ${}_2c^B$ and $Y = {}_2c_0$, ${}_2c_0^R$, ${}_2c_0^B$ respectively. The inclusions are strict.

Proof. (i) The first part is obvious. To show the inclusions are strict, consider the following example.

Example 3. Let $X = c$, $M(x) = x$, $q(x) = \sup\limits_i |x^i|$ and

$$p_{nk} = \begin{cases} 3 & \text{for } n \text{ odd and all } \ k \in N, \\ 2 & \text{otherwise.} \end{cases}$$

Let the sequence $< a_{nk} >$ be defined by

$$a_{nk} = \begin{cases} (n+k, n+k, n+k, \ldots) & \text{for } n \text{ odd and all } \ k \in N, \\ (n, n, n, \ldots) & \text{otherwise.} \end{cases}$$

Then

$$\Delta a_{nk} = \begin{cases} (-1,-1,-1,-1,\ldots) & \text{for } n \text{ odd and all } \ k \in N, \\ (1,1,1,1,\ldots) & \text{otherwise.} \end{cases}$$

Then $< a_{nk} > \in {}_2\ell_\infty(M,\Delta,p,q)$ but $< a_{nk} > \notin Z(M,\Delta,p,q)$, for $Z = {}_2c^R$, ${}_2c_0^R$.

(ii) We prove ${}_2c(M,p,q) \subseteq {}_2c_0(M,\Delta,p,q)$. Since $\sup\limits_{n,k} \frac{p_{nk}}{p_{n+1,k}} < \infty$, $\sup\limits_{n,k} \frac{p_{nk}}{p_{n,k+1}} < \infty$, we have

$$\frac{p_{nk}}{p_{n+1,k}} \le K_1, \quad \frac{p_{nk}}{p_{n,k+1}} \le K_2, \ \text{ for some } \ K_1 > 0, \ \ K_2 > 0.$$

$$\Rightarrow p_{nk} \le K_1 p_{n+1,k}, \quad p_{nk} \le K_2 p_{n,k+1} \tag{6}$$

Also

$$\begin{aligned} p_{nk} &\le K_2 . p_{n,k+1} \\ &\le K_2 K_1 p_{n+1,k+1} \quad \text{for all } \ n,k \in N \ \ Eq.(6) \\ &= K_3 p_{n+1,k+1} \quad \text{(say), for all } \ n,k \in N. \end{aligned} \tag{7}$$

Let $< a_{nk} > \in {}_2c(M,p,q)$. Then for some $\rho > 0$,

$$\left[M\left(q\left(\frac{a_{nk} - L}{\rho} \right) \right) \right]^{p_{nk}} \to 0, \ \text{ as } \ n \to \infty, \ \ k \to \infty.$$

Let $r = 4\rho$. Now,

$$\left[M\left(q\left(\frac{\Delta a_{nk}}{r}\right)\right)\right]^{p_{nk}} = \left[M\left(q\left(\frac{a_{nk} - a_{n+1,k} - a_{n,k+1} + a_{n+1,k+1}}{r}\right)\right)\right]^{p_{nk}}$$
$$\leq D^2\left[\left\{\frac{1}{4}M\left(q\left(\frac{a_{nk} - L}{\rho}\right)\right)\right\}^{p_{nk}} + \left\{\frac{1}{4}M\left(q\left(\frac{a_{n,k+1} - L}{\rho}\right)\right)\right\}^{p_{nk}}\right.$$
$$\left. + \left\{\frac{1}{4}M\left(q\left(\frac{a_{n,k+1} - L}{\rho}\right)\right)\right\}^{p_{nk}} + \left\{\frac{1}{4}M\left(q\left(\frac{a_{n+1,k+1} - L}{\rho}\right)\right)\right\}^{p_{nk}}\right]$$
$$\to 0, \text{ as } n \to \infty, \ k \to \infty \text{ and for some } r > 0. \qquad \text{[using (6) and (7)]}$$

Hence $< a_{nk} > \in {}_2c_0(M, \Delta, p, q)$. Thus ${}_2c(M, p, q) \subseteq {}_2c_0(M, \Delta, p, q)$. Similarly it can be proved that ${}_2c^R(M, p, q) \subseteq {}_2c_0^R(M, \Delta, p, q)$. ■

To show the strict inclusions, consider the following example.

Example 4. Let $X = c$, $M(x) = x^2$ and $q(x) = \sup_i |x^i|$. Let the sequence $< a_{nk} > = < (a_{nk}^i) >$ be defined by

$$a_{nk} = (n + k, n + k, n + k, n + k, \ldots) \quad \text{for all } n, k \in N.$$

Clearly $< a_{nk} > \in {}_2c(M, \Delta, p, q)$, but $< a_{nk} > \notin {}_2c(M, p, q)$.

Proposition 3. *The spaces $Z(M, \Delta, p, q)$, for $Z = {}_2c^R$, ${}_2c_0^R$ are nowhere dense subset of ${}_2\ell_\infty(M, \Delta, p, q)$.*

Proof. The proof is clear from the Proposition 2 (i) and Theorem 3. ■

Proposition 4. *Let M_1 and M_2 be Orlicz functions.*

(i) Then $Z(M_2, \Delta, p, q) \subseteq Z(M_1, \Delta, p, q)$, for $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ if $M_1(x) \leq M_2(x)$, for all $x \in [0, \infty)$.

(ii) Then $Z(M_1, \Delta, p, q) \cap Z(M_2, \Delta, p, q) \subseteq Z(M_1 + M_2, \Delta, p, q)$, for $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ and ${}_2\ell_\infty$.

Proof. (i) The proof is obvious.

(ii) Consider the case $Z = {}_2c$. Let $< a_{nk} > \in {}_2c(M_1, \Delta, p, q) \cap {}_2c(M_2, \Delta, p, q)$. Then for some $\rho_1, \rho_2 > 0$,

$$\left[M_1\left(q\left(\frac{\Delta a_{nk} - L}{\rho_1}\right)\right)\right]^{p_{nk}} < \frac{\varepsilon}{2D} \text{ for all } n \geq n_0, \ k \geq k_0, \ (n_0, \ k_0 \in N).$$
$$\left[M_2\left(q\left(\frac{\Delta a_{nk} - L}{\rho_2}\right)\right)\right]^{p_{nk}} < \frac{\varepsilon}{2D} \text{ for all } n \geq n_0', \ k \geq k_0', \ (n_0', \ k_0' \in N).$$

Let $\rho = \max\{\rho_1, \rho_2\}$, $n_0'' = \max\{n_0, n_0'\}$, $k_0'' = \max\{k_0, k_0'\}$.

Now for $n \geq n_0''$, $k \geq k_0''$ and for some $\rho > 0$,

$$\left[(M_1+M_2)\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{p_{nk}} \leq D\left[M_1\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{p_{nk}} + D\left[M_2\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{p_{nk}} < D\left(\frac{\varepsilon}{2D}+\frac{\varepsilon}{2D}\right) = \varepsilon$$

Thus $< a_{nk} > \in {}_2c(M_1+M_2,\Delta,p,q)$. Hence the proof. Similarly it can be proved for the other spaces. ∎

Proposition 5. *(i) If* $0 < \inf p_{nk} \leq p_{nk} < 1$, *then* $Z(M,\Delta,p,q) \subseteq Z(M,\Delta,q)$, *for* $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ *and* ${}_2\ell_\infty$.

(ii) If $1 < p_{nk} \leq \sup p_{nk} < \infty$, *then* $Z(M,\Delta,q) \subseteq Z(M,\Delta,p,q)$, *for* $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ *and* ${}_2\ell_\infty$.

Proof. (i) The result follows from the following inequality.

$$M\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right) \leq \left[M\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{p_{nk}}.$$

(ii) The result follows from the following inequality.

$$\left[M\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{p_{nk}} \leq M\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right).$$

∎

Proposition 6. *If* $0 < p_{nk} \leq t_{nk} < \infty$, *then* $Z(M,\Delta,p,q)$, *for* $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ *and* ${}_2\ell_\infty$.

Proof. The result follows from the following inequality.

$$\left[M\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{t_{nk}} \leq \left[M\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right]^{p_{nk}}.$$

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