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ON SOME GENERALIZED NEW TYPE DIFFERENCE SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION IN A SEMINORMED SPACE

Abstract. The idea of difference sequence spaces were introduced by Kizmaz [6] and generalized by Et. and Colak [4]. Later Tripathy, Esi and Tripathy [15] introduced the notion of the new difference operator Δ_m^n for $n, m \in N$. In this paper we introduced some new type of generalized difference sequence spaces defined by a modulus function and the new type of statistically convergent generalized difference sequence space. We study their different properties and obtain some inclusion relations involving these new type difference sequence spaces.

Key words: modulus function, de la Vallee-Poussin means, paranorm.

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1. Introduction

The difference sequence space $Z(\Delta)$ was introduced by Kizmaz [6] as follows:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty$, c, and c_0 where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$. Later, these difference sequence spaces were generalized by Et and Colak [4] as follows:

Let $r \in N$ be fixed, then

$$Z(\Delta^r) = \{(x_k) : (\Delta^r x_k) \in Z\}$$

for $Z = \ell_\infty$, c, and c_0 where $\Delta^r x_k = \Delta^{r-1} x_k - \Delta^{r-1} x_k + 1$ and $\Delta^0 x_k = x_k$ for all $k \in N$.

The generalized difference has the following binomial representation:

$$\Delta^r x_k = \sum_{k=1}^{n} (-1)^\nu \binom{n}{r} x_{k+v}, \text{ for all } k \in N.$$

Recently, on generalizing this difference operator, Tripathy, Esi and Tripathy [15] have introduced a new type of generalized difference operator as follows:

Let $r, m \in N$ be fixed, then

$$Z(\Delta_m^r) = \{(x_k) : (\Delta_m^r x_k) \in Z\},\ \text{for } Z = \ell_\infty, c, \text{and } c_0.$$

where $\Delta_m^r = \Delta_m^{r-1}x_k - \Delta_m^{r-1}x_k + 1$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$. This generalized difference notion has the following binomial representation:

$$\Delta_m^r x_k = \sum_{k=1}^{n} (-1)^\nu \binom{r}{\nu} x_{k+mv}$$

The notion of modulus function was introduced by Nakano [13]. The Notion was further investigated by Ruckle [14] and many others. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that
(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x+y) \le f(x) + f(y)$,
(iii) f is continuous, and ,
(iv) f is continuous from right at 0.

It is immediate from (ii) and (iv) that f is continuous on $[0,\infty)$. Also from condition (ii), we have $f(nx) \le nf(x)$ for all $n \in N$ and so $n^{-1}f(x) \le f(xn^{-1})$, for all $n \in N$. A modulus function may be bounded (for example, $f(x) = x(1+x)^{-1}$) or unbounded(for example, $f(x) = x$). Ruckle [14], Maddox [10], Esi[2] and several authors used a modulus f to construct some sequence spaces.

Remark. If f is a modulus function, then the composition $f^s = f.f....f$ (s times) is also a modulus function, where s is a positive integer.

Let $p = (p_k)$ be a sequence of positive real numbers. We have the following well known inequality, which will be used throughout this paper:

$$|a_k + b_k|^{p_k} \le D(|a_k|^{p_k} + |b_k|^{p_k}) \tag{1}$$

where a_k and b_k complex numbers, $D = \max\{1, 2^{H-1}\}$ and $H = \sup_k p_k < \infty$ (one may refer to maddox [11]).

Spaces of strongly summable sequencees were studied at the initial stage by Kuttner [7], Maddox [9] and others. The class of sequences those are strongly Cesàro summable with respect to a modulus was introduced by Maddox [10] as an extension of the definition of strongly Cesàro summable sequences. Connor [1] further extended this definition to a definition of strongly A-summability with respect to a modulus when A is non-negative regular matrix.

Let $\Lambda = (\lambda_i)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{i+1} \le \lambda_i + 1$, for all $i \in N$.

The generalized de la Vallee-Poussin means is defined by $t_i(x) = \lambda_i^{-1} \sum_{k \in I_i} x_k$, where $I_i = [i - \lambda_i + 1, i]$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_i(x) \to L$, as $i \to \infty$ (see for instance Leindler [8]).

2. Definitions and preliminaries

Throughout E will represent a seminormed space, seminormed by q. We define $w(E)$ to be the vector space of all E-valued sequences. Let f be a modulus function, $p = (p_k)$ be any sequence of positive real numbers, $A = (a_{nk})$ be a non-negative matrix such that $\sup_n \sum_{k=1}^{\infty} a_{nk} < \infty$ and $r, m \in N$ be fixed.

We define the following sets of sequences in this article:

$$[V_\lambda^E, A, \Delta_m^r, f, p]_0 = \left\{ x \in w(E) : \lim_{i \to \infty} \lambda_i^{-1} \sum_{k \in I_i} a_{nk} [f(q(\Delta_m^r\ x_k))]^{p_k} = 0, \text{ uniformly in } n \right\}$$

$$[V_\lambda^E, A, \Delta_m^r, f, p]_1 = \left\{ x \in w(E) : \lim_{i \to \infty} \lambda_i^{-1} \sum_{k \in I_i} a_{nk} [f(q(\Delta_m^r\ x_k - L))]^{p_k} = 0, \text{ uniformly in } n, \text{ for some L} \right\}.$$

$$[V_\lambda^E, A, \Delta_m^r, f, p]_\infty = \left\{ x \in w(E) : \sup_n \sup_i \lambda_i^{-1} \sum_{k \in I_i} a_{nk} [f(q(\Delta_m^r\ x_k))]^{p_k} < \infty \right\}.$$

If $x \in [V_\lambda^E, A, \Delta_m^r, f, p]_1$ then we write $x \to L([V_\lambda^E, A, \Delta_m^r, f, p]_1)$ and L will be called $[\Lambda, A, \Delta_m^r, E, f, p]_1$-limit of $x = (x_k)$ with respect to the modulus function f.

For $m = 1$, these spaces are denoted by $[V_\lambda^E, A, \Delta^r, f, p]_Z$, for $Z = 0, 1, \infty$ respectively.

We define

$$[V_\lambda^E, \Delta_m^r, f, p]_1 = \left\{ x \in w(E) : \lim_{i \to \infty} \lambda_i^{-1} \sum_{k \in I_i} [f(q(\Delta_m^r\ x_k - L))]^{p_k} = 0, \text{ for some L} \right\}.$$

Similarly $[V_\lambda^E, \Delta_m^r, f, p]_0$ and $[V_\lambda^E, \Delta_m^r, f, p]_\infty$ can be defined.

For $E = C$, the set of complex numbers, $q(x) = |x|; f(x) = x; p_k = 1$, for all $k \in N$; $r = 0$, $m = 0$ the spaces $[V_\lambda^E, \Delta_m^r, f, p]_Z$, for $Z = 0, 1, \infty$ represent the spaces $[V, \lambda]_Z$, for $Z = 0, 1, \infty$. These spaces are called as λ-strongly summable to zero, λ-stromgly summable and λ-strongly bounded by the de la Vallee-Poussin method. In the special case, where $\lambda_i = i$, for all $i = 1, 2, 3, ...$ the sets $[V, \lambda]_0, [V, \lambda]$ and $[V, \lambda]_\infty$ reduce to the sets w_0, w and w_∞ introduced and studied by Maddox [9].

3. Main results

In this section we prove the result of this article.

Theorem 1. *Let the sequence $p = (p_k)$ be bounded. Then the sequence spaces $[V_\lambda^E, A, \Delta_m^r, f, p]_Z$ are linear spaces over the complex field C, for $Z = 0, 1$ and ∞.*

Proof. We prove the theorem for the class of sequences $[V_\lambda^E, A, \Delta_m^r, f, p]_\infty$. For the other cases the theorem can be proved following similar techniques. Let $x, y \in [V_\lambda^E, A, \Delta_m^r, f, p]_\infty$ and $\alpha, \beta \in C$. Then there exist positive integers M_1 and M_2 such that $|\alpha| \le M_1$ and $|\beta| \le M_2$. We have

$$\begin{aligned}
&\lambda_i^{-1} \sum_{k \in I_i} a_{nk}[f(q(\Delta_m^r\ (\alpha x_k + \beta y_k))]^{p_k} \\
&\quad \le \lambda_i^{-1} \sum_{k \in I_i} a_{nk}[f(|\alpha| q(\Delta_m^r\ x_k)) + f(|\beta| q(\Delta_m^r\ y_k))]^{p_k} \\
&\quad \le D(M_1)^H \lambda_i^{-1} \sum_{k \in I_i} a_{nk}[f(q(\Delta_m^r\ x_k))]^{p_k} \\
&\qquad + D(M_2)^H \lambda_i^{-1} \sum_{k \in I_i} a_{nk}[f(q(\Delta_m^r\ y_k))]^{p_k} \\
&\qquad \to 0, \text{ uniformly in } n.
\end{aligned}$$

This proves that $[V_\lambda^E, A, \Delta_m^r, f, p]_\infty$ is a linear space. ■

Theorem 2. *Let f be a modulus function, then*

$$[V_\lambda^E, A, \Delta_m^r, f, p]_0 \subset [V_\lambda^E, A, \Delta_m^r, f, p]_1 \subset [V_\lambda^E, A, \Delta_m^r, f, p]_\infty.$$

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x \in [V_\lambda^E, A, \Delta_m^r, f, p]_1$. Let $x_k \to L([V_\lambda^E, A, \Delta_m^r, f, p]_1)$, then there exists positive integer M_1 such that $q(L) \le M_1$. Then we have

$$\begin{aligned}
\lambda_i^{-1} \sum_{k \in I_i} a_{nk}[f(q(\Delta_m^r\ x_k))]^{p_k} &\le D\lambda_i^{-1} \sum_{k \in I_i} a_{nk}[f(q(\Delta_m^r\ x_k - L))]^{p_k} \\
&\quad + D(M_1, f(1))^H \lambda_i^{-1} \sum_{k \in I_i} a_{nk}
\end{aligned}$$

Thus $x \in [V_\lambda^E, A, \Delta_m^r, f, p]_\infty$, since $x \in [V_\lambda^E, A, \Delta_m^r, f, p]_1$.

This complete the proof. ■

Theorem 3. *Let $p = (p_k) \in \ell_\infty$, then $[V_\lambda^E, A, \Delta_m^r, f, p]_\infty$ is a paranormed space with*

$$g(x) = \sup_i \left(\lambda_i^{-1} \sum_{k \in I_i} a_{nk} [f(q(\Delta_m^r \ x_k))]^{p_k} \right)^{\frac{1}{M}}$$

where $M = \max(1, \sup_k p_k)$.

Proof. From Theorem 1, for each $x \in [V_\lambda^E, A, \Delta_m^r, f, p]_0$, $g(x)$ exists. Clearly, $g(-x) = g(x)$. It is trivial that $\Delta_m^r x_k = \theta$ for $x = \overline{\theta}$. Hence, we get $g(\overline{\theta}) = 0$. By Minkowski's inequality, we have $g(x+y) \leq g(x) + g(y)$. Now we show that the scalar multiplication is continuous. Let α be any fixed complex number. By definition of f, we have $x \to \theta$ implies,$g(\alpha x) \to 0$. Similarly we have x fixed and $\alpha \to 0$ implies $g(\alpha x) \to 0$. Finally $x \to \theta$ and $\alpha \to 0$ implies $g(\alpha x) \to 0$.

This completes the proof. ∎

Theorem 4. *If $r \geq 1$, then the inclusion $[V_\lambda^E, A, \Delta_m^{r-1}, f, p]_Z \subset [V_\lambda^E, A, \Delta_m^r, f, p]_Z$ is strict for $Z = 0, 1$ and ∞. In general, $[V_\lambda^E, A, \Delta_m^j, f, p]_Z \subset [V_\lambda^E, A, \Delta_m^r, f, p]_Z$, for $j = 0, 1, 2, ...r-1$ and the inclusions are strict, for $Z = 0, 1$ and ∞.*

Proof. The result follows from the following inequality:

$$\lambda_i^{-1} \sum_{k \in l_i} a_{nk} [f(q(\Delta_m^r x_k))]^{p_k} \leq D \lambda_i^{-1} \sum_{k \in l_i} a_{nk} [f(q(\Delta_m^{r-1} x_k))]^{p_k} + D \lambda_i^{-1} \sum_{k \in l_i} a_{nk} [f(q(\Delta_m^{r-1} x_{k+1}))]^{p_k}.$$

Proceeding inductively, we have $[V_\lambda^E, A, \Delta_m^j, f, p]_Z \subset [V_\lambda^E, A, \Delta_m^r, f, p]_Z$, for $j = 0, 1, 2, ..., r-1$. The inclusion is strict follows from the following examples. ∎

Example 1. Let $E = C, q(x) = |x|$; $\lambda_i = 1$, for all $i \in N$; $p_k = 3$, for all $k \in N$. Let $f(x) = x$, for all $x \in [0, \infty)$; $a_{nk} = k^{-2}$, for all $n, k \in N$; $m = 1, r \geq 1$. Then consider the sequence $x = (x_k)$ defined by $x_k = k^r$, for all $k \in N$. We have $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1} x_k = (-1)^r r!(k + (r-1)2^{-1})$, for all $k \in N$. Hence, $(x_k) \in [V_\lambda^C, A, \Delta^r, f, p]_Z$ for $Z = 1, \infty$ but $(x_k) \notin [V_\lambda^C, A, \Delta^{r-1}, f, p]_Z$, for $Z = 1, \infty$.

Example 2. In the above example,if one considers $p_k = 2$, for all $k \in N$, then $(x_k) = (k^r) \in [V_\lambda^C, A, \Delta^r, f, p]_0$, but $(x_k) \notin [V_\lambda^C, A, \Delta^{r-1}, f, p]_0$.

Theorem 5. *Let f be a modulus function, then* (*a*) *Let* $0 < p_k \leq q_k$, *for all* $k \in N$ *and* $(q_k p_k^{-1})$ *be bounded, then* $[V_\lambda^E, A, \Delta_m^r, f, q]_1 \subset [V_\lambda^E, A, \Delta_m^r, f, p]_1$.

(*b*) *If* $0 < \inf_k p_k < p_k \leq 1$, *for all* k, *then* $[V_\lambda^E, A, \Delta_m^r, f, p]_1 \subset [V_\lambda^E, A, \Delta_m^r, f]_1$.

(*c*) *If* $1 \leq p_k < \sup_k p_k < \infty$, then $[V_\lambda^E, A, \Delta_m^r, f]_1 \subset [V_\lambda^E, A, \Delta_m^r, f, p]_1$.

Proof. (*a*) Following the technique applied in page 351, in the discussions after Theorem 5 by Maddox[9], one can easily prove this part.

The proofs of the parts (b) and (c) are consequence of part (a). ■

Theorem 6. *Let* f *be a modulus function and* s *be a positive integer. Then,*

$$[V_\lambda^E, A, \Delta_m^r, f, q]_\infty \subset [V_\lambda^E, A, \Delta_m^r, f, p]_\infty$$

Proof. Let $\varepsilon > 0$ be given and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $y_k = f^{s-1}(q(\Delta_m^r x_k - L))$ and consider

$$\sum_{k \in I_r} a_{nk}[f(y_k)]^{p_k} = \sum_{\substack{k \in I_r \\ y_k \leq \delta}} a_{nk}[f(y_k)]^{p_k} + \sum_{\substack{k \in I_r \\ y_k > \delta}} a_{nk}[f(y_k)]^{p_k}.$$

Since f is continuous, we have

$$\sum_{\substack{k \in I_r \\ y_k \leq \delta}} a_{nk}[f(y_k)]^{p_k} \leq \varepsilon^H \sum_{\substack{k \in I_r \\ y_k \leq \delta}} a_{nk} \tag{2}$$

and for $y_k > \delta$, we use the fact that, $y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}$ and so, by the definition of f, we have for $y_k > \delta$,

$$f(y_k) < 2f(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{\lambda_i} \sum_{\substack{k \in I_r \\ y_k \leq \delta}} a_{nk}[f(y_k)]^{p_k} \leq \max(1, (2f(1)\delta^{-1})^H) \frac{1}{\lambda_i} \sum_{\substack{k \in I_i \\ y_k \leq \delta}} a_{nk} y_k^{p_k} \tag{3}$$

From (2) and (3), we obtain $[V_\lambda^E, A, \Delta_m^r, f, q]_\infty \subset [V_\lambda^E, A, \Delta_m^r, f, p]_\infty$. ■

4. Statistical convergence

The notion of statistical convergence was introduced by Fast[5] and studied by various authors. Mursaleen[12] introduced the new concept of λ-statistical convergence as follows:

A sequence $x = (x_k)$ is said to be λ-statistically convergent or s_λ-convergent to L if for every $\varepsilon > 0$,

$$\lim_{i\to\infty} \frac{1}{\lambda_i} |\{k \in I_i : |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write S_λ-lim $x = L$ or $x_k \to L(S_\lambda)$. S_λ denotes the class of all λ-statistically convergent sequences.

Definition. A sequence $x = (x_k)$ is said to be $(\lambda, \Delta_m^r, E, p)$-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{i\to\infty} \lambda_i^{-1} |\{k \in I_i : [q(\Delta_m^r x_k - L)]^{p_k} \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $[S_\lambda^E, \Delta_m^r, p] - \lim x = L$ or $x_k \to L([S_\lambda^E, \Delta_m^r, p])$. In the case $p_k = 1$, for all $k \in N$ and $m = 1$, $[S_\lambda^E, \Delta_m^r, p]$ reduces to $S_\lambda(\Delta^r, E)$, which is studied by Et, Altin and Altinok [3].

For $\lambda_i = i$, for all $i \in N$, the space $[S_\lambda^E, \Delta_m^r, p]$ is denoted by $[S^E, \Delta_m^r, p]$. The following definition will be used in this section:

$$\ell_\infty(\Delta_m^r, E, p) \ = \ \left\{ x \in w(E) : \sup_k [q(\Delta_m^r x_k)]^{p_k} < \infty \right\}.$$

■

Theorem 7. (a) *$[V_\lambda^E, \Delta_m^r, p]_1 \subset [S_\lambda^E, \Delta_m^r, p]$ and the inclusion is strict.*
(b) *If $x \in \ell_\infty(\Delta_m^r, E, p) \cap [S_\lambda^E, \Delta_m^r, p]$, then $x \in [V_\lambda^E, \Delta_m^r, p]_1$.*
(c) *$\ell_\infty(\Delta_m^r, E, p) \cap [S_\lambda^E, \Delta_m^r, p] = [V_\lambda^E, \Delta_m^r, p]_1 \cap \ell_\infty(\Delta_m^r, E, p)$.*

Proof. (a) Let $\varepsilon > 0$ and $x_k \to [V_\lambda^E, \Delta_m^r, p]_1$. Then we have

$$\lambda_i^{-1} \frac{1}{\lambda_i} \sum_{k\in I_i} (q(\Delta_m^r x_k - L))^{p_k} \geq \varepsilon^H |\{k \in I_i : (q(\Delta_m^r x_k - L))^{p_k} \geq \varepsilon\}|.$$

Hence $x_k \to [S_\lambda^E, \Delta_m^r, p]$. ■

The inclusion relation is strict follows from the following example.

Example 3. Let $E = C$ and consider the sequence $x = (x_k)$ such that $\Delta_m^r x_k = k$, for all $k = j^2$, $j \in N$ and $\Delta_m^r x_k = 0$, otherwise. Then $x \notin \ell_\infty(\Delta_m^r, E, p)$ and $x \notin [V_\lambda^E, \Delta_m^r, p]_1$ but $x_k \to 0([S_\lambda^E, \Delta_m^r, p])$.

(b). Suppose that $x_k \to L([S_\lambda^E, \Delta_m^r, p])$ and $x \in \ell_\infty(\Delta_m^r, E, p)$, say $\sup_k [q(\Delta_m^r x_k - L)]^{p_k} \leq T$. Given $\varepsilon > 0$, let

$$G_i = \{k \in I_i : [q(\Delta_m^r x_k - L)]^{p_k} \geq \varepsilon\} \text{ and} \tag{4}$$
$$H_i = \{k \in I_i : [q(\Delta_m^r x_k - L)]^{p_k} < \varepsilon\}$$

Then we have

$$\lambda_i^{-1}\sum_{k\in I_i}(q(\Delta_m^r x_k - L))^{p_k} = \lambda_i^{-1}\sum_{k\in G_i}(q(\Delta_m^r x_k - L))^{p_k} + \lambda_i^{-1}\sum_{k\in H_i}(q(\Delta_m^r x_k - L))^{p_k} \le T\lambda_i^{-1}|G_1| + \varepsilon^H. \tag{5}$$

Hence $x_k \to L([V_\lambda^E, \Delta_m^r, p]_1)$.

(c) The proof follows from (a) and (b).

Theorem 8. *If* $\liminf\limits_{i\to\infty} \frac{\lambda_i}{i} > 0$, *then* $[S^E, \Delta_m^r, p]) \subset [S_\lambda^E, \Delta_m^r, p])$.

Proof. For given $\varepsilon > 0$, we get

$$\{k \le i : [q(\Delta_m^r x_k - L)]^{p_k} \ge \varepsilon\} \supset G_1, \text{ for } G_1 \text{ refer Eq(4)}.$$

Thus

$$\begin{aligned} i^{-1}|\{k \le i : [q(\Delta_m^r x_k - L)]^{p_k} \ge \varepsilon\}| &\ge i^{-1}|G_i| \\ &= \frac{\lambda_i}{i}\frac{1}{\lambda_i}|G_i|. \end{aligned}$$

Hence $x \in [S_\lambda^E, \Delta_m^r, p]$. ■

Theorem 9. *Let f be a modulus and* $0 < h = \inf\limits_k p_k \le p_k \le \sup\limits_k p_k = H < \infty$. *Then*

$$[V_\lambda^E, \Delta_m^r, f, p]_1 \subset [S_\lambda^E, \Delta_m^r, p].$$

Proof. Let $x \in [V_\lambda^E, \Delta_m^r, f, p]_1$ and $\varepsilon > 0$ be given

$$\lambda_i^{-1}\sum_{k\in I_i}[f(q(\Delta_m^r x_k - L))]^{p_k} = \lambda_i^{-1}\sum_{k\in G_i}[f(q(\Delta_m^r x_k - L))]^{p_k} + \lambda_i^{-1}\sum_{k\in H_i}[f(q(\Delta_m^r x_k - L))]^{p_k},$$

where G_i and H_i are as in Eq(4).

$$\ge \lambda_i^{-1}\sum_{k\in G_i}[f(q(\Delta_m^r x_k - L))]^{p_k} \ge \lambda_i^{-1}\sum_{k\in G_i}[f(\varepsilon)]^{p_k}$$

$$\ge \lambda_i^{-1}\sum_{k\in G_i}\min([f(\varepsilon)]^h, [f(\varepsilon)]^H) \ge \lambda_i^{-1}|G_i|\min([f(\varepsilon)]^h, [f(\varepsilon)]^H).$$

Hence $x \in [S_\lambda^E, \Delta_m^r, f, p]$. ■

Theorem 10. *Let f be a bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then*

$$[V_\lambda^E, \Delta_m^r, f, p]_1 \supset [S_\lambda^E, \Delta_m^r, p].$$

Proof. Suppose that f is bounded. Let $\varepsilon > 0$ be given. Since f is bounded, there exists an integer T such that $f(x) < T$ for all $x \geq 0$. Then

$$\lambda_i^{-1} \sum_{k \in I_i} [f(q(\Delta_m^r x_k - L))]^{p_k} = \lambda_i^{-1} \sum_{k \in G_i} [f(q(\Delta_m^r x_k - L))]^{p_k} + \lambda_i^{-1} \sum_{k \in H_i} [f(q(\Delta_m^r x_k - L))]^{p_k},$$

where G_i and H_i are as in Eq(4).

$$\leq \lambda_i^{-1} \sum_{k \in G_i} \max[T^h, T^H] + \lambda_i^{-1} \sum_{k \in H_i} [f(\varepsilon)]^{p_k}$$

$$\leq \max[T^h, T^H] \lambda_i^{-1} |G_i| + \max([f(\varepsilon)]^h, [f(\varepsilon)]^H).$$

■

Hence $x \in [V_\lambda^E, \Delta_m^r, f, p]_1$.

Theorem 11. *Let f be bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then*

$$[V_\lambda^E, \Delta_m^r, f, p]_1 = [S_\lambda^E, \Delta_m^r, p].$$

Proof. Let f be bounded, by Theorem 9 and Theorem 10, we have

$$[V_\lambda^E, \Delta_m^r, f, p]_1 \supset [S_\lambda^E, \Delta_m^r, p].$$

Conversely, suppose that f is unbounded. Then there exists a sequence (z_k) of positive numbers with $f(z_k) = k^2$ for $k \in N$. If we choose $\Delta_m^r x_j = z_k$, for all $j = k^2, j \in N$ and $\Delta_m^r = 0$ otherwise. Then we have

$$\lambda_i^{-1} |\{k \in I_i : |\Delta_m^r x_k|^{p_k} \geq \varepsilon\}| \leq (\lambda_{i-1})^{\frac{1}{2}} \lambda_i^{-1},$$

for all i and so $x \in [S_\lambda^E, \Delta_m^r, p]$, but $x \notin [V_\lambda^E, \Delta_m^r, p]_1$, for $E = C$. This contradicts to $[V_\lambda^E, \Delta_m^r, f, p]_1 = [S_\lambda^E, \Delta_m^r, p]$. ■

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