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A GENERALIZATION OF SOME RESULTS ON MULTI-VALUED WEAKLY PICARD MAPPINGS IN $b-$METRIC SPACE

Abstract. In this paper, we establish some convergence results in a complete $b-$metric space for the Picard iteration associated to two multi-valued weak contractions by employing the concepts of monotone and comparison functions. Our results generalize and extend those of Berinde and Berinde [8], Daffer and Kaneko [15] and Nadler [27]. Theorem 2.1 in our paper generalizes Theorem 5 of Nadler [27] and a recent result of Berinde and Berinde [8], it also extends, improves and unifies several classical results pertainning to single and multi-valued contractive mappings in the fixed point theory. Also, Theorem 2.3 is a generalization and extension of Theorem 5 of Nadler [27] as well as Theorem 4 of Berinde and Berinde [8].

Key words: multi-valued weak contraction; single-valued contractive mappings.

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1. Introduction

Let (X, d) be a complete metric space and $CB(X)$ denote the family of all nonempty closed and bounded subsets of X. For $A,\ B \subset X$, define the distance between A and B by $D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$, the diameter of A and B by $\delta(A, B) = \sup\{d(a, b) \mid a \in A,\ b \in B\}$, and the Hausdorff-Pompeiu metric on $CB(X)$ by

$$H(A, B) = \max\{\sup\{d(a, B) \mid a \in A\},\ \sup\{d(b, A) \mid b \in B\}\}$$

$H(A, B)$ is induced by d.

Let $P(X)$ be the family of all nonempty subsets of X and $T : X \to P(X)$ a multi-valued mapping. Then, an element $x \in X$ such that $x \in T(x)$ is called a *fixed point of* T. Denote the set of all the fixed points of T by Fix (T), that is, Fix $(T) = \{x \in X \mid x \in T(x)\}$.

Markins [25] and Nadler [27] initiated the study of fixed point theorems for multi-valued operators. The celebrated Banach's fixed point theorem is extended to the following result of Nadler [27] from the single-valued maps to the multi-valued contractive maps.

Theorem 1 (**Nadler [27]**). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a set-valued α-contraction, that is, a mapping for which there exists a constant $\alpha \in (0, 1)$, such that*

$$H(Tx, Ty) \leq \alpha d(x, y), \ \forall \ x, \ y \in X. \tag{1}$$

Then T has at least one fixed point.

For the Banach's fixed point theorem and its various generalizations in single-valued case, we refer to Agarwal et al [1], Banach [2], Berinde [3, 4, 5, 6, 7] and some other references in the reference section of this paper.

Apart from Markins [25] and Nadler [27], several other papers have been devoted to the treatment of multi-valued operators and these include Berinde and Berinde [8], Ciric [12], Ciric and Ume [13, 14], Daffer and Kaneko [15], Itoh [18], Kaneko [20, 21], Kubiaczyk and Ali [23], Lim [24], Mizoguchi [26] and some others in the reference section. The following definitions shall be required in the sequel.

Definition 1. *A function $\varphi : I\!R_+ \to I\!R_+$ is called (c)-comparison if it satisfies:*

(i) φ is monotone increasing;

(ii) $\varphi^n(t) \to 0$ as n $\to \infty$, $\forall$ t > 0 (φ^{n} *stands for the nth iterate of* φ);

(iii) $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ *for all* $t > 0$.

We say that φ is a comparison function if it satisfies (i) and (ii) only. See [3, 4] and [33] for detail.

Remark 1. Every comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\varphi(t) < t$.

Definition 2. *Let (X, d) be a metric space and $T : X \to P(X)$ a multi-valued operator. T is said to be a multi-valued weakly Picard (MWP) Operator if and only if for each $x \in X$ and any $y \in T(x)$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that*

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$ for all $n = 0, 1, \cdots$;

(iii) the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent and its limit is a fixed point of T.

Remark 2. A sequence $\{x_n\}_{n=0}^{\infty}$ satisfying conditions (i) and (ii) in Definition 2 will be called a *sequence of successive approximations* of T, starting from (x, y) or a *Picard iteration* associated to T or a *(Picard) orbit* of T at the initial point x_0.

Definition 3. *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *a multi-valued operator.* T *is said to be a multi-valued weak contraction or a multi-valued* (θ, L)*-contraction if and only if there exist two constants* $\theta \in (0, 1)$ *and* $L \geq 0$ *such that*

$$H(Tx, Ty) \leq \theta d(x, y) + LD(y, Tx), \ \forall\ x, y \in X. \tag{2}$$

For Definition 3 and Definition 3, see [8].

We state the following results on multi-valued operators:

Theorem 2 (**Berinde and Berinde [8]**)**.** *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *a multi-valued* (θ, L)*-weak contraction. Then,*

(i) *Fix* $(T) \neq \phi$;

(ii) *for any* $x_0 \in X$*, there exists an orbit* $\{x_n\}_{n=0}^{\infty}$ *of* T *at the point* x_0 *that converges to a fixed point* u *of* T*, for which the following estimates hold:*

$$d(x_n, u) \leq \frac{h^n}{1-h} d(x_1, x_0), \quad n = 0, 1, \cdots,$$

$$d(x_n, u) \leq \frac{h}{1-h} d(x_n, x_{n-1}), \quad n = 1, 2, \cdots,$$

for a certain constant $h < 1$.

By replacing the term $\theta d(x, y)$ in condition (2) by $\alpha(d(x, y))d(x, y)$, where the function $\alpha : [0, \infty) \to [0, 1)$ satisfies $\limsup_{r \to t^+} \alpha(r) < 1$, for every $t \in [0, \infty)$, then the authors obtained the following result:

Theorem 3 (**Berinde and Berinde [8]**)**.** *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *a generalized multi-valued* $(\alpha,\ L)$*-weak contraction, that is, a mapping for which there exists a function* $\alpha : [0, \infty) \to [0, 1)$ *satisfying* $\limsup_{r \to t^+} \alpha(r) < 1$*, for every* $t \in [0, \infty)$*, such that*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + LD(y, Tx), \ \forall\ x, y \in X. \tag{3}$$

Then T *has at least one fixed point.*

Definition 4 (**Czerwik [11]**)**.** *Let* X *be a (nonempty) set and* $s \geq 1$ *a real number. A function* $d : X \times X \to I\!R_+$ *is said to be a b-metric if* $\forall\ x, y, z \in X$,

(i) $d(x, y) = 0$ iff x = y;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x,z) \leq s[d(x,y)+d(y,z)]$.
The pair (X,d) is called a $b-$metric space. In fact, the class of $b-$metric spaces is effectively larger than that of metric spaces, since a $b-$metric is a metric when $s=1$.

In this paper, we obtain more general results than those of Berinde and Berinde [8] using the following two general contractive definitions:

Definition 5. *Let (X,d) be a $b-$metric space and $T : X \to P(X)$ a multi-valued operator. T is said to be a generalized multi-valued (ψ,φ)-weak contraction if and only if there exist a continuous monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0)=0$ and a continuous (c)-comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$(\star) \quad H(Tx,Ty) \leq q^{-1}[\psi(d(x,y)) + \varphi(D(y,Tx))],\ q>1,\ \forall\ x,y \in X.$$

We also have that T is a generalized multi-valued ϕ-weak contraction if and only if there exist a function $\alpha : [0,\infty) \to [0,1)$ and two continuous monotone increasing functions $\phi_1, \phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi_1(0)=1$ and $\phi_2(0)=0$ such that

$$(\star\star)\ H(Tx,Ty) \leq [\alpha(d(x,y))d(x,y)]^{\phi_1(D(y,Tx))} + \phi_2(D(y,Tx)),\ \forall\ x,y \in X,$$

where $\limsup\limits_{r\to t^+} \alpha(r) < 1$, for every $t \in [0,\infty)$.

Remark 3. *(i)* If in condition $(\star)$, $\psi(u) = q\theta u$, $\theta \in (0,1)$, $q\theta < 1$, $\forall\ u \in \mathbb{R}_+$ and $\varphi(v) = qLv$, $L \geq 0$, $\forall\ v \in \mathbb{R}_+$, then we obtain condition (2), which was employed in the proof of Theorem 1.7 by Berinde and Berinde [8] (Theorem 1.7 is Theorem 3 of Berinde and Berinde [8].

(ii) In condition $(\star)$, if $\psi(u) = q\theta u$, $q\theta < 1$, $\theta \in (0,1)$, $\forall\ u \in \mathbb{R}_+$ and $\varphi_2(v) = 0$, $\forall\ v \in \mathbb{R}_+$, then we obtain Theorem 1.1 which is Theorem 5 of Nadler [27].

(iii) In a similar manner, the condition $(\star\star)$ reduces to that employed by Berinde and Berinde [8] if $\phi_1(u) = 1$, $\forall\ u \in \mathbb{R}_+$ and $\phi_2(v) = Lv$, $L \geq 0$, $\forall\ v \in \mathbb{R}_+$, while we obtain the contractive condition in Corollary 2.2 of Daffer and Kaneko [15] when $\phi_2(v) = 0$, $\forall\ v \in \mathbb{R}_+$.

However, we shall require the following Lemma in the sequel.

Lemma 1. *Let (X,d) be a metric space. Let $A, B \subset X$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that*

$$d(a,b) \leq qH(A,B).$$

Lemma 1 is contained in Berinde and Berinde [8], Ciric [12] and Rus [32] in a metric space setting.

Lemma 2 (Nadler [27]). *: Let $A, B \subset CB(X)$ and let $a \in A$. Then, there exists $b \in B$ such that*

$$d(a,b) \leq H(A,B) + \eta.$$

Remark 4. The constants α and α^k, $k \geq 1$, play the role of η in (1). We shall employ Lemma 2 in the proof of Theorem 2.4 in the sequel.

2. Main results

In this section, we shall establish our main results:

Theorem 4. *Let (X,d) be a complete $b-$metric space with continuous $b-$metric and $T : X \to CB(X)$ a generalized multi-valued $(\psi,\ \varphi)$-weak contraction. Suppose that $\psi : I\!R_+ \to I\!R_+$ is a continuous (c)-comparison function and $\varphi : I\!R_+ \to I\!R_+$ is a continuous monotone increasing function such that $\varphi(0) = 0$. Then,*
(i) Fix $(T) \neq \phi$;
(ii) for any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point x^* of T;
(iii) the a priori and the a posteriori error estimates are given by

$$d(x_n, x^*) \leq s \sum_{k=0}^{\infty} \psi^{k+n}(d(x_0, x_1)), \quad s \geq 1,\ n = 1, 2, \cdots, \tag{4}$$

$$d(x_n, x^*) \leq s \sum_{k=0}^{\infty} \psi^{k}(d(x_{n-1}, x_n)), \quad s \geq 1,\ n = 1, 2, \cdots, \tag{5}$$

respectively.

Proof. Let $q > 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$, that is, $x_1 \in Tx_1$, which implies that Fix $(T) \neq \phi$.

Let $H(Tx_0, Tx_1) \neq 0$. Then, we have by Lemma 1 that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq qH(Tx_0, Tx_1),$$

so that by $(\star)$ we have

$$\begin{aligned} d(x_1, x_2) &\leq q\,q^{-1}[\psi(d(x_0, x_1)) + \varphi(D(x_1, Tx_0))] \\ &= \psi(d(x_0, x_1)) + \varphi(D(x_1, x_1)) \\ &= \psi(d(x_0, x_1)). \end{aligned}$$

If $H(Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$, that is, $x_2 \in Tx_2$. Let $H(Tx_1, Tx_2) \neq 0$. Again, by Lemma 1.12, there exists $x_3 \in Tx_2$ such that

$$\text{(6)}\quad \begin{aligned} d(x_2, x_3) &\leq qH(Tx_1, Tx_2) \leq q\,q^{-1}[\psi(d(x_1, x_2)) + \varphi(D(x_2, Tx_1))] \\ &= \psi(d(x_1, x_2)) + \varphi(D(x_2, x_2)) \\ &= \psi(d(x_1, x_2)) \leq \psi^2(d(x_0, x_1)) \end{aligned}$$

By induction, we obtain

$$\text{(7)}\qquad d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

Therefore, we have by the property (iii) of Definition 4 that

$$\text{(8)}\quad \begin{aligned} d(x_n, x_{n+p}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})] \\ &\leq s[\psi^n(d(x_0, x_1)) + \psi^{n+1}(d(x_0, x_1)) + \cdots + \psi^{n+p-1}(d(x_0, x_1))] \end{aligned}$$

$$\text{(9)}\qquad d(x_n, x_{n+p}) = s\sum_{k=n}^{n+p-1} \psi^k(d(x_0, x_1))$$

From (9), we have

$$\text{(10)}\quad \begin{aligned} d(x_n, x_{n+p}) &\leq s\sum_{k=n}^{n+p-1} \psi^k(d(x_0, x_1) \\ &= s\left[\sum_{k=0}^{n+p-1} \psi^k(d(x_0, x_1) - \sum_{k=0}^{n-1} \psi^k(d(x_0, x_1)\right] \to 0 \text{ as n} \to \infty. \end{aligned}$$

We therefore have from (10), that for any $x_0 \in X$, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since (X, d) is a complete $b-$metric space, then $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in X$. That is,

$$\text{(11)}\qquad \lim_{n\to\infty} x_n = x^*.$$

Therefore, by $(\star)$, we have that

$$\text{(12)}\quad \begin{aligned} D(x^*, Tx^*) &\leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\ &\leq s[d(x^*, x_{n+1}) + H(Tx_n, Tx^*)] \\ &\leq sd(x^*, x_{n+1}) + sq^{-1}[\psi(d(x_n, x^*)) + \varphi(D(x^*, Tx_n))] \end{aligned}$$

By using (11), the continuity of the functions ψ, φ and the fact that $x_{n+1} \in Tx_n$, then $\varphi(D(x^*, Tx_n)) \to 0$ as $n \to \infty$ and $\psi(d(x_n, x^*)) \to 0$ as $n \to \infty$. It follows from (12) that $D(x^*, Tx^*) = 0$ as $n \to \infty$. Since Tx^* is closed, then $x^* \in Tx^*$.

To prove the a priori error estimate in (4), we have from (9) that

$$d(x_n, x_{n+p}) \leq s \sum_{k=n}^{n+p-1} \psi^k(d(x_0, x_1)) = s \sum_{k=0}^{p-1} \psi^{n+k}(d(x_0, x_1)),$$

from which it follows by the continuity of the $b-$metric that

$$d(x_n, x^*) = d(x^*, x_n) = \lim_{p\to\infty} d(x_{n+p}, x_n) \leq s \sum_{k=0}^{\infty} \psi^{n+k}(d(x_0, x_1)),$$

giving the result in (4). To prove the a posteriori estimate in (5), we get by condition ($\star$) and Lemma 1 that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq qH(Tx_{n-1}, Tx_n) \\ &\leq q\, q^{-1}[\psi(d(x_{n-1}, x_n)) + \varphi(D(x_n, Tx_{n-1}))] \\ &= \psi(d(x_{n-1}, x_n)) + \varphi(D(x_n, x_n)) = \psi(d(x_{n-1}, x_n)). \end{aligned}$$

Also, we have

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \leq \psi^2(d(x_{n-1}, x_n)),$$

so that in general, we obtain

$$d(x_{n+k}, x_{n+k+1}) \leq \psi^{k+1}(d(x_{n-1}, x_n)), \quad k = 0, 1, \cdots . \tag{13}$$

Using (13) in (8) yields

$$\begin{aligned} d(x_n, x_{n+p}) &\leq s[\psi(d(x_{n-1}, x_n)) + \psi^2(d(x_{n-1}, x_n)) + \cdots \\ &\qquad + \psi^{p-1}(d(x_{n-1}, x_n))] \\ &= s \sum_{k=0}^{p-1} \psi^k(d(x_{n-1}, x_n)) \qquad (14) \end{aligned}$$

Again, by taking limits in (14) as $p \to \infty$ and using the continuity of the $b-$metric, we have

$$d(x_n, x^*) = d(x^*, x_n) = \lim_{p\to\infty} d(x_{n+p}, x_n) \leq s \sum_{k=0}^{\infty} \psi^k(d(x_{n-1}, x_n)),$$

giving the required a posteriori error estimate. ■

Remark 2. Theorem 4 is a generalization and extension of Theorem 2 (which is itself Theorem 3 of Berinde and Berinde [8]). It is also a generalization and extension of Theorem 1 (which is Theorem 5 of Nadler [27]). Indeed, Theorem 4 is a generalization and extension of a multitude of results in the literature pertainning to the single-valued and multi-valued cases.

Theorem 5. *Let (X, d) be a complete $b-$metric space with continuous $b-$metric and $T : X \to CB(X)$ a generalized multi-valued ϕ-weak contraction. Suppose that there exist a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \alpha(r) < 1$, for every $t \in [0, \infty)$ and two continuous monotone increasing functions $\phi_1, \phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi_1(0) = 1$ and $\phi_2(0) = 0$. Then, T has at least one fixed point.*

Proof. The theorem is proved using the idea of Berinde and Berinde [8] as well as Daffer and Kaneko [15]. Suppose that $x_0 \in X$ and $x_1 \in Tx_0$. We choose a positive integer N_1 such that

$$\alpha^{N_1}(d(x_0, x_1)) \leq [1 - \alpha(d(x_0, x_1))]d(x_0, x_1). \tag{15}$$

By Lemma 2, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + \alpha^{N_1}(d(x_0, x_1)). \tag{16}$$

Using $(\star\star)$ and (15) in (16), then we have

$$\begin{aligned} d(x_1, x_2) &\leq [\alpha(d(x_0, x_1))d(x_0, x_1)]^{\phi_1(D(x_1, Tx_0))} \\ &\quad + \phi_2(D(x_1, Tx_0)) + \alpha^{N_1}(d(x_0, x_1)) \\ &= \alpha(d(x_0, x_1))d(x_0, x_1) + \alpha^{N_1}(d(x_0, x_1)) \leq d(x_0, x_1). \end{aligned}$$

Now, we choose again a positive integer N_2, $N_2 > N_1$ such that

$$\alpha^{N_2}(d(x_1, x_2)) \leq [1 - \alpha(d(x_1, x_2))]d(x_1, x_2). \tag{17}$$

Since $Tx_2 \in CB(X)$, by Lemma 2 again, we can select $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + \alpha^{N_2}(d(x_1, x_2)). \tag{18}$$

Again, using $(\star\star)$ and (17) in (18), then we get

$$\begin{aligned} d(x_2, x_3) &\leq [\alpha(d(x_1, x_2))d(x_1, x_2)]^{\phi_1(D(x_2, Tx_1))} \\ &\quad + \phi_2(D(x_2, Tx_1)) + \alpha^{N_2}(d(x_1, x_2)) \\ &= \alpha(d(x_1, x_2))d(x_1, x_2) + \alpha^{N_2}(d(x_1, x_2)) \leq d(x_1, x_2). \end{aligned}$$

By induction, since $Tx_k \in CB(X)$, for each k, we may choose a positive integer N_k such that

$$\alpha^{N_k}(d(x_{k-1}, x_k)) \leq [1 - \alpha(d(x_{k-1}, x_k))]\, d(x_{k-1}, x_k). \tag{19}$$

By selecting $x_{k+1} \in Tx_k$ such that

$$d(x_k, x_{k+1}) \leq H(Tx_{k-1}, Tx_k) + \alpha^{N_k}(d(x_{k-1}, x_k)), \tag{20}$$

so that using ($\star\star$) and (19) in (20) yield

$$d(x_k, x_{k+1}) \le d(x_{k-1}, x_k). \tag{21}$$

Let $d_k = d(x_k, x_{k-1})$, $k = 1, 2, \cdots$. The inequality relation (21) shows that the sequence $\{d_k\}$ of nonnegative numbers is decreasing. Therefore, $\lim\limits_{k\to\infty} d_k$ exists. Thus, let $\lim\limits_{k\to\infty} d_k = c \ge 0$.

We now prove that the Picard iteration or orbit $\{x_k\} \subset X$ so generated is a Cauchy sequence. By condition on α, for $t = c$ we have $\limsup\limits_{t\to c^+} \alpha(t) < 1$. For $k \ge k_0$, let $\alpha(d_k) < h$, where $\lim\limits_{t\to c^+} \sup \alpha(t) < h < 1$. Using (20), we have by deduction that $\{d_k\}$ satisfies the recurrence inequality:

$$d_{k+1} \le d_k \alpha(d_k) + \alpha^{N_k}(d_k), \ k = 1, 2, \cdots . \tag{22}$$

Using induction in (22) leads to

$$d_{k+1} \le \Pi_{j=1}^{k} \alpha(d_j) d_1 + \sum_{m=1}^{k-1} \Pi_{j=m+1}^{k} \alpha(d_j) \alpha^{N_m}(d_m) + \alpha^{N_k}(d_k), \quad k \ge 1. \tag{23}$$

We now find a suitable upper bound for the right-hand side of (23), using the fact that $\alpha < 1$ as follows:

$$\begin{aligned} d_{k+1} &\le \Pi_{j=1}^{k} \alpha(d_j) d_1 + \sum_{m=1}^{k-1} \Pi_{j=m+1}^{k} \alpha(d_j) \alpha^{N_m}(d_m) + \alpha^{N_k}(d_k) \\ &< d_1 h^k + \sum_{m=1}^{k-1} h^{k-m} h^{N_m} + h^{N_k} = d_1 h^k + h^k \sum_{m=1}^{k-1} h^{N_m - m} + h^{N_k} \\ &\le C_1 h^k + C_2 h^k + C_3 h^k = C_4 h^k, \ \text{where } C_4 = C_1 + C_2 + C_3. \end{aligned} \tag{24}$$

and C_1, C_2, C_3, C_4 are constants.

Now, for $k \ge k_0$ and $p \in \mathbb{N}$, we have by using (24) and the repeated application of the triangle inequality that

$$\begin{aligned} d(x_k, x_{k+p}) &\le s[d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{k+p-1}, x_{k+p})] \\ &= s[d_{k+1} + d_{k+2} + \cdots + d_{k+p}] \\ &\le s[C_4(h^k + h^{k+1} + \cdots + h^{k+p-1})] \\ &= C_4 \left(\frac{1-h^p}{1-h}\right) h^k s = C_5 h^k s, \end{aligned} \tag{25}$$

where C_5 is a constant.

Since $0 < h < 1$, then the right-hand side of (25) tends to 0 as $k \to \infty$, showing that $\{x_k\}$ is a Cauchy sequence. Therefore, $x_k \to u \in X$ as $k \to \infty$ since X is a complete $b-$metric space. So,

$$
\begin{aligned}
(26) \qquad D(u,Tu) &\leq s[d(u,x_k) + d(x_k,Tu)] \leq s[d(u,x_k) + H(Tx_{k-1},Tu)] \\
&\leq sd(u,x_k) + s\left[\alpha(d(x_{k-1},u))d(x_{k-1},u)\right]^{\phi_1(D(u,Tx_{k-1}))} \\
&\quad + \phi_2(D(u,Tx_{k-1})) \\
&< sd(u,x_k) + s\left[h \ d(x_{k-1},u)\right]^{\phi_1(D(u,Tx_{k-1}))} \\
&\quad + s\phi_2(D(u,Tx_{k-1})), \quad s \geq 1.
\end{aligned}
$$

By using the fact that $x_k \in Tx_{k-1}$ and $x_k \to u$ as $k \to \infty$, we have $D(u,Tx_{k-1}) \to 0$ as $k \to \infty$. We therefore, have by the continuity of ϕ_j $(j = 1,2)$ that $\phi_1(D(u,Tx_{k-1})) \to 1$ as $k \to \infty$ and $\phi_2(D(u,Tx_{k-1})) \to 0$ as $k \to \infty$. Hence, since the right-hand side terms of (26) tend to zero as $k \to \infty$, we have $u \in Tu$. Using the continuity of the $b-$metric in (25) as $p \to \infty$, we obtain an error estimate

$$
d(x_k,u) \ = \ \lim_{p\to\infty} d(x_k,x_{k+p}) \ \leq \ C_5 h^k s, \quad k \geq k_0, \ \ s \geq 1,
$$

for the Picard iteration process under condition $(\star\star)$. ∎

Remark 3. Theorem 5 generalizes and extends Theorem 4 of Berinde and Berinde [8], Theorem 2.1 of Daffer and Kaneko [15] and some related results in kaneko [20, 21] as well as Nadler's fixed point theorem [27]. Similar results in single-valued case are also extended by Theorem 5.

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