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GENERALIZED GROWTH OF ENTIRE HARMONIC FUNCTIONS

Abstract. Let $H(x)$, $x = (x_1, x_2, \ldots, x_n)$, be an entire harmonic function in $\Re^n$.Fryant and Shankar [1] had obtained growth properties of H explicitly in terms of its Fourier coefficients. In this paper, we obtain the characterizations of generalized order and type and introduce the generalized lower order for H.Special case of functions of slow growth has also been considered. Our results generalize some of the results obtained in [1].

Key words: harmonic function, spherical harmonics, entire function, Fourier coefficients, order and type, generalized order and generalized type, lower order.

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1. Introduction

Let H_k denote the set of all homogeneous harmonic polynomials in the n variables $x_1, x_2, \ldots x_n$, having real coefficients. Then H_k is a vector space of dimension d_k where

$$d_k = (n + 2k - 2) \frac{(n + k - 3)!}{k!\,(n - 2)!}$$

(see [9, p.145]) with respect to the inner product

$$< f, g > = \frac{1}{\omega_{n-1}} \int\limits_{|x|=1} f(x)\, g(x)\, d\sigma_1,$$

where ω_{n-1} denotes the area of the unit sphere $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2} = 1$ and $d\sigma_1$ is the element of surface area on this sphere. Let $\{Y_k^m\}_{m=1}^{d_k}$ be an orthonormal basis for H_k.These orthonormal, homogeneous and harmonic polynomials Y_k^m are called spherical harmonics of degree k and satisfy the orthogonality relation (see [9, p.141]):

$$\int\limits_{|x|=1} Y_k^{\mu}(x)\, Y_j^{\nu}(x) d\sigma_1 = 0 \qquad \text{if} \quad j \neq k,$$

Let H be harmonic in a neighborhood Ω of the origin in $\Re^n$, that is, H satisfies the Laplace equation

$$\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \cdots + \frac{\partial^2 H}{\partial x_n^2} = 0$$

throughout Ω. Then on the sphere $\{x : |x| \leq \eta,\ 0 < \eta < \infty\}$ contained in Ω, the function H has the Fourier series expansion in terms of the spherical harmonics as

$$H(x) = \sum_{k=0}^{\infty} \sum_{m=1}^{d_k} a_{km}\, r^k Y_k^m(x/r), \qquad r = |x| \tag{1}$$

where

$$a_{km} = \frac{1}{\omega_{n-1}\, \rho^{n+2k-1}} \int\limits_{|x|=\eta} H(x)\, Y_k^m(x)\, d\sigma$$

and $d\sigma = \eta^{n-1} d\sigma_1$ is the element of surface area on the sphere $|x| = \eta$. The series (1) converges in the L^2 norm on the sphere $|x| = \eta$ and converges uniformly and absolutely on compact subsets of the interior $|x| < \eta$.

Let us put $|a_k| = \left[\sum\limits_{i=1}^{d_k} (a_{ki})^2\right]^{1/2}$. Fryant and Shankar [1] obtained radius of convergence of the disk of uniform convergence of the series (1) in terms of the sequence $\{a_k\}$. Thus we have

Theorem A. *[1, Theorem 1] The series*

$$\sum_{k=0}^{\infty} \sum_{m=1}^{d_k} a_{km}\, r^k Y_k^m(x/r), \qquad r = |x|,$$

converges absolutely and uniformly on compact subsets of the disk $|x| < R$ where

$$R^{-1} = \limsup_{k \to \infty} |a_k|^{1/k}$$

and further, such convergence obtains within no larger ball centered at the origin. The series(1) represents an entire function when $R = \infty$.

Let $M(r, H) = M(r) = \max\limits_{|x|=r} |H(x)|$. On the lines of usual definitions of order and type for an entire function of a complex variable, Fryant and Shankar [1] defined the growth parameters for an entire harmonic function $H(x)$. Thus the order ρ of $H(x)$ is defined to be

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}, \tag{2}$$

and for $0 < \rho < \infty$, the type T of $H(x)$ is defined as

$$T = \limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}}. \tag{3}$$

Fryant and Shankar [1, Theorems 3 and 4] obtained characterizations of the order and type in terms of the coefficients $\{a_k\}$. However, they did not consider the lower order of H as well as the further classification of entire harmonic functions which are of fast growth or zero order. In this note, we obtain the characterizations of growth parameters for these classes of functions. We use the generalized functions as given by Seremeta [6] and Shah [7].

2. Generalized order and generalized type

We first give some definitions.

Let $\phi : [a, \infty) \to R$ be a real valued function such that
(i) $\phi(x) > 0$,
(ii) $\phi(x)$ is differentiable $\forall\, x \in [a, \infty)$,
(iii) $\phi(x)$ is strictly increasing, and
(iv) $\phi(x) \to \infty$ as $x \to \infty$.
Further, for every real valued function $\gamma(x)$ such that $\gamma(x) \to 0$ as $x \to \infty$, ϕ satisfies

$$\lim_{x\to\infty} \frac{\phi\,[(1+\gamma(x)\,)\,x]}{\phi(x)} = 1. \tag{4}$$

Then ϕ is said to belong to the class L^0. The function $\phi(x)$ is said to belong to the class Λ if $\phi(x) \in L^0$ and in place of (4), satisfies the stronger condition

$$\lim_{x\to\infty} \frac{\phi\,(cx)}{\phi(x)} = 1, \tag{5}$$

for all c, $0 < c < \infty$. Functions ϕ satisfying (5) are also called slowly increasing functions (see [5]).

Let $f(z)$ be an entire function, its maximum modulus function being given by

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

Using the generalized functions of the class L^0 and Λ, Seremeta [6], obtained the following characterizations:

Theorem B. *Let $\alpha(t) \in \Lambda$, $\beta(t) \in L^0$. Set $F(t,c) = \beta^{-1}[c\,\alpha(t)]$. If $d\,F(t,c)/d\ln t = O(1)$ as $t \to \infty$ for all c, $0 < c < \infty$, then for the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$,*

$$\limsup_{r\to\infty} \frac{\alpha\,(\ln M(r,f))}{\beta\,(\ln r)} = \limsup_{n\to\infty} \frac{\alpha\,(n)}{\beta\left(\ln |c_n|^{-1/n}\right)}. \tag{6}$$

Theorem C. *Let $\alpha(t) \in L^0$, $\beta(t) \in L^0$, $\gamma(t) \in L^0$. Let ρ be a fixed number, $0 < \rho < \infty$. Set $F(t,\sigma,\rho) = \gamma^{-1}\left\{\left[\beta^{-1}\,(\sigma\,\alpha(t))\right]^{1/\rho}\right\}$. Suppose that for all σ, $0 < \sigma < \infty$, F satisfies: (a) If $\gamma(t) \in \Lambda$and $\alpha(t) \in \Lambda$, then $dF(t,\sigma,\rho)/d\ln t = O(1)$ as $t \to \infty$ (b) If $\gamma(t) \in L^0 - \Lambda$ or $\alpha(t) \in L^0 - \Lambda$, then $\lim_{t\to\infty} d\ln F(t,\sigma,\rho)/d\ln t = 1/\rho$. Then we have*

$$\limsup_{r\to\infty} \frac{\alpha\,(\ln M(r,f))}{\beta\,[(\gamma(r))^{\rho}]} = \limsup_{n\to\infty} \frac{\alpha\,(n/\rho)}{\beta\left\{\left[\gamma\left(e^{1/\rho}\,|c_n|^{-1/n}\right)\right]^{\rho}\right\}}. \tag{7}$$

Later, S.M. Shah [7] called the left hand quantity in (6) as the generalized order $\rho(\alpha,\beta,f)$ and introduced the generalized lower order $\lambda(\alpha,\beta,f)$ as

$$\lambda(\alpha,\beta,f) = \liminf_{r\to\infty} \frac{\alpha\,(\ln M(r,f))}{\beta\,(\ln r)}.$$

Further, Shah proved that

Theorem D. *[7, Theorem 2] Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function. Set $F(t) = \beta^{-1}(\alpha(t))$. Let, for some function $\psi(t)$ tending to ∞ (howsoever slowly) as $t \to \infty$,*

$$\frac{\beta(t\psi(t))}{\beta(e^t)} \to 0 \quad as \quad t \to \infty, \tag{8}$$

$$\frac{dF(t)}{d(\log t)} = 0(1) \quad as \quad t \to \infty, \tag{9}$$

(10) $\quad |c_n/c_{n+1}|$ *is ultimately a non decreasing function of n.*

Then

$$\lambda(\alpha,\beta,f) = \liminf_{n\to\infty} \frac{\alpha\,(n)}{\beta\left(\ln |c_n|^{-1/n}\right)}.$$

In the context of the entire harmonic function $H(x)$ defined above, we introduce the following functions of a complex variable z :

$$g(z) \ = \ \omega_{n-1}^{-1} \sum_{k=0}^{\infty} |a_k| \sqrt{d_k} z^k \quad \text{and} \quad h(z) \ = \ \sum_{k=0}^{\infty} |a_k|^2 z^{2k}.$$

Then

$$M_g(r) \ = \ \max_{|z|=r} |g(z)| \ = \ \omega_{n-1}^{-1} \sum_{k=0}^{\infty} |a_k| \sqrt{d_k} r^k \quad \text{and}$$
$$M_h(r) \ = \ \max_{|z|=r} |h(z)| \ = \ \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

Fryant and Shankar [1, Theorems 2 and 3] showed that if $H(x)$ is entire then $g(z)$ and $h(z)$ are also entire. Further, we have

$$M_h(r) \ \le \ M^2(r) \ \le \ M_g^2(r). \tag{11}$$

For the entire harmonic function $H(x)$, we define the generalized order and generalized lower order as

$$\rho \ \equiv \ \rho(\alpha, \beta, H) \ = \ \limsup_{r\to\infty} \frac{\alpha \ (\ln M(r))}{\beta \ (\ln r)},$$
$$\lambda \ \equiv \ \lambda\,(\alpha, \beta, H) \ = \ \liminf_{r\to\infty} \frac{\alpha \ (\ln M(r))}{\beta \,(\ln r)}\ .$$

Further, when ρ is a non-zero, finite number, we define the generalized type as

$$T \ \equiv \ T(\alpha, \beta, H, \rho) \ = \ \limsup_{r\to\infty} \frac{\alpha \ (\ln M(r))}{\beta \ [(\gamma(r))^{\rho}]}.$$

Here, the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$ pertain to the functions defined as in Theorems B and C above.

We now prove

Theorem 1. *Let $H(x)$ be an entire harmonic function. Then its generalized order is given as*

$$\rho \ = \ \limsup_{k\to\infty} \frac{\alpha\,(k)}{\beta \left(\ln |a_k|^{-1/k}\right)}. \tag{12}$$

Proof. From (11), we have

$$\ln\, M_h(r) \ \le \ 2 \,\ln\, M(r) \ \le \ 2 \,\ln\, M_g(r). \tag{13}$$

Since α is a monotonic increasing function, we have

$$\frac{\alpha\,(\ln M_h(r))}{\beta\,(\ln r)} \le \frac{\alpha\,(2\ln M(r))}{\beta\,(\ln r)} \le \frac{\alpha\,(2\,\ln M_g(r))}{\beta\,(\ln r)}\,. \tag{14}$$

From (6), we have for the entire function $h(z)$,

$$\begin{aligned}\limsup_{r\to\infty}\frac{\alpha\,(\ln M_h(r))}{\beta\,(\ln r)} &= \limsup_{k\to\infty}\frac{\alpha(2k)}{\beta\left(\ln|a_k|^{-2}\right)}\\ &= \limsup_{k\to\infty}\frac{\alpha\,(k)}{\beta\,\left(\ln|a_k|^{-1/k}\right)}.\end{aligned} \tag{15}$$

Similarly, for the entire function $g(z)$, we get

$$\limsup_{r\to\infty}\frac{\alpha\,(\ln M_g(r))}{\beta\,(\ln r)} = \limsup_{k\to\infty}\frac{\alpha\,(k)}{\beta\left(\frac{1}{k}\ln\{\omega_{n-1}^{-1}\,d_k^{1/2}\,|a_k|^{-1}\}\right)}.$$

Now $d_k = (n+2k-2)\,\dfrac{(n+k-3)!}{k!\,(n-2)!} \cong \dfrac{2k^{n-2}\,e^{3-n}}{(n-2)|}$ (using Stirling's formula) for all large values of k. Hence we get

$$\frac{1}{k}\,\ln\left\{\omega_{n-1}^{-1}\,d_k^{1/2}\,|a_k|^{-1}\right\} \cong \frac{1}{k}\,\ln|a_k|^{-1}\,.$$

Thus

$$\limsup_{r\to\infty}\frac{\alpha\,(\ln M_g(r))}{\beta\,(\ln r)} = \limsup_{k\to\infty}\frac{\alpha\,(k)}{\beta\,\left(\ln|a_k|^{-1/k}\right)}. \tag{16}$$

Since $\alpha(t)\in\Lambda$, on combining the inequalities in (14), (15) and (16), we finally get (12). This proves Theorem 1. ■

Next we prove

Theorem 2. *Let $H(x)$ be an entire harmonic function such that (i) $|a_k/a_{k+1}|$ is ultimately a non decreasing function of k, (ii) functions $\alpha(t)$ and $\beta(t)$ satisfy the conditions (8) and (9). Then*

$$\lambda \equiv \lambda(\alpha,\beta,H) = \liminf_{r\to\infty}\frac{\alpha\,(\ln M(r))}{\beta\,(\ln r)} = \liminf_{k\to\infty}\frac{\alpha\,(k)}{\beta\,\left(\ln|a_k|^{-1/k}\right)}.$$

Proof. As in Theorem 1, since $\alpha(t)\in\Lambda$, we get from (13)

$$\liminf_{r\to\infty}\frac{\alpha\,(\ln M_h(r))}{\beta\,(\ln r)} = \liminf_{r\to\infty}\frac{\alpha\,(\ln M(r))}{\beta\,(\ln r)} = \liminf_{r\to\infty}\frac{\alpha\,(\ln M_g(r))}{\beta\,(\ln r)}. \tag{17}$$

From the definition of d_k, we find that

$$\frac{d_k}{d_{k+1}} = \frac{(k+1)(n+2k-2)}{(n+2k)(n+k-2)}$$

which is a non-decreasing function of k for $n \geq 2$.

Thus if (i) above holds then the ratio $|a_k \sqrt{d_k} \,/\, a_{k+1} \sqrt{d_{k+1}}\,|$ is also non -decreasing for all large values of k.Consequently, we have from Theorem D,

$$\liminf_{r\to\infty} \frac{\alpha\,(\ln M_g(r))}{\beta\,(\ln r)} = \liminf_{k\to\infty} \frac{\alpha\,(k)}{\beta\left(\ln |\omega_{n-1}^{-1}\sqrt{d_k}\,a_k|^{-1/k}\right)} = \liminf_{k\to\infty} \frac{\alpha\,(k)}{\beta\left(\ln |a_k|^{-1/k}\right)}. \tag{18}$$

Again, under the assumption (i) above,$|a_k/a_{k+1}|^2$ will be a non -decreasing function of k . Hence applying Theorem D to the entire function $h(z)$, we get

$$\liminf_{r\to\infty} \frac{\alpha\,(\ln M_h(r))}{\beta\,(\ln r)} = \liminf_{k\to\infty} \frac{\alpha\,(2k)}{\beta\left(\frac{1}{2k}\,\ln |a_k|^{-2}\right)} = \liminf_{k\to\infty} \frac{\alpha\,(k)}{\beta\left(\ln |a_k|^{-1/k}\right)}. \tag{19}$$

Using the relations (17), (18) and (19), we get the desired result and Theorem 2 is proved. ■

Remark 1. Taking $\alpha(t) = \beta(t) = \ln t$ in Theorem 1, we get Theorem 3 of [1]. Similarly, on choosing $\alpha(t) = \beta(t) = \ln t$ in Theorem 2 above we get the coefficient characterization for the classical lower order of $H(x)$.

Now we shall give the characterization of generalized type.

Theorem 3. *Let $H(x)$ be an entire harmonic function of generalized order ρ, $0 < \rho < \infty$. Let $\alpha(t) \in \Lambda$, $\beta(t) \in L^0$, $\gamma(t) \in \Lambda$. Set $F(t,\sigma,\rho) = \gamma^{-1}\{[\beta^{-1}(\sigma\,\alpha(t))]^{1/\rho}\}$. Suppose that for all σ, $0 < \sigma < \infty$, $dF(t,\sigma,\rho)/d\ln t = O(1)$ as $t \to \infty$. Then we have*

$$\limsup_{r\to\infty} \frac{\alpha\,(\ln M(r))}{\beta\,[(\gamma(r))^{\rho}]} = \limsup_{k\to\infty} \frac{\alpha\,(k)}{\beta\left\{\left[\gamma\left(|a_k|^{-1/k}\right)\right]^{\rho}\right\}}. \tag{20}$$

Proof. We consider the entire functions

$$g(z) = \omega_{n-1}^{-1} \sum_{k=0}^{\infty} |a_k|\sqrt{d_k} z^k, \quad \text{and} \quad h(z) = \sum_{k=0}^{\infty} |a_k|^2 z^{2k}.$$

As before, we have

$$\ln M_h(r) \;\le\; 2\ln M(r) \;\le\; 2\ln M_g(r).$$

Hence

$$\alpha(\ln M_h(r)) \;\le\; \alpha(2\ln M(r)) \;\le\; \alpha(2\ln M_g(r)).$$

Since $\alpha(t) \in \Lambda$, we have from the above inequalities,

$$\limsup_{r\to\infty} \frac{\alpha\,(\ln M_h(r))}{\beta\,[(\gamma(r))^{\rho}]} \;\le\; \limsup_{r\to\infty} \frac{\alpha\,(\ln M(r))}{\beta\,[(\gamma(r))^{\rho}]} \;\le\; \limsup_{r\to\infty} \frac{\alpha\,(\ln M_g(r))}{\beta\,[(\gamma(r))^{\rho}]}. \tag{21}$$

Considering the entire function $h(z)$, from Theorem C above, we have

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\alpha\,(\ln M_h(r))}{\beta\,[(\gamma(r))^{\rho}]} &= \limsup_{k\to\infty} \frac{\alpha(k/\rho)}{\beta\left\{\left[\,\gamma\left(e^{1/\rho}(|a_k|^2)^{-1/2k}\right)\right]^{\rho}\right\}} \\
&= \limsup_{k\to\infty} \frac{\alpha(k)}{\beta\left\{\left[\gamma\left(|a_k|^{-1/k}\right)\right]^{\rho}\right\}}
\end{aligned}$$

since $\alpha(t) \in \Lambda$ and $\gamma(t) \in \Lambda$. Now, as shown in the proof of Theorem 1, $(d_k)^{1/2k} \to 1$ as $k \to \infty$. Hence for the entire function $g(z)$, we have

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\alpha\,(\ln M_g(r))}{\beta\,[(\gamma(r))^{\rho}]} &= \limsup_{k\to\infty} \frac{\alpha(k/\rho)}{\beta\left\{\left[\,\gamma\left(e^{1/\rho}(|a_k|\,\omega_{n-1}^{-1}\,\sqrt{d_k})^{-1/k}\right)\right]^{\rho}\right\}} \\
&= \limsup_{k\to\infty} \frac{\alpha\,(k)}{\beta\left\{\left[\gamma\left(|a_k|^{-1/k}\right)\right]^{\rho}\right\}},
\end{aligned}$$

since $\alpha(t) \in \Lambda$ and $\gamma(t) \in \Lambda$. Hence from the inequalities in (21), we get (20). This proves Theorem 3. ■

Remark 2. It is to be noted that in our assumptions, we have taken the functions $\alpha(t)$ and $\gamma(t)$ to be in the class Λ. This is necessary in view of the inequalities in (13). Hence, let us take $\alpha(t) = \ln^{(p-2)}(t)$ and $\gamma(t) = \ln^{(q-1)}(t)$ and $\beta(t) = t$, $p > q > 1$ and

$$\ln^{(k)}(t) \;=\; \underbrace{\ln\ln\ldots\ln}_{k\,\text{times}}(t)$$

Then from (20) we obtain

$$\limsup_{r\to\infty} \frac{\ln^{(p-1)} M(r)}{\left(\ln^{(q-1)} r\right)^{\rho}} \;=\; \limsup_{k\to\infty} \frac{\ln^{(p-2)}(k)}{\left(\ln^{(q-1)} |a_k|^{-1/k}\right)^{\rho}}.$$

The above formula characterizes the (p, q)- type of the entire harmonic function (see [3],Theorem 1) corresponding to the case $p > q > 1$. The case $p = q = 2$ can be considered in a manner similar to [1, Theorem 4].

3. Entire harmonic functions of slow growth

In this section we consider harmonic functions of slow growth. The generalized order and lower generalized order studied above leave an important case, that is, when $\alpha(t) = \beta(t)$. This represents the case of entire functions of slow growth and the coefficient formulae derived above are not valid for this case as the assumptions made in Theorem B and Theorem D on the functions $F(t, c)$ or $F(t)$ can not hold. To overcome this difficulty, Kapoor and Nautiyal [4] introduced a new class of functions. Thus a function $\phi(t) \in \Omega$ if $\phi(t)$ satisfies (4) and there exists a function $\delta(t) \in \Lambda$ and t_0, K_1 and K_2 such that for all $t > t_0$,

$$0 < K_1 \leq \frac{d(\phi(t))}{d(\delta(\ln t))} \leq K_2 < \infty. \tag{22}$$

Further a function $\phi(t) \in \overline{\Omega}$ if $\phi(t)$ satisfies (4) and

$$\lim_{t\to\infty} \frac{d(\phi(t))}{d(\ln(t))} = K, \quad 0 < K < \infty. \tag{23}$$

Kapoor and Nautiyal [4, p 66] showed that Ω, $\overline{\Omega} \subseteq \Lambda$ and $\Omega \cap \overline{\Omega} = \Phi$.

Let $\alpha(t) \in \Omega$ or $\overline{\Omega}$. Then following Kapoor & Nautiyal [4, p. 66], for the entire harmonic functions $H(x)$ we define the generalized order ρ^* and generalized lower order λ^* as

$$\rho^* = \rho(\alpha, \alpha, H) = \limsup_{r\to\infty} \frac{\alpha(\ln M(r))}{\alpha(\ln r)} ,$$

$$\lambda^* = \lambda(\alpha, \alpha, H) = \liminf_{r\to\infty} \frac{\alpha(\ln M(r))}{\alpha(\ln r)}.$$

It is to be noted that if the function $\alpha(t) \in \overline{\Omega}$ then ρ^* and λ^* reduce to the case of ordinary case of functions of slow growth i.e. $\rho(2, 2)$ and $\lambda(2, 2)$, (see [2]).

Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be an entire function. Then we have [4, Theorem 4]

$$\rho(\alpha, \alpha, f) = \begin{cases} \max\{1, L^*\} & \text{if} \quad \alpha(t) \in \Omega \\ 1 + L^* & \text{if} \quad \alpha(t) \in \overline{\Omega} \end{cases} \tag{24}$$

$$L^* = \limsup_{n\to\infty} \frac{\alpha(k)}{\alpha(\ln |c_k|^{-1/k})}.$$

Further

$$\lambda(\alpha, \alpha, f) = \begin{cases} \max\{1, l^*\} & \text{if} \quad \alpha(t) \in \Omega \\ 1 + l^* & \text{if} \quad \alpha(t) \in \overline{\Omega} \end{cases} \tag{25}$$

where

$$l^* = \liminf_{n\to\infty} \frac{\alpha(k)}{\alpha(\ln|c_k|^{-1/k})}$$

and the sequence $|c_k/c_{k+1}|$ is ultimately a non decreasing function of k.

Now we state

Theorem 4. *Let $H(x)$ be an entire harmonic function of generalized order ρ^* . Then*

$$\rho^* = \begin{cases} \max\{1, L^{**}\} & \text{if} \quad \alpha(t) \in \Omega, \\ 1 + L^{**} & \text{if} \quad \alpha(t) \in \overline{\Omega}, \end{cases} \tag{26}$$

where

$$L^{**} = \limsup_{k\to\infty} \frac{\alpha(k)}{\alpha(\ln|a_k|^{-1/k})}.$$

Further, the generalized lower order λ^* is given by

$$\lambda^* = \begin{cases} \max\{1, l^{**}\} & \text{if} \quad \alpha(t) \in \Omega \\ 1 + l^{**} & \text{if} \quad \alpha(t) \in \overline{\Omega} \end{cases} \tag{27}$$

where

$$l^{**} = \liminf_{k\to\infty} \frac{\alpha(k)}{\alpha(\ln|a_k|^{-1/k})}$$

and the sequence $|a_k/a_{k+1}|$ is ultimately a non -decreasing function of k.

The proof of above results follow on the lines of proofs of Theorems 1 & 2 and (24) and (25), since $\Omega\,,\,\overline{\Omega}\,\subseteq\,\Lambda$. Hence we omit the details.

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