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SOME CLASSES OF DIFFERENCE SEQUENCES OF FUZZY REAL NUMBERS

Abstract. In this article we disscuss some properties of the classes of difference sequences $c^F(\Delta)$, $c_0^F(\Delta)$ and $\ell_\infty^F(\Delta)$ of fuzzy real numbers, like solidness, symmetricity, sequence algebra, convergence free, nowhere denseness and prove some inclusion results.

Key words: fuzzy real number, difference sequence, solid, symmetric, convergence free, sequence algebra.

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1. Introduction

Kizmaz [3] studied the classical difference sequence spaces $c(\Delta)$, $c_0(\Delta)$ and $\ell_\infty(\Delta)$. The notion is defined as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c$, c_0 and ℓ_∞, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on R, the real line. For $X, Y \in D$ define

$$X \leq Y, \text{ if } a_1 \leq b_1 \text{ and } a_2 \leq b_2,$$
$$d(X,Y) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$.

It is known that (D, d) is a complete metric space. Also "$\leq$" is a partial order in D.

A *fuzzy real number* X is a fuzzy set on R, i.e. a mapping $X : R \to I$ $(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

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If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

The *α-cut* or *α-level set*, $[X]^\alpha$ of the fuzzy real number X, for $0 < \alpha \leq 1$, defined by $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$.

The *strong α-cut* of the fuzzy real number X, for $0 \leq \alpha \leq 1$ is the set $\{t \in R : X(t) > \alpha\}$.

By 0-*cut* or 0-*level set* of the fuzzy real number X, we mean the closure of the strong 0-cut.

A fuzzy real number X is said to be *upper-semi continuous* if, for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$. Throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The set R of all real numbers can be embedded in $R(I)$. For $r \in R$, $\overline{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1\,, & \text{for} \quad t = r\,, \\ 0\,, & \text{for} \quad t \neq r. \end{cases}$$

The arithmetic operations for α-level sets are defined as follows:

Let $X, Y \in R(I)$ and α-level sets be $[X]^\alpha = [a_1^\alpha, b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha, b_2^\alpha]$, $\alpha \in [0,1]$. Then

$$\begin{aligned} [X \oplus Y]^\alpha &= [a_1^\alpha + a_2^\alpha,\, b_1^\alpha + b_2^\alpha]\,, \\ [X - Y]^\alpha &= [a_1^\alpha - b_2^\alpha,\, b_1^\alpha - a_2^\alpha]\,, \\ [X \otimes Y]^\alpha &= \left[\min_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha,\ \max_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha\right] \\ \text{and} \ \ [Y^{-1}]^\alpha &= \left[\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha}\right], \quad 0 \notin Y. \end{aligned}$$

The *absolute value*, $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and Seikkala [2])

$$|X|(t) \ = \ \begin{cases} \max\left(X(t), X(-t)\right), & \text{for} \quad t \geq 0\,, \\ 0\,, & \text{for} \quad t < 0. \end{cases}$$

A fuzzy real number X is called *non-negative* if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by

$$\bar{d}(X, Y) \ = \ \sup_{0 \leq \alpha \leq 1} d\left([X]^\alpha, [Y]^\alpha\right).$$

Then $\bar{d}$ defines a metric on $R(I)$. For $X, Y \in R(I)$ define

$$X \leq Y, \text{ if } [X]^\alpha \leq [Y]^\alpha, \text{ for any } \alpha \in [0,1].$$

A sequence (X_k) of fuzzy real numbers is said to be *convergent* to the fuzzy real number X_0 if, for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \geq n_0$.

A fuzzy real number sequence (X_k) is said to be *bounded* if $|X_k| \leq \mu$, for some $\mu \in R^*(I)$; equivalently, (X_k) is *bounded* if $\sup\limits_k \bar{d}\,(X_k, \bar{0}) < \infty$.

2. Definitions and preliminaries

Savas [6] studied the classes of difference sequences $c^F(\Delta)$ and $\ell_\infty^F(\Delta)$ of fuzzy real numbers.

A fuzzy real number difference sequence $\Delta X = (\Delta X_k)$ is said to be *convergent* to a fuzzy real number X, written as $\lim\limits_{k\to\infty} \Delta X_k = X$ if, for every $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\bar{d}\,(\Delta X_k, X) < \varepsilon, \text{ for all } k > n_0.$$

A fuzzy real number difference sequence $\Delta X = (\Delta X_k)$ is said to be *bounded* if $|\Delta X_k| \leq \mu$, for some $\mu \in R^*(I)$; equivalently, (ΔX_k) is *bounded* if $\sup\limits_k \bar{d}\,(\Delta X_k, \bar{0}) < \infty$.

For $r \in R$ and $X \in R(I)$ the scalar product rX is defined by

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{for } r \neq 0, \\ \bar{0}, & \text{for } r = 0. \end{cases}$$

A class of sequences E^F is said to be *normal* (or *solid*) if $(Y_k) \in E^F$, whenever $|Y_k| \leq |X_k|$, for all $k \in N$ and $(X_k) \in E^F$.

A class of sequences E^F is said to be *monotone* if E^F contains the canonical pre-images of all its step sets.

Let $K = \{k_1 < k_2 < k_3 < \cdots\} \subseteq N$ and E^F be a class of sequences. A *K-step set* of E^F is a class of sequences $\lambda_k^{E^F} = \big\{(X_{k_n}) \in w^F : (X_n) \in E^F\big\}$.

A *canonical* pre-image of a sequence $(X_{k_n}) \in \lambda_k^{E^F}$ is a sequence $(Y_n) \in w^F$ defined as follows:

$$Y_n = \begin{cases} X_n, & \text{for } n \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

A *canonical pre-image of a step set* $\lambda_k^{E^F}$ is a set of canonical pre-images of all elements in $\lambda_k^{E^F}$, i.e. Y is in canonical pre-image $\lambda_k^{E^F}$ if and only if Y is canonical pre-image of some $X \in \lambda_k^{E^F}$.

From the above definitions we have the following remarks.

Remark 1. A class of sequences E^F is solid $\Rightarrow E^F$ is monotone.

A class of sequences E^F is said to be *symmetric* if $(X_{\pi(n)}) \in E^F$, whenever $(X_k) \in E^F$, where π is a permutation of N.

A class of sequences E^F is said to be *sequence algebra* if $(X_k \otimes Y_k) \in E^F$, whenever $(X_k), (Y_k) \in E^F$.

A class of sequences E^F is said to be *convergence free* if $(Y_k) \in E^F$, whenever $(X_k) \in E^F$ and $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

Throughout the article w^F, c^F, c_0^F and ℓ_∞^F denote the classes of *all*, *convergent*, *null* and *bounded sequences* of fuzzy real numbers respectively. Similarly $c^F(\Delta)$, $c_0^F(\Delta)$ and $\ell_\infty^F(\Delta)$ denote the classes of *convergent*, *null* and *bounded difference sequences* of fuzzy real numbers.

It is clear that $c^F(\Delta)$, $c_0^F(\Delta)$ and $\ell_\infty^F(\Delta)$ are closed under addition and scalar multiplication.

Remark 2. For the crisp set we have (x_k) converges to L implies (Δx_k) converges to 0. But for the fuzzy real numbers, when (X_k) converges to X (a fuzzy real number) then (ΔX_k) converges to Z (a fuzzy real number), where area bounded by the curve Z and the real line is double the area of the curve bounded by X and the real line. Further, the nature of the curve will be symmetric about the membership line, i.e. the line $t = 0$. Hence the α-cuts of Z will be of the type $[Z]^\alpha = [-a, a]$, for some crisp $a \in R_+ \cup \{0\}$, the set of non-negative real numbers. This is clear from the following example.

Example 1. Consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} (t - 5 + 3k^{-1}), & \text{for} \quad 5 - 3k^{-1} \le t \le 6 - 3k^{-1}, \\ (3 - 3^{-1}t - k^{-1}) & \text{for} \quad 6 - 3k^{-1} < t \le 9 - 3k^{-1}, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then $[X_k]^\alpha = \big[(5 + \alpha - 3k^{-1}),\ 3(3 - \alpha - k^{-1})\big]$ and $[\Delta X_k]^\alpha = [\{4\alpha - 4 - 3k^{-1} + 3(k+1)^{-1}\big\}, \{4 - 4\alpha - 3k^{-1} + 3(k+1)^{-1}\}]$ i.e. $X_k \to X$ as $k \to \infty$, where $[X]^\alpha = [5 + \alpha, 3(3 - \alpha)]$ for all $\alpha \in (0, 1]$ and $\Delta X_k \to Z$ as $k \to \infty$, where $[Z]^\alpha = [4\alpha - 4, 4 - 4\alpha)]$ for all $\alpha \in (0, 1]$.

Here, the width of each α-cut in $[Z]^\alpha$ is double the corresponding α-cut in $[X]^\alpha$. So the area bounded by the curve Z and the real line is double the area of the curve bounded by X and the real line.

Lemma 1 (**Savas [6], Theorem 1**). *$\ell_\infty^F(\Delta)$ and $c^F(\Delta)$ are complete metric spaces with the metric*

$$\rho(X, Y) = \bar{d}(X_1, Y_1) + \sup_k \bar{d}(\Delta X_k, \Delta Y_k),$$

where $X = (X_k)$ and $Y = (Y_k)$ are in $\ell_\infty^F(\Delta)$ or $c^F(\Delta)$.

3. Main results

Theorem 1. *The classes of sequences $c^F(\Delta)$, ${c_0}^F(\Delta)$ and $\ell_\infty^F(\Delta)$ are neither monotone nor solid.*

Proof. The result follows from the following two examples. ■

Example 2. Consider the sequence $(X_k) \in {c_0}^F(\Delta) \subset c^F(\Delta)$, defined by

$$X_k(t) = \begin{cases} 1-k(t-1), & \text{for} \quad 1 \le t \le 1+k^{-1}, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then for all $\alpha \in (0,1]$ we have,

$$[\Delta X_k]^\alpha = \left[(\alpha-1)(k+1)^{-1}, (1-\alpha)k^{-1}\right], \text{ i.e. } \Delta X_k \to \bar{0}, \text{ as } k \to \infty.$$

Thus $(X_k) \in {c_0}^F(\Delta) \subset c^F(\Delta)$.

Let $J = \{k \in N : k = 2i-1, i \in N\}$ be a subset of N and let $\overline{({c_0}^F(\Delta))}_J$ be the canonical pre-image of the J-step set $\left({c_0}^F(\Delta)\right)_J$ of ${c_0}^F(\Delta)$, defined as follows:

$(Y_k) \in \overline{({c_0}^F(\Delta))}_J$, the canonical pre-image of $(X_k) \in {c_0}^F(\Delta)$ implies

$$Y_k = \begin{cases} X_k, & \text{for} \quad k \in J, \\ \bar{0}\,, & \text{for} \quad k \notin J. \end{cases}$$

Now for all $\alpha \in (0,1]$ we have,

$$[Y_k]^\alpha = \begin{cases} \left[1, 1+(1-\alpha)k^{-1}\right], & \text{for} \quad k \in J, \\ [0,0]\,, & \text{for} \quad k \notin J. \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} \left[1, 1+(1-\alpha)k^{-1}\right], & \text{for} \quad k \in J, \\ \left[(\alpha-1)(k+1)^{-1}-1, -1\right], & \text{for} \quad k \notin J. \end{cases}$$

Thus $(Y_k) \notin c^F(\Delta) \left(\supset {c_0}^F(\Delta)\right)$. Therefore, ${c_0}^F(\Delta)$ and $c^F(\Delta)$ are not monotone.

The classes ${c_0}^F(\Delta)$ and $c^F(\Delta)$ are not solid follows from Remark 1.

Example 3. Consider the sequence $(X_k) \in \ell_\infty^F(\Delta)$ defined by

$$X_k(t) = \begin{cases} t-(k-1), & \text{for} \quad k-1 \le t \le k, \\ (k+1-t), & \text{for} \quad k < t \le k+1 \\ 0\,, & \text{otherwise.} \end{cases}$$

Then for all $\alpha \in (0,1]$ we have, $[\Delta X_k]^\alpha = [2\alpha - 3, 1 - 2\alpha]$, for all $k \in N$. Thus $(X_k) \in \ell_\infty^F(\Delta)$.

Let $J = \{k \in N : k = 2i - 1, i \in N\}$ be a subset of N and let $\overline{(\ell_\infty^F(\Delta))}_J$ be the canonical pre-image of the J-step set $\left(\ell_\infty^F(\Delta)\right)_J$ of $\ell_\infty^F(\Delta)$, defined as follows:

$(Y_k) \in \overline{(\ell_\infty^F(\Delta))}_J$ the canonical pre-image of $(X_k) \in \ell_\infty^F(\Delta)$ implies

$$Y_k = \begin{cases} X_k, & \text{for} \quad k \in J, \\ \bar{0}, & \text{for} \quad k \notin J. \end{cases}$$

Now for all $\alpha \in (0,1]$ we have,

$$[Y_k]^\alpha = \begin{cases} [(k-1+\alpha),(k+1-\alpha)], & \text{for} \quad k \in J, \\ [0,0], & \text{for} \quad k \notin J. \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} [(k-1+\alpha),(k+1-\alpha)] & \text{for} \quad k \in J, \\ [-(k+2-\alpha),-(k+\alpha)] & \text{for} \quad k \notin J. \end{cases}$$

Therefore, $(Y_k) \notin \ell_\infty^F(\Delta)$ and thus the class $\ell_\infty^F(\Delta)$ is not monotone. The class $\ell_\infty^F(\Delta)$ is not solid follows from Remark 1.

Theorem 2. *The classes of sequences* $c^F(\Delta)$, ${c_0}^F(\Delta)$ *and* $\ell_\infty^F(\Delta)$ *are not convergence free.*

Proof. The result follows from the following example.

Example 4. Consider the sequence $(X_k) \in {c_0}^F(\Delta) \subset c^F(\Delta) \subset \ell_\infty^F(\Delta)$, defined as follows:

For $k = i^2$, $i \in N$, $\quad X_k = \bar{0}$.

Otherwise,

$$X_k(t) = \begin{cases} 1, & \text{for} \quad 0 \le t \le k^{-1}, \\ 0, & \text{otherwise}. \end{cases}$$

Then for all $\alpha \in (0,1]$ we have,

$$[X_k]^\alpha = \begin{cases} [0,0], & \text{for} \quad k = i^2, \ i \in N, \\ [0,k^{-1}], & \text{otherwise}. \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[-(k+1)^{-1}, 0\right], & \text{for} \quad k = i^2, \ i \in N, \\ \left[0, k^{-1}\right], & \text{for} \quad k = i^2 - 1, \ i \in N, \text{ with } i > 1, \\ \left[-(k+1)^{-1}, k^{-1}\right], & \text{otherwise}. \end{cases}$$

Hence $\Delta X_k \to \bar{0}$ as $k \to \infty$. Thus $(X_k) \in {c_0}^F(\Delta) \subset c^F(\Delta) \subset \ell_\infty^F(\Delta)$.
Let (Y_k) be defined as follows:
For $k = i^2$, $i \in N$, $Y_k = \bar{0}$.
Otherwise,

$$Y_k(t) = \begin{cases} 1, & \text{for} \quad 0 \le t \le k, \\ 0, & \text{otherwise.} \end{cases}$$

Now for all $\alpha \in (0,1]$ we have,

$$[Y_k]^\alpha = \begin{cases} [0,0], & \text{for} \quad k = i^2, \ i \in N, \\ [0,k], & \text{otherwise.} \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} [-(k+1),0], & \text{for} \quad k = i^2, \ i \in N, \\ [0,k], & \text{for} \quad k = i^2 - 1, \ i \in N, \ \text{with } i > 1, \\ [-(k+1),k], & \text{otherwise.} \end{cases}$$

Thus $(Y_k) \notin \ell_\infty^F(\Delta) \left(\supset c^F(\Delta) \supset {c_0}^F(\Delta)\right)$.
Therefore, the classes $c^F(\Delta)$, ${c_0}^F(\Delta)$ and $\ell_\infty^F(\Delta)$ are not convergence free. ■

Theorem 3. *The classes of sequences $c^F(\Delta)$, ${c_0}^F(\Delta)$ and $\ell_\infty^F(\Delta)$ are not symmetric.*

Proof. The result follows from the following two examples.

Example 5. Defined the unbounded sequence $(X_k) \in {c_0}^F(\Delta) \subset c^F(\Delta)$ as

$$X_1(t) = \begin{cases} 1, & \text{for} \quad -1 \le t \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

and for $k \ge 2$,

$$X_k(t) = \begin{cases} 1, & \text{for} \quad -\left(\sum_{r=1}^{k-1}\left(\frac{1}{2r}\right) + \frac{1}{k}\right) \le t \le -\sum_{r=1}^{k-1}\left(\frac{1}{2r}\right), \\ 0, & \text{otherwise.} \end{cases}$$

For each $\alpha \in (0,1]$ we have $[X_1]^\alpha = [-1,0]$ and for $k \ge 2$,

$$[X_k]^\alpha = \left[-\left(\sum_{r=1}^{k-1}\left(\frac{1}{2r}\right) + \frac{1}{k}\right), -\sum_{r=1}^{k-1}\left(\frac{1}{2r}\right)\right].$$

Then for all $k \in N$ and for all $\alpha \in (0,1]$ we have

$$\begin{aligned}[\Delta X_k]^\alpha &= \left[-\left\{k^{-1}-(2k)^{-1}\right\},\left\{(2k)^{-1}+(k+1)^{-1}\right\}\right] \\ &= \left[-(2k)^{-1},\left\{(2k)^{-1}+(k+1)^{-1}\right\}\right].\end{aligned}$$

Hence $\Delta X_k \to \bar{0}$, as $k \to \infty$. Thus $(X_k) \in {c_0}^F(\Delta) \subset c^F(\Delta)$.

Let the sequence (Y_k) be a rearrangement of the sequence (X_k), defined as follows:

$$(Y_k) = (X_1,\ X_2,\ X_4,\ X_3,\ X_9,\ X_5,\ X_{16},\ X_6,\ X_{25},\ X_7\ \ldots).$$

i.e. $(Y_k) = X_{\left(\frac{k+1}{2}\right)^2}$, for all k odd,

$= X_{\left(n+\frac{k}{2}\right)}$, for all k even and n satisfies

$$n(n-1) < \tfrac{k}{2} \le n(n+1), \quad n \in N.$$

Then for $k=1$ we have,

$$[\Delta Y_1]^\alpha = [X_1]^\alpha - [X_2]^\alpha = [-0.5,1], \quad \text{for each } \alpha \in (0,1].$$

For all k odd with $k>1$ and $n \in N$, satisfying $n(n-1) < \frac{k+1}{2} \le n(n+1)$, we have

$$\begin{aligned}[\Delta Y_k]^\alpha &= \left[X_{\left(\frac{k+1}{2}\right)^2}\right]^\alpha - \left[X_{\left(n+\frac{k+1}{2}\right)}\right]^\alpha \\ &= \left[-\left\{\sum_{r=\left(n+\frac{k+1}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{2r}+\frac{1}{\left(\frac{k+1}{2}\right)^2}\right\},\right. \\ &\qquad \left. -\left\{\sum_{r=\left(n+\frac{k+1}{2}\right)}^{\left(\frac{k+1}{2}\right)^2-1}\frac{1}{2r}\right\}+\frac{1}{\left(n+\frac{k+1}{2}\right)}\right], \quad \text{for all } \alpha \in (0,1].\end{aligned}$$

For all k even and $n \in N$, satisfying $n(n-1) < \frac{k}{2} \le n(n+1)$, we have

$$\begin{aligned}[\Delta Y_k]^\alpha &= \left[X_{\left(n+\frac{k}{2}\right)}\right]^\alpha - \left[X_{\left(\frac{k+2}{2}\right)^2}\right]^\alpha \\ &= \left[\left\{\sum_{r=\left(n+\frac{k}{2}\right)}^{\left(\frac{k+2}{2}\right)^2-1}\frac{1}{2r}\right\}-\frac{1}{\left(n+\frac{k}{2}\right)},\right. \\ &\qquad \left.\left\{\sum_{r=\left(n+\frac{k}{2}\right)}^{\left(\frac{k+2}{2}\right)^2-1}\frac{1}{2r}+\frac{1}{\left(\frac{k+2}{2}\right)^2}\right\}\right], \quad \text{for all } \alpha \in (0,1].\end{aligned}$$

Here it is observed that the values of (ΔY_k) increases with

$$\Delta Y_4(t) = \begin{cases} 1\,, & \text{for} \quad 0.2759 \le t \le 0.7200, \\ 0\,, & \text{otherwise.} \end{cases}$$

for all $k > 3$ and k even and decreases for $k > 3$ and k odd. Therefore the sequence can not converge to a point.

Thus $(Y_k) \notin c^F(\Delta) \big(\supset {c_0}^F(\Delta)\big)$ and hence $c^F(\Delta)$ and ${c_0}^F(\Delta)$ are not symmetric.

Example 6. Consider the sequence $(X_k) \in \ell_\infty^F(\Delta)$, defined by

$$X_k(t) = \begin{cases} 1\,, & \text{for} \quad k \le t \le k + 2^{-1}, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then for all $\alpha \in (0,1]$ we have

$$[\Delta X_k]^\alpha = [-1.5, -0.5], \quad \text{for all } \ k \in N.$$

Thus $(X_k) \in \ell_\infty^F(\Delta)$.

Let the sequence (Y_k) be a rearrangement of the sequence (X_k) defined as follows:

$$(Y_k) = (X_1,\ X_2,\ X_4,\ X_3,\ X_9,\ X_5,\ X_{16},\ X_6,\ X_{25},\ X_7\ \ldots).$$

i.e. $(Y_k) = X_{\left(\frac{k+1}{2}\right)^2}$, for all k odd,

$= X_{\left(n+\frac{k}{2}\right)}$, for all k even and n satisfies

$$n(n-1) < \tfrac{k}{2} \le n(n+1), \quad n \in N.$$

Then for all k odd and $n \in N$, satisfying $n(n-1) < \frac{k+1}{2} \le n(n+1)$, we have

$$\text{(1)} \quad [\Delta Y_k]^\alpha = \left[X_{\left(\frac{k+1}{2}\right)^2}\right]^\alpha - \left[X_{\left(n+\frac{k+1}{2}\right)}\right]^\alpha$$
$$= \left[\left\{\left(\frac{k+1}{2}\right)^2 - \left(\frac{3k+1}{2}\right) - \frac{1}{2}\right\},\right.$$
$$\left.\left\{\left(\frac{k+1}{2}\right)^2 - \left(\frac{3k+1}{2}\right) + \frac{1}{2}\right\}\right], \quad \text{for all } \ \alpha \in (0,1],$$

and for all k even and $n \in N$, satisfying $n(n-1) < \frac{k}{2} \leq n(n+1)$, we have

$$
\begin{aligned}
(2)\quad [\Delta Y_k]^\alpha &= \left[X_{(n+\frac{k}{2})}\right]^\alpha - \left[X_{(\frac{k+2}{2})^2}\right]^\alpha \\
&= \left[\left\{\left(n+\frac{k}{2}\right) - \left(\frac{k+2}{2}\right)^2 - \frac{1}{2}\right\},\right. \\
&\qquad \left.\left\{\left(n+\frac{k}{2}\right) - \left(\frac{k+2}{2}\right)^2 + \frac{1}{2}\right\}\right], \text{ for all } \alpha \in (0,1],
\end{aligned}
$$

From equation (1) and (2) it is clear that $([\Delta Y_k]^\alpha)$ is unbounded, for all $\alpha \in (0,1]$.

Thus $(Y_k) \notin \ell_\infty^F(\Delta)$. Therefore the class $\ell_\infty^F(\Delta)$ is not symmetric. ■

Theorem 4. *The class of sequences* ${c_0}^F(\Delta) \cap \ell_\infty^F(\Delta)$ *is sequence algebra.*

Proof. Let $(X_k), (Y_k) \in \ell_\infty^F(\Delta)$ such that $(\Delta X_k), (\Delta Y_k) \in {c_0}^F$. Then we have

$$\Delta(X_k \otimes Y_k) = Y_k \otimes \Delta X_k + X_{k+1} \otimes \Delta Y_k \to \bar{0}, \text{ as } k \to \infty.$$

Hence the result. ■

Theorem 5. *The classes of sequences* $c^F(\Delta)$ *and* $\ell_\infty^F(\Delta)$ *are not sequence algebra.*

Proof. The result follows from the following example.

Example 7. Consider the two sequences $(X_k), (Y_k) \in c^F(\Delta) \subset \ell_\infty^F(\Delta)$, defined by

$$X_k(t) = \begin{cases} 1, & \text{for } k-1 \leq t \leq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$Y_k(t) = \begin{cases} 1, & \text{for } k-1 \leq t \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Then for each $\alpha \in (0,1]$, we have

$$[X_k]^\alpha = [k-1, k+1] \text{ and } [Y_k]^\alpha = [k-1, k].$$

Therefore, for all $k \in N$ we have

$\Delta X_k = X$, where

$$X(t) = \begin{cases} 1\,, & \text{for} \quad -3 \le t \le -1, \\ 0\,, & \text{otherwise.} \end{cases}$$

and $\Delta Y_k = Y$, where

$$Y(t) = \begin{cases} 1\,, & \text{for} \quad -2 \le t \le 0, \\ 0\,, & \text{otherwise.} \end{cases}$$

Thus (X_k) and $(Y_k) \in c^F(\Delta) \subset \ell_\infty^F(\Delta)$.

Now for each $\alpha \in (0,1]$ we have

$$[\Delta(X_k \otimes Y_k)]^\alpha = \left[(k-1)^2, k(k+1)\right] - \left[k^2, (k+1)(k+2)\right] = [-(k+1), k].$$

i.e. $(X_k \otimes Y_k) \notin \ell_\infty^F(\Delta) \ \left(\supset c^F(\Delta)\right)$.

Hence the result. ■

Theorem 6. (a) ${c_0}^F \subset {c_0}^F(\Delta)$ *and the inclusion is strict.*
(b) $c^F \subset c^F(\Delta)$ *and the inclusion is strict.*

Proof. (a) Let us consider a sequence $(X_k) \in {c_0}^F$. Clearly (from Remark 2) we have $\Delta X_k \to \bar{0}$, as $k \to \infty$ and hence ${c_0}^F \subset {c_0}^F(\Delta)$.

The strictness of the inclusion follows from the following example.

Example 8. Consider the sequence (X_k) defined in Example 2. Then for each $\alpha \in (0,1]$ we have

$$[X_k]^\alpha = \left[1, 1 + (1-\alpha)k^{-1}\right]$$

i.e. $X_k \to \bar{1}$, as $k \to \infty$ and $\Delta X_k \to \bar{0}$, as $k \to \infty$.

Thus $(X_k) \notin {c_0}^F$, but $(X_k) \in {c_0}^F(\Delta)$. Hence the inclusion is strict.

(b) Consider a sequence $(X_k) \in c^F$. Then $\Delta X_k \to X$, as $k \to \infty$, where X is of particular type, defined by $[X]^\alpha = [-a, a]$, for some crisp $a \in R+ \cup \{0\}$, the set of non-negative real numbers and for each $\alpha \in (0,1]$ (refer to Remark 2).

Hence $c^F \subset c^F(\Delta)$. The inclusion is strict follows from the following example.

Example 9. Consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} 1\,, & \text{for} \quad k \le t \le k+1, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then for each $\alpha \in (0,1]$ we have $[X_k]^\alpha = [k, k+1]$ for all $k \in N$ and $\Delta X_k = X$ for all $k \in N$, where

$$X(t) = \begin{cases} 1, & \text{for} \quad -2 \le t \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(X_k) \notin c^F$, but $(X_k) \in c^F(\Delta)$. Therefore the inclusion is proper. ■

Theorem 7. *The classes of sequences $c^F(\Delta)$ and ${c_0}^F(\Delta)$ are nowhere dense subsets of $\ell_\infty^F(\Delta)$.*

Proof. From lemma we have $c^F(\Delta)$ and ${c_0}^F(\Delta)$ are closed subsets of the complete metric space $\ell_\infty^F(\Delta)$. Also ${c_0}^F(\Delta)$ and $c^F(\Delta)$ are proper subsets of $\ell_\infty^F(\Delta)$, which follows from the following example.

Example 10. Consider the sequence (X_k) defined as follows:

For k even,

$$X_k(t) = \begin{cases} 1, & \text{for} \quad 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

and for k odd,

$$X_k(t) = \begin{cases} 1 + k(t+1), & \text{for} \quad -(1+k^{-1}) \le t \le -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for each $\alpha \in (0,1]$ we have

$$[X_k]^\alpha = \begin{cases} [0,1], & \text{for } k \text{ even,} \\ \left[-\left(1+k^{-1}\right), -1\right], & \text{for } k \text{ odd.} \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[1, 2+(k+1)^{-1}\right], & \text{for } k \text{ even,} \\ \left[-\left(2+k^{-1}\right), -1\right], & \text{for } k \text{ odd.} \end{cases}$$

Thus $(\Delta X_k) \notin c^F \ \left(\supset {c_0}^F\right)$, but $(\Delta X_k) \in \ell_\infty^F$. Hence the result. ■

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