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GENERALIZED KÖTHE-TOEPLITZ DUAL OF SOME DOUBLE SEQUENCE SPACES

Abstract. In this article we introduce the notion of $\eta-$dual of double sequence spaces. We find the $\eta-$ dual of some double sequence spaces. We verify the perfectness of different double sequence spaces relative to $\eta-$ dual.

Key words: dual space, perfect space, ℓ_r-space, bounded variation, regular convergence.

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1. Introduction

The notion for duals for sequence spaces introduced by Köthe and Toeplitz [8]. Later on it was studied by Maddox [10], Lascarides [9], Okutoyi [13], Chandra and Tripathy [3] and many others.It is also found in the monographs of Köthe [7], Maddox [11], Cook [4] and Kamthan and Gupta [6].

The notion of $\alpha-$ duals is generalized by Chandra and Tripathy [3] on introducing the notion of $\eta-$ duals for sequence spaces.

The notion of double sequences is found in Browmich [2]. Hardy [5] introduced the notion of bounded variation double sequences. Later on it was investigated by Moricz [12], Tripathy [15], Patterson [14], Basarir and Sonalcan [1] and Tripathy, Choudhary and Sarma [16] and many others.

Throughout the article ${}_2w$, ${}_2\ell_\infty$, ${}_2c$, ${}_2c^R$, ${}_2c_0$, ${}_2c_0^R$, ${}_2\ell_1$, ${}_2\ell_p$, ${}_2\phi$, ${}_2bv$, ${}_2\sigma$, ${}_2w_p$ denote the spaces of *all*, *bounded*, *convergent in Pringsheim's sense*, *regularly convergent*, *null in Pringsheim's sense*, *regularly null*, *absolutely summable*, *p-absolutely summable*, *finite*, *bounded variation*, *eventually alternating* and *strongly p-Cesàro summable* double sequence spaces respectively. Throughout the paper a double sequences will be denoted by $< a_{mn} >$ and sums without limit means that the summation is from $m = 1$ to ∞ and $n = 1$ to ∞.

The $\alpha-$dual of a subset E of ${}_2w$ is defined as

$$E^\alpha = \{< y_{mn} > \in {}_2w : < x_{mn}y_{mn} > \in {}_2\ell_1 \text{ for all } < x_{mn} > \in E\}.$$

2. Definitions and preliminaries

We list some of the double sequences, whose $\eta-$ dual will be obtained in this article.

$_2\ell_\infty = \{< a_{mn} > \in \ _2w : \sup\limits_{m,n}|a_{mn}| < \infty\}$.

$_2c = \{< a_{mn} > \in \ _2w : a_{mn} \to L$ as $\min(m,n) \to \infty$ for some $L \in C\}$.

$$_2c^R = \{< a_{mn} > \in \ _2c : (a) \lim_{n\to\infty} a_{mn} = L_m \in C, \text{ for some } L_m \in C \text{ for each } m \in N,$$

$$(b) \lim_{m\to\infty} a_{mn} = J_n \in C, \text{for some } J_n \in C \text{ for each } n \in N\}.$$

$_2c_0 = \{< a_{mn} > \in \ _2w : a_{mn} \to 0$ as $\min(m,n) \to \infty\}$.

$_2c_0^R = \{< a_{mn} > \in \ _2w : a_{mn} \to 0$ as $\max(m,n) \to \infty\}$.

$$_2bv = \{< a_{mn} > \in \ _2w : \sum|\Delta_m a_{m,n}| < \infty, \ \sum|\Delta_n a_{m,n}| < \infty \text{ and } \sum\sum|\Delta_{m,n}a_{m,n}| < \infty\}, \text{ where}$$

$$\Delta_m a_{m,n} = a_{m,n} - a_{m+1,n}, \qquad \Delta_n a_{m,n} = a_{m,n} - a_{m,n+1},$$

$$\Delta_{m,n}a_{m,n} = \Delta_n a_{m,n} - \Delta_n a_{m+1,n}.$$

We define $_2bv_0 = \ _2bv \cap \ _2c_0$.

$$_2w_p = \{< a_{ij} > \in \ _2w : \lim_{\substack{m\to\infty\\ n\to\infty}} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij} - L|^p = 0\}$$

The sequence spaces convergent in Pringsheim's sense and null in Pringsheim's sense contain some unbounded sequences.

The spaces $_2c^R$, $_2c_0^R$, $_2c^R$, $_2c \cap \ _2\ell_\infty$, $_2c_0 \cap \ _2\ell_\infty$ and $_2\ell_\infty$ are normed linear spaces normed by

$$\|A\| = \| < a_{nk} > \| = \sup_{m,n}|a_{mn}|.$$

From the above definition it is clear that

$$_2c^R \subset \ _2c \cap \ _2\ell_\infty \subset \ _2\ell_\infty(_2c)$$

and

$$_2c_0^R \subset \ _2c_0 \cap \ _2\ell_\infty \subset \ _2\ell_\infty(_2c_0).$$

Definition 1. *The space of all eventually alternating double sequences is defined by*

$$ {}_2\sigma \;=\; \{< a_{mn} > \in \; {}_2w : a_{mn} = -a_{m,n+1} \text{ for all } n \geq n_0 \\ \text{and } a_{mn} = -a_{m+1,n} \text{ for all } m \geq m_0\}. $$

Definition 2. *Let E be a nonempty subset of ${}_2w$ and $r \geq 1$, then η dual of E is defined by*

$$ E^\eta \;=\; \{< a_{nk} > \in \; {}_2w : \sum_m \sum_n |a_{mn} b_{mn}|^r < \infty \text{ for all } < b_{mn} > \in E\} $$

The space E is said to be η-reflexive if $E^{\eta\eta} = E$. Taking $r = 1$ in this definition we get E^α, i.e. α dual of $E \subset \; {}_2w$.

The proof of the following result is obvious in view of the definition of η-dual of double sequences.

Lemma 1. *(i) E^η is a linear subspace of ${}_2w$ for every $E \subset \; {}_2w$.*
(ii) $E \subset F$ implies $E^\eta \supset F^\eta$.
(iii) $E \subset E^{\eta\eta}$ for every $E, F \subset \; {}_2w$.
The following result is the analogue of the Hölder's inequality for double sequences.

Lemma 2 (Hölder's inequality for double sequences). *If a_{ij}, b_{ij} are positive real numbers then*

$$ \sum_i \sum_j a_{ij} b_{ij} \;\leq\; \left\{ \sum_i \left(\sum_j a_{ij}^p \right) \right\}^{\frac{1}{p}} \left\{ \sum_i \left(\sum_j a_{ij}^q \right) \right\}^{\frac{1}{q}}. $$

Proof. We have

$$ \sum_i \sum_j a_{ij} b_{ij} \;=\; \sum_i \left(\sum_j a_{ij} b_{ij} \right) \;\leq\; \sum_i \left\{ \left(\sum_j a_{ij}^p \right)^{\frac{1}{p}} \left(\sum_j b_{ij}^q \right)^{\frac{1}{q}} \right\} \\ \leq\; \left\{ \sum_i \left(\sum_j a_{ij}^p \right) \right\}^{\frac{1}{p}} \left\{ \sum_i \left(\sum_j b_{ij}^q \right) \right\}^{\frac{1}{q}}. $$

■

3. Main results

Theorem 1. *$({}_2\ell_r)^\eta = {}_2\ell_\infty$ and $({}_2\ell_\infty)^\eta = {}_2\ell_r$. The spaces ${}_2\ell_r$ and ${}_2\ell_\infty$ are perfect spaces.*

Proof. Let $< a_{mn} > \in {}_2\ell_\infty$. We have

$$\sum_m \sum_n |a_{mn} b_{mn}|^r < \infty \text{ for all } < b_{mn} > \in {}_2\ell_r.$$

Hence ${}_2\ell_\infty \subseteq ({}_2\ell_r)^\eta$.

Conversely, let $< a_{mn} > \notin {}_2\ell_\infty$. Then there exists a single sequence $< a_{i,n_i} >$ such that $a_{i,n_i} \geq i^s$ for some $s > 0$. Consider the double sequence $< b_{mn} >$ defined by

$$b_{mn} = \begin{cases} i^{-s}, & \text{if } m = i,\ n = n_i,\ i \in N \\ 0, & \text{otherwise.} \end{cases}$$

Then $< b_{mn} > \in {}_2\ell_r$ but $< a_{mn} b_{mn} > \notin {}_2\ell_r$. Hence $({}_2\ell_r)^\eta \subseteq {}_2\ell_\infty$. The proof for the case $({}_2\ell_\infty)^\eta = {}_2\ell_r$ is easy so omitted.

This completes the proof of the theorem. ■

Theorem 2. *$({}_2c^R)^\eta = ({}_2c_0^R)^\eta = {}_2\ell_r$. The spaces ${}_2c^R$ and ${}_2c_0^R$ are not perfect.*

Proof. We first show that $({}_2c_0^R)^\alpha = {}_2\ell_1$. Since ${}_2c_0^R \subseteq {}_2\ell_\infty$ so ${}_2\ell_1 = ({}_2\ell_\infty)^\alpha \subseteq ({}_2c_0^R)^\alpha$. Next we show that $({}_2c_0^R)^\alpha \subseteq {}_2\ell_1$. Let $< a_{nk} > \notin {}_2\ell_1$.

Then we can find sequences $< m_i >$ and $< n_i >$ of naturals with $m_0 = n_0 = 1$ such that

$$\sum_{m=1}^{m_i} \sum_{n=1}^{n_i} |a_{mn}| - \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{i-1}} |a_{mn}| > \frac{1}{(i+1)^{\frac{1}{2}}}, \quad i = 0, 1, 2, 3, \ldots$$

Define the sequence $< b_{mn} >$ by

$$b_{mn} = (i+1)^{-\frac{1}{3}} \text{ for } m_{i-1} < m \leq m_i \text{ and } n_{i-1} < n \leq n_i, \text{ for all } i \in N.$$

Then $< b_{mn} > \in {}_2c_0^R$.

Now,

$$\begin{aligned} \sum_{m=1}^\infty \sum_{n=1}^\infty |a_{mn} b_{mn}| &= \sum_{i=0}^\infty \left(\sum_{m=1}^{m_i} \sum_{n=1}^{n_i} |a_{mn} b_{mn}| - \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{i-1}} |a_{mn} b_{mn}| \right) \\ &= \sum_{i=0}^\infty \frac{1}{(i+1)^{\frac{1}{3}}} \left(\sum_{m=1}^{m_i} \sum_{n=1}^{n_i} |a_{mn}| - \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{i-1}} |a_{mn}| \right) \\ &> \sum_{i=0}^\infty \frac{1}{(i+1)^{\frac{1}{3}} (i+1)^{\frac{1}{2}}} = \infty. \end{aligned}$$

Thus $< a_{nk} > \in ({}_2c_0^R)^\alpha$. Hence we have $({}_2c_0^R)^\alpha \subseteq {}_2\ell_1$. Thus $({}_2c_0^R)^\alpha = {}_2\ell_1$. Using this one can easily show that $({}_2c_0^R)^\eta = {}_2\ell_r$.

Next $({}_2c^R)^\eta = {}_2\ell_r$ follows from Theorm 1 and the inclution relation

$${}_2c_0^R \subseteq {}_2c^R \subseteq {}_2\ell_\infty^R$$

The spaces are not perfect follows from Theorem 1. This completes the proof of the theorem. ■

The following result is immediate from the above result.

Corollary 1. $({}_2c \cap {}_2\ell_\infty)^\eta = ({}_2c_0 \cap {}_2\ell_\infty)^\eta = {}_2\ell_r$. *The spaces* ${}_2c \cap {}_2\ell_\infty$ *and* ${}_2c_0 \cap {}_2\ell_\infty$ *are not perfect.*

Theorem 3. $({}_2bv)^\eta = ({}_2bv_0)^\eta = {}_2\ell_r$. *The spaces* ${}_2bv$ *and* ${}_2bv_0$ *are not perfect.*

Proof. We have ${}_2bv \subseteq {}_2\ell_\infty$. Hence ${}_2\ell_r = ({}_2\ell_\infty)^\eta \subseteq ({}_2bv_0)^\eta$. Now, we show that $({}_2bv_0)^\eta \subseteq {}_2\ell_r$.

Let $< b_{nk} > \notin {}_2\ell_r$. Then we can find a sequence (n_k) of naturals with $n_1 = 1$ such that

$$\sum_{m=1}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} |b_{mn}|^r > k^r \text{ for all } k = 1, 2, 3, 4, \dots .$$

Consider the sequence $< a_{mn} >$ defined by

$$a_{mn} = k^{-1} \text{ if } n_k \le n < n_{k+1}, \text{ for all } k = 1, 2, 3, 4, \dots .$$

Then

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\Delta a_{mn}| &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\sum_{n=n_k}^{n_{k+1}-1} |\Delta a_{mn}| \right) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\sum_{n=n_k}^{n_{k+1}-1} |a_{mn} - a_{m,n+1} - a_{m+1,n} + a_{m+1,n+1}| \right) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} \left| \frac{1}{k} - \frac{1}{k+1} - \frac{1}{k} + \frac{1}{k+1} \right| = 0. \end{aligned}$$

Hence $< a_{mn} > \in {}_2bv_0$.

Now,

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn} b_{mn}|^r &= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} |a_{mn} b_{mn}|^r \\ &= \sum_{k=1}^{\infty} \frac{1}{k^r} \sum_{m=1}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} |b_{mn}|^r = \sum_{k=1}^{\infty} \frac{1}{k^r} k^r = \infty. \end{aligned}$$

Thus we arrive at a contradiction. Hence $({}_2bv_0)^\eta \subseteq {}_2\ell_r$. Therefore $({}_2bv_0)^\eta = {}_2\ell_r$. The rest of the proof follows from the inclusion ${}_2bv_0 \subset {}_2bv \subset {}_2\ell_\infty$ and Theorem 1. ■

Theorem 4. $({}_2\sigma)^\eta = {}_2\ell_r$. *The space* ${}_2\sigma$ *is not perfect.*

Proof. We have ${}_2\sigma \subseteq {}_2\ell_\infty \Rightarrow {}_2\ell_r \subseteq ({}_2\sigma)^\eta$. Let $< b_{mn} > \in ({}_2\sigma)^\eta$. Then $\sum_m\sum_n |a_{mn}b_{mn}|^r < \infty$ for all $< a_{mn} > \in {}_2\sigma$.

Consider the sequence $< a_{mn} >$ defined by $a_{mn} = 1 = -a_{m+1,n} = -a_{m,n+1}$ for all $m, n \in N$. Then $< a_{mn} > \in {}_2\sigma$.

$$\Rightarrow \sum_m\sum_n |b_{mn}|^r < \infty \Rightarrow < b_{mn} > \in {}_2\ell_r \Rightarrow ({}_2\sigma)^\eta \subseteq {}_2\ell_r.$$

Hence $({}_2\sigma)^\eta = {}_2\ell_r$. The rest of the proof follows from Theorem 1. The proof of the following result is easy in view of the definition and the properties of the spaces ${}_2w$ and ${}_2\phi$. ■

Theorem 5. $({}_2w)^\eta = {}_2\phi$, $({}_2\phi)^\eta = {}_2w$. *The spaces* ${}_2w$ *and* ${}_2\phi$ *are perfect.*

Theorem 6. $({}_2w_p \cap {}_2\ell_\infty)^\eta = {}_2\ell_r$. *The space* ${}_2w_p \cap {}_2\ell_\infty$ *is not perfect.*

Proof. Clearly ${}_2\ell_r \subseteq ({}_2w_p \cap {}_2\ell_\infty)^\eta$. Conversely, let $< a_{mn} > \notin {}_2\ell_r$, then $\sum_m\sum_n |a_{mn}|^r = \infty$. Consider the sequence $< b_{mn} >$ defined by

$$b_{mn} = 1, \quad \text{for all } m, n \in N.$$

Then $< b_{mn} > \in {}_2w_p \cap {}_2\ell_\infty$, but $\sum_m\sum_n |a_{mn}b_{mn}|^r = \infty$. Hence $< a_{mn} > \notin ({}_2w_p \cap {}_2\ell_\infty)^\eta$. Therefore $({}_2w_p \cap {}_2\ell_\infty)^\eta \subseteq {}_2\ell_r$. Thus $({}_2w_p \cap {}_2\ell_\infty)^\eta = {}_2\ell_r$. The rest of the proof is obvious. ■

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