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COINCIDENCE AND COMMON FIXED POINT THEOREMS VIA R-WEAK COMMUTATIVITY OF TYPE (A_T)

Abstract. We prove common fixed point theorems for two pairs of hybrid mappings satisfying implicit relations in complete metric spaces using the concept of $R-$weak commutativity of type A_T and we correct errors of [1], [3] and [8]. Our theorems generalize results of [1-3], [8], [12-16] and [21].

Key words: hybrid mappings, common fixed point, $R-$weakly commuting of type A_T, metric space.

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1. Introduction and preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \subset X$, $D(x, A) = \inf\{d(x, y), y \in A\}$.

Let $CB(X)$ be the set of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric with respect to d defined by

$$H(A, B) = \max\left\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(A, b)\right\} \quad \text{for all } A, B \in CB(X).$$

It is well known that $(CB(X), H)$ is a metric space and if (X, d) is complete, then $(CB(X), H)$ is also complete

Lemma 1 ([9]). *If $A, B \in CB(X)$ and $k > 1$, then for each $a \in A$, there exists $b \in B$ such that $d(a, b) \leq kH(A, B)$.*

Let $f : X \to X$ be a single-valued mapping and $T : X \to CB(X)$ be a multi-valued mapping.

Definition 1. 1) *A point $x \in X$ is said to be a coincidence point of f and T if $fx \in Tx$. We denote by $C(f, T)$ the set of all coincidence points of f and T.*

2) *A point $x \in X$ is a fixed point of T if $x \in Tx$.*

Definition 2. 1) *f and T are said to be commuting [4] in X if for all $x \in X$, $fTx = Tfx$.*

2) *f and T are said to be weakly commuting on X [17, 18] if for all $x \in X$, $fTx \in CB(X)$ and*

$$H(fTx, Tfx) \ \le \ D(fx, Tx)$$

3) *f and T are said to be compatible [5, 7] if for all $x \in X$, $fTx \in CB(X)$ and*

$$\lim_{n\to\infty} H(fTx_n, Tfx_n) \ = \ 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$ for some $t \in X$ and $A \in CB(X)$.

Commuting implies weakly commuting implies compatible, but the converse is not true in general. See [7].

Let $T : X \to X$ be a single-valued mapping and $F : X \to CB(X)$ be a multi-valued mapping.

Definition 3 ([10], [19])**.** 1) *T and F are said to be R-weakly commuting at $x \in X$ if $TFx \in CB(X)$ and there exists an $R > 0$ such that*

$$H(TFx, FTx) \ \le \ RD(Tx, Fx). \tag{1}$$

2) *T and F are said to be pointwise R-weakly commuting on X if for all $x \in X$, $TFx \in CB(X)$ and (1) holds for some $R > 0$.*

Definition 4 ([6])**.** *T and F are said to be R-weakly commuting of type (A_T) at $x \in X$ if there exists an $R > 0$ such that*

$$D(TTx, FTx) \ \le \ RD(Tx, Fx). \tag{2}$$

T and F are said to be R-weakly commuting of type (A_T) on X if for all $x \in X$, (2) holds.

Remark 1. If F is a single-valued mapping, then the definition of R-weak commutativity of type (A_T) reduces to that of Pathak et. al [11].

If T and F are compatible, then they are R-weakly commuting of type (A_T), but the converse is not true in general, see [6].

The following theorem was proved by [8].

Theorem 1. *Let (X, d) be a complete metric space, $S, T : X \to X$ and $F, G : X \to CB(X)$ satisfying*

$$F(X) \subset S(X) \quad \textit{and} \quad G(X) \subset T(X), \tag{3}$$

(4) *The pairs (T, F) and (S, G) are R-weakly commuting of type (A_T) at their coincidence points.*

$$(5) \qquad H(Fx, Gy) \leq a \frac{D^2(Fx, Sy) + D^2(Gy, Tx)}{D(Fx, Sy) + D(Gy, Tx)} + bd(Tx, Sy),$$

for all $x, y \in X$, $x \neq y$, $Fx \neq Fy$ and $Gx \neq Gy$, where $a, b > 0$ and $a + 2b < 1$, whenever $D(Fx, Sy) + D(Gy, Tx) \neq 0$ and $H(Fx, Gy) = 0$ whenever $D(Fx, Sy) + D(Gy, Tx) = 0$. Then, there exists $z \in X$ such that $z = Tz = Sz \in Fz \cap Gz$.

This theorem generalizes Theorems 3.1 and 3.2 of [1].

In [13] and [14], the study of fixed points for mappings satisfying implicit relations was introduced and the study of a pair of hybrid mappings satisfying implicit relations was initiated in [15].

It is our purpose in this paper to prove coincidence and common fixed point theorems for two pairs of hybrid mappings satisfying implicit relations using the concept of $R-$weak commutativity of type A_T which generalize the results of [1-3], [8], [12-16] and [21].

2. Implicit relations

Let Φ_6 the family of all real continuous mappings $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1) : ϕ is increasing in variable t_1 and decreasing in variables t_3, t_4, t_5 and t_6.

(ϕ_2) : there exists $0 \leq h < 1$ and $k > 1$ such that

(ϕ_a) : $u \leq kt$ and $\phi(t, v, v, u, u + v, 0) \leq 0$ or

(ϕ_b) : $u \leq kt$ and $\phi(t, v, u, v, 0, u + v) \leq 0$

implies $u \leq hv$.

Example 1. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, $a, c > 0$, $b \geq 0$ and $a + 2b + 2c < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{a + 2b + 2c}$, $u \leq kt$ and $\phi(t, v, v, u, u + v, 0) = t - av - b(v + u) - c(u + v) \leq 0$. Then, $u \leq kt \leq kav + kb(v + u) + kc(u + v)]$ and so $u \leq hv$, where $h = \dfrac{k(a + b + c)}{1 - (kb + kc)} < 1$. Similarly, $u \leq kt$ and $\phi(t, v, u, v, 0, u + v) \leq 0$ implies $u \leq hv$.

Example 2. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\}$, $0 < a < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{a}$, $u \le kt$ and $\phi(t, v, v, u, u+v, 0) = t - a\max\{v, u, \dfrac{u+v}{2}\} \le 0$. Then, $u \le kt \le ka\max\{v, u, \dfrac{u+v}{2}\} = ka\max\{v, u\}$. If $u > 0$ and $u \ge v$, it follows that $u \le kau < u$ which is a contradiction and so $u \le hv$, where $h = ka < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t, v, u, v, 0, u+v) \le 0$ implies $u \le hv$.

Example 3. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\max\{t_2^2, t_3t_4, t_5t_6, t_3t_5, t_4t_6\}^{\frac{1}{2}}$, $0 < a < \dfrac{1}{\sqrt{2}}$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{a\sqrt{2}}$, $u \le kt$ and $\phi(t, v, v, u, u+v, 0) = t - a\max\{v^2, uv, v(u+v)\}^{\frac{1}{2}} \le 0$. Then, $u \le kt \le ka\max\{v^2, uv, v(u+v)\}^{\frac{1}{2}}$. If $u > 0$ and $u \ge v$, it follows that $u \le ka\sqrt{2}u < u$ which is a contradiction and so $u \le hv$, where $h = ka\sqrt{2} < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t, v, u, v, 0, u+v) \le 0$ implies $u \le hv$.

Example 4. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 + \dfrac{t_1}{1 + t_5t_6} - at_2^2 - bt_3^2 - ct_4^2$, $a > 0$, $b, c \ge 0$ and $a + b + c < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{\sqrt{a+b+c}}$, $u \le kt$ and $\phi(t, v, v, u, u+v, 0) = t^2 + t - av^2 - bv^2 - cu^2 \le 0$. Then, $t^2 \le av^2 + bv^2 + cu^2$ and $u^2 \le k^2t^2 \le k^2\left(av^2 + bv^2 + cu^2\right)$. It follows that $u \le h_1v$, where $h_1 = k\sqrt{\dfrac{a+b}{1-k^2c}} < 1$. Similarly, $u \le kt$ and $\phi(t, v, u, v, 0, u+v) \le 0$ implies $u \le h_2v$, where $h_2 = k\sqrt{\dfrac{a+c}{1-k^2b}} < 1$. If $h = \max\{h_1, h_2\}$, then $u \le hv$.

Example 5. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^p - \max\{at_2t_3^{p-1}, at_2^{p-1}t_4, at_3^{p-1}t_4, ct_5^{p-1}t_6\}$, $p \ge 2$, $0 < a < 1$ and $c \ge 0$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{\sqrt[p]{a}}$, $u \le kt$ and $\phi(t, v, v, u, u+v, 0) = t^p - \max\{av^p, av^{p-1}u\} \le 0$. Then, $u^p \le k^pt^p \le k^p\max\{av^p, av^{p-1}u\}$. If $u > 0$ and $u \ge v$, it follows that $u^p \le ak^pu^p < u^p$ which is a contradiction and so $u \le hv$, where $h = k\sqrt[p]{a} < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t, v, u, v, 0, u+v) \le 0$ implies $u \le hv$.

Example 6. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b[a\max\{t_2, t_3, t_4, \dfrac{t_5+t_6}{2}\} - (1-a)\max\{t_2^2, t_3t_4, t_5t_6, \frac{1}{2}t_3t_6, \frac{1}{2}t_4t_5\}^{\frac{1}{2}}]$, $0 < b < 1$ and $0 \le a < 1$.

(ϕ_1) : Obviously.

(ϕ_2) : Let $1 < k < \dfrac{1}{b}$, $u \le kt$ and $\phi(t, v, v, u, u+v, 0) = t - b[a \max\{v, u, \dfrac{u+v}{2}\} - (1-a)\max\{v^2, uv, \frac{1}{2}u(u+v)\}^{\frac{1}{2}}] \le 0$. Then, $u \le kt \le kb[a \max\{v, u, \dfrac{u+v}{2}\} + (1-a)\max\{v^2, uv, \frac{1}{2}u(u+v)\}^{\frac{1}{2}}]$. If $u > 0$ and $u \ge v$, it follows that $u \le kbu < u$ which is a contradiction and so $u \le hv$, where $h = kb < 1$. If $u = 0$, therefore $u \le hv$. Similarly, $u \le kt$ and $\phi(t, v, u, v, 0, u+v) \le 0$ implies $u \le hv$.

Example 7. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b\dfrac{t_5^2 + t_6^2}{t_5 + t_6} - c(t_3 + t_4)$, $t_5 + t_6 \ne 0$, $a, b > 0$, $c \ge 0$ and $a + 2b + 2c < 1$.

Example 8. $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b\dfrac{t_3^2 + t_4^2}{t_3 + t_4} - c(t_5 + t_6)$, $t_3 + t_4 \ne 0$, $a, b, c > 0$ and $a + 2b + 2c < 1$.

They follow as in Example 1 since $\dfrac{t_5^2 + t_6^2}{t_5 + t_6} \le t_5 + t_6$ and $\dfrac{t_3^2 + t_4^2}{t_3 + t_4} \le t_3 + t_4$ if $t_5 + t_6 \ne 0$ and $t_3 + t_4 \ne 0$.

3. Main results

Theorem 2. *Let (X, d) be a metric space, $S, T : X \to X$ and $F, G : X \to CB(X)$ satisfying (3)*

$$\begin{aligned}(6)\qquad \phi(H(Fx, Gy), d(Tx, Sy), D(Tx, Fx), D(Sy, Gy),\\ D(Tx, Gy), D(Sy, Fx)) \le 0\end{aligned}$$

for all $x, y \in X$, where $\phi \in \Phi_6$, whenever $D(Tx, Gy) + D(Sy, Fx) \ne 0$ and $H(Fx, Gy) = 0$ whenever $D(Tx, Gy) + D(Sy, Fx) = 0$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then

a) there exists $q, p \in X$ such that $Tq \in Fq$ and $Sp \in Gp$.

Further, if the pair (T, F) is R-weakly commuting of type (A_T) and (S, G) is R-weakly commuting of type (A_S) at their coincidence points,

b) there exists $z \in X$ such that $Tz \in Fz$ and $Sz \in Gz$.

c) In the case (b), if $Sz = Tz$, then $Sz = Tz \in Fz \cap Gz$.

d) In the case (c), if $Sz = Tz = z$, then z is a common fixed point of S, T, F and G.

Proof. First, assume that there exists $q, p \in X$ such that $D(Sp, Fq) + D(Tq, Gp) = 0$. So, $D(Sp, Fq) = 0$ and $D(Tq, Gp) = 0$ which implies that $Sp \in Fq$ and $Tq \in Gp$. Since $H(Fq, Gp) = 0$, it follows that $D(Tq, Fq) \le H(Fq, Gp) = 0$ and hence $Tq \in Fq$. In a similar manner, we get $Sp \in Gp$.

Now, assume that $D(Tx, Gy) + D(Sy, Fx) \neq 0$ for all $x, y \in X$. Let $x_0 \in X$ be an arbitrary point. By (3) and Lemma 1, we define a sequence $\{y_n\}$ in X by

$$y_{2n} = Tx_{2n} \in Gx_{2n-1}, \qquad y_{2n+1} = Sx_{2n+1} \in Fx_{2n}$$

and

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\leq kH(Fx_{2n}, Gx_{2n-1}), \\ d(y_{2n+1}, y_{2n+2}) &\leq kH(Fx_{2n}, Gx_{2n+1}), \text{ for } n = 1, 2, \dots . \end{aligned}$$

Using (6) and (ϕ_1), we have

$$\begin{aligned} 0 &\geq \phi(H(Fx_{2n}, Gx_{2n-1}), d(Tx_{2n}, Sx_{2n-1}), D(Tx_{2n}, Fx_{2n}), \\ &\quad D(Sx_{2n-1}, Gx_{2n-1}), D(Tx_{2n}, Gx_{2n-1}), D(Sx_{2n-1}, Fx_{2n})) \\ &\geq \phi(H(Fx_{2n}, Gx_{2n-1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\quad d(y_{2n-1}, y_{2n}), 0, d(y_{2n-1}, y_{2n+1})) \\ &\geq \phi(H(Fx_{2n}, Gx_{2n-1}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\ &\quad d(y_{2n-1}, y_{2n}), 0, d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})). \end{aligned}$$

By (ϕ_b), we obtain

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}).$$

In the same manner, applying (6) we get

$$\begin{aligned} 0 &\geq \phi(H(Fx_{2n}, Gx_{2n+1}), d(Tx_{2n}, Sx_{2n+1}), D(Tx_{2n}, Fx_{2n}), \\ &\quad D(Sx_{2n+1}, Gx_{2n+1}), D(Tx_{2n}, Gx_{2n+1}), D(Sx_{2n+1}, Fx_{2n})) \\ &\geq \phi(H(Fx_{2n}, Gx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ &\quad d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), 0) \\ &\geq \phi(H(Fx_{2n}, Gx_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\ &\quad d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0). \end{aligned}$$

Therefore

$$d(y_{2n+1}, y_{2n+2}) \leq hd(y_{2n}, y_{2n+1}).$$

and so

$$d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n).$$

Then, $\{y_n\}$ is a Cauchy sequence in X. Assume that $S(X)$ is complete. Then, $\{y_{2n+1}\}$ converges to $z \in S(X)$ and so there exists $p \in X$ such that $z = Sp$. Also, $\{y_{2n}\}$ converges to z since

$$d(y_{2n}, z) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, z)$$

Applying (6) and (ϕ_1) we have

$$(7)\qquad \begin{aligned} 0 \ \geq\ & \phi(H(Fx_{2n},Gp),d(Tx_{2n},Sp),D(Tx_{2n},Fx_{2n}),\\ & D(Sp,Gp),D(Tx_{2n},Gp),D(Sp,Fx_{2n}))\\ \geq\ & \phi(D(y_{2n+1},Gp),d(y_{2n},z),d(y_{2n},y_{2n+1}),\\ & D(Sp,Gp),D(y_{2n},Gp),d(Sp,y_{2n+1})). \end{aligned}$$

Letting $n\to\infty$ we get

$$\phi(D(Sp,Gp),0,0,D(Sp,Gp),D(Sp,Gp),0)\ \leq\ 0.$$

By (ϕ_a) we obtain $Sp\in Gp$. As $G(X)\subset T(X)$, there exists $q\in X$ such that $z=Sp=Tq$.

Using (6) and (ϕ_1) we have

$$(8)\qquad \begin{aligned} 0 \ \geq\ & \phi(H(Fq,Gx_{2n-1}),d(Tq,Sx_{2n-1}),D(Tq,Fq),\\ & D(Sx_{2n-1},Gx_{2n-1}),D(Tq,Gx_{2n-1}),D(Sx_{2n-1},Fq))\\ \geq\ & \phi(D(Fq,y_{2n}),d(Tq,y_{2n-1}),D(Tq,Fq),\\ & d(y_{2n-1},y_{2n}),d(Tq,y_{2n}),D(y_{2n-1},Fq)). \end{aligned}$$

Letting $n\to\infty$ we get

$$\phi(D(Fq,Tq),0,D(Fq,Tq),0,0,D(Fq,Tq))\ \leq\ 0.$$

By (ϕ_b) we obtain $Tq\in Fq$. Since T and F are R-weakly commuting of type (A_T) at $q\in C(F,T)$, there exists an $R>0$ such that $D(TTq,FTq)\leq RD(Tq,Fq)$ and so $Tz\in Fz$. In the same manner, $Sz\in Gz$. If $Sz=Tz$, then $Sz=Tz\in Fz\cap Gz$ and if $Sz=Tz=z$, then z is a common fixed point of S,T,F and G.

Suppose that $T(X)$ is complete. Therefore, $\{y_{2n}\}$ converges to $z\in T(X)$ and so there exists $q\in X$ such that $z=Tq$. Applying (6) and (ϕ_1) we have the inequality (8). Letting $n\to\infty$ we get

$$\phi(D(Fq,Tq),0,D(Fq,Tq),0,0,D(Fq,Tq))\ \leq\ 0.$$

By (ϕ_b) we obtain $Tq\in Fq$. As $F(X)\subset S(X)$, there exists $p\in X$ such that $z=Sp=Tq$.

Using (6) and (ϕ_1) we get the inequality (7). Letting $n\to\infty$ we get

$$\phi(D(Sp,Gp),0,0,D(Sp,Gp),D(Sp,Gp),0)\ \leq\ 0.$$

By (ϕ_a) we obtain $Sp\in Gp$. The rest of the proof follows as in the case $S(X)$ is complete. ∎

Corollary 1. *Let (X, d) be a metric space, $S, T : X \to X$ and $F, G : X \to CB(X)$ satisfying (3) and*

$$\begin{aligned} H(Fx, Gy) \le\ & ad(Tx, Sy) + b(D(Tx, Fx) + D(Sy, Gy)) \\ & + c(D(Tx, Gy) + D(Sy, Fx)) \end{aligned}$$

for all $x, y \in X$, where $a, c > 0$, $b \ge 0$ and $a + 2b + 2c < 1$. Suppose that one of $S(X)$ or $T(X)$ is complete. Then, (a) holds. Further, if the pair (T, F) is R-weakly commuting of type (A_T) and (S, G) is R-weakly commuting of type (A_S) at their coincidence points, therefore the conclusions (b), (c) and (d) of Theorem 2 hold.

Proof. It follows from Theorem 2 and Example 1. ■

Remark 2. In Theorems of [1] and [8], to prove that $z = Tz$, the authors used: "$Tx_{2n} \in Gx_{2n-1}$ and $Tz \in Fz$ implies that $d(Tx_{2n}, Tz) \le H(Gx_{2n-1}, Fz)$" which is false because "$a \in A$ and $b \in B$ implies $d(a, b) \le H(A, B)$" is not true in general as it shown by the following example.

Example 9. Let $d(x, y) = |x - y|$, $A = [0, \frac{1}{2}]$ and $B = [\frac{1}{4}, 1]$. We have $0 \in A$ and $1 \in B$, but $d(0, 1) = 1 > H(A, B) = \frac{1}{2}$. Therefore, Theorem 1.7 of [8] is false as it is proved by the following example.

Example 10. Let $(X, d) = ([1, \infty), |.|)$, $Sx = Tx = x^2 + 1$ and $Fx = Gx = [2, x + 3]$ for all $x \in X$. It is easy to verify that for all $x, y \in X$

$$d(Sx, Sy) = \left|x^2 - y^2\right| \ge 2\,|x - y| = H(Fx, Fy)$$

and hence

$$\begin{aligned} H(Fx, Fy) &\le \frac{1}{2}d(Tx, Ty) \\ &\le \frac{1}{2}d(Tx, Ty) + \frac{1}{8}\frac{D^2(Ty, Fx) + D^2(Tx, Fy)}{D(Ty, Fx) + D(Tx, Fy)} \end{aligned}$$

if $D(Ty, Fx) + D(Tx, Fy) \ne 0$ and the other conditions of Theorem 1.7 of [8] are satisfied, but S and F have no common fixed point.

The following corollary is the correct form of Theorem 1.7 of [8].

Corollary 2. *Let (X, d)be a complete metric space, $T, S : X \to X$ and $F, G : X \to CB(X)$satisfying (3) and*

$$H(Fx, Gy) \le ad(Tx, Sy) + c\,\frac{D^2(Sy, Fx) + D^2(Tx, Gy)}{D(Sy, Fx) + D(Tx, Gy)}$$

for all $x, y \in X$, where $a, c > 0$ and $a + 2c < 1$, whenever $D(Tx, Gy) + D(Sy, Fx) \neq 0$ and $H(Fx, Gy) = 0$ whenever $D(Tx, Gy) + D(Sy, Fx) = 0$. Then, (a) holds. Further, if the pair (T, F) is R -weakly commuting of type (A_T) and (S, G) is R-weakly commuting of type (A_S) at their coincidence points, therefore the conclusions (b), (c) and (d) of Theorem 2 hold.

Proof. It follows from the fact that $\dfrac{D^2(Sy, Fx) + D^2(Tx, Gy)}{D(Sy, Fx) + D(Tx, Gy)} \leq D(Sy, Fx) + D(Tx, Gy)$ if $D(Tx, Gy) + D(Sy, Fx) \neq 0$ and Corollary 1. ■

Remark 3. In [16] Remark 3 and [8] Remark 5, we have: "the conditions in the hypothesis of Theorem 3.1 of [1] and Theorem 1.7 of [8], $x \neq y, Fx \neq Fy$ and $Gx \neq Gy$ are necessary since the theorem fails for F and G taken as constant mappings". This is demonstrated by the following example.

Example 11. Let $X = \{0, 1\}, Tx = 1 - x$ and $Fx = Gx = \{0, 1\}$ for all $x \in X$. It is easy to verify that the mappings satisfy all the hypothesis except $x \neq y, Fx \neq Fy$.

Remark 4. 1) In Example 11, we have $T(0) \in F(0)$ and $T(1) \in F(1)$; i.e., T and F have coincidence points. Since $T^2(0) \neq T(0)$ and $T^2(1) \neq T(1)$, T and F have no common fixed point

2) In theorems of [1], [3] and [8], $x \neq y, Fx \neq Fy$ and $Gx \neq Gy$ are not necessary as it is shown by the following example.

3) In Theorem 1 of [21], S and g are compatible should be the pairs (S, f) and (T, G) are compatible and in Corollary 2, g should be replaced by f and the pair (S, f) is compatible should be added.

4) In [16], the authors made the following remark. It is not yet known whether their theorem remains true if one of the mappings f and T is not continuous and Theorem 2 of [20] yields that the answer is affirmative.

Example 12. Let $X = \{0, 1, \frac{1}{2}\}$, $Tx = 1 - x$ and $Fx = Gx = \{0, \frac{1}{2}, 1\}$ for all $x \in X$. It is easy to verify that the mappings satisfy the conditions of theorems of [1], [3] and [8] except $x \neq y$, $Fx \neq Fy$, but $T(\frac{1}{2}) = \frac{1}{2} \in F(\frac{1}{2})$ and so $\frac{1}{2}$ is a common fixed point of T and F.

As $x \neq y, Fx \neq Fy$ and $Gx \neq Gy$ are not necessary, it follows that theorem of [1] and Theorems 3.2 and 3.3 of [3] part (a) are false, it suffices to take Example 3.8 for [1] and $X = \{0, 1\}, Tx = 1 - x, Sx = Ix = Jx = x$ and $Fx = Gx = \{0, 1\}$ for all $x \in X$ for [3].

We can also prove the following theorem which generalizes Theorems 3.2 and 3.3 of [3].

Theorem 3. *Let* (X, d) *be a metric space,* $S, T, f, g : X \to X$ *and* $F, G : X \to CB(X)$ *satisfying*

$$F(X) \subset Tg(X) \quad \textit{and} \quad G(X) \subset Sf(X)$$

$$\phi(H(Fx, Gy), d(Sfx, Tgy), D(Sfx, Fx), D(Tgy, Gy), D(Sfx, Gy), D(Fx, Tgy)) \le 0$$

for all $x, y \in X$ *, where* $\phi \in \Phi_6$*, whenever* $D(Sfx, Gy) + D(Fx, Tgy) \neq 0$ *and* $H(Fx, Gy) = 0$ *whenever* $D(Sfx, Gy) + D(Fx, Tgy) = 0$*. Suppose that one of* $S(X)$ *or* $T(X)$ *is complete. Then*

a) There exists $p, q \in X$ *such that* $Sfp \in Fp$ *and* $Tgq \in Gq$*.*

Further, if (Sf, F) *is R-weakly commuting of type* A_{Sf} *and* (Tg, G) *is R-weakly commuting of type* A_{Tg} *at their coincidence points, therefore*

b) There exists $z \in X$ *such that* $Sfz \in Fz$ *and* $Tgz \in Gz$*.*

c) In the case (b), if $Sfz = Tgz$*, then* $Sfz = Tgz \in Fz \cap Gz$*.*

d) In the case (c), if $Sfz = Tgz = z$*,* (S, f)*,*(Sf, S)*,* (T, g)*,* (Tg, T) *commute,* $S^2z = Sz$*,* $f^2z = fz$*,* $T^2z = Tz$ *and* $g^2z = gz$*, then* z *is a common fixed point of* f*,* S*,* T*,* g*,* Sf*,* Tg*,* F *and* G*.*

The following theorem generalizes theorems of Popa [13-16].

Theorem 4. *Let* (X, d) *be a metric space,* $S, T : X \to X$ *and* $F, G : X \to CB(X)$ *satisfying (3) and*

$$\phi(H(Fx, Gy), d(Tx, Sy), D(Tx, Fx), D(Sy, Gy), D(Tx, Gy), D(Sy, Fx)) \le 0$$

for all $x, y \in X$*, where* $\phi \in \Phi_6$*. Suppose that one of* $S(X)$ *or* $T(X)$ *is complete. Then, (a) holds. Further, if* (S, G) *is R-weakly commuting of type* A_S *and* (T, F) *is R-weakly commuting of type* A_T *at their coincidence points, therefore the conclusions (b), (c) and (d) of Theorem 2 hold.*

Theorem 5. *Let* $\{F_n\}_{n \ge 1}$ *be a sequence of mappings from a metric space* (X, d) *into* $CB(X)$ *and* $S, T : X \to X$ *satisfying*

$$F_1(X) \subset S(X) \quad \textit{and} \quad F_n(X) \subset T(X), \quad n > 1 \tag{9}$$

$$\phi(H(F_1x, F_ny), d(Tx, Sy), D(Tx, F_1x), D(Sy, F_ny), D(Tx, F_ny), D(Sy, F_1x)) \le 0$$

for all $x, y \in X$*, where* $\phi \in \Phi_6$*, whenever* $D(Tx, F_ny) + D(F_1x, Sy) \neq 0$ *and* $H(F_1x, F_ny) = 0$ *whenever* $D(Tx, F_ny) + D(F_1x, Sy) = 0$*. Suppose that one of* $S(X)$ *or* $T(X)$ *is complete. Then* *a) There exists* $p, q \in X$ *such that* $Sp \in F_n p$ *and* $Tq \in F_1 q$, $n > 1$.

Further, if pair (T, F_1) *is R-weakly commuting of type* (A_T) *and* (S, F_n) *is R-weakly commuting of type* (A_S) *at their coincidence points for* $n > 1$, *therefore*

b) There exists $z \in X$ *such that* $Tz \in F_1 z$ *and* $Sz \in F_n z$.

c) In the case (b), if $Sz = Tz$, *then* $Sz = Tz \in F_1 z \cap F_n z$.

d) In the case (c), if $Sz = Tz = z$, *then* z *is a common fixed point of* T_n, F *and* G.

The following theorem generalizes theorems of Popa [13-16] and Djoudi and Aliouche [2].

Theorem 6. *Let* $\{F_n\}_{n\geq 1}$ *be a sequence of mappings from a metric space* (X, d) *into* $CB(X)$ *and* $S, T : X \to X$ *satisfying (9) and*

$$\phi(H(F_1x, F_ny), d(Tx, Sy), D(Tx, F_1x), D(Sy, F_ny), D(Tx, F_ny), D(Sy, F_1x)) \leq 0$$

for all $x, y \in X$, *where* $\phi \in \Phi_6$. *Suppose that one of* $S(X)$ *or* $T(X)$ *is complete. Then, (a) holds. Further, if* (T, F_1) *is* $R-$*commuting of type* A_T *and* (S, F_n) *is R-weakly commuting of type* A_S *at their coincidence points for* $n > 1$, *therefore the conclusions (b), (c) and (d) of Theorem 5 hold.*

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