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LIPSCHITZ-CONTINUITY OF THE SOLUTION MAP OF SOME NONCONVEX SECOND-ORDER DIFFERENTIAL INCLUSIONS

Abstract. We prove the Lipschitz dependence on the initial condition of the solution set of a nonconvex second-order differential inclusions by applying the contraction principle in the space of selections of the multifunction instead of the space of solutions.

Key words: cosine family of operators, fixed point, contractive set-valued map.

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1. Introduction

In this note we consider the second-order differential inclusion

$$x'' \in Ax + F(t,x), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map and A is the infinitesimal generator of a strongly continuous cosine family of operators $\{C(t);\ t \in R\}$ on a separable Banach space X.

Existence of solutions and qualitative properties of the solutions of problem (1) have been obtained in [1, 2] etc. via fixed point techniques. In [5] a Filippov type existence result is obtained by a different approach which consists in the application of the contraction principle in the space of selections of the multifunction instead of the space of solutions.

This approach allows to obtain also some qualitative properties of the solution set. More precisely, in the present paper we study the properties of the map that associates to a given initial condition $(x_0, y_0) \in X \times X$ the set of mild solutions of problem (1) starting from (x_0, y_0) and the main purpose is to prove that this solution map depends Lipschitz-continuously on the initial condition.

Our results may be considered as extensions to second-order differential inclusions of the form (1) of previous results ([4]) obtained for semilinear differential inclusions of the form

$$x' \in Ax + F(t,x), \qquad x(0) = x_0,$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t); t \geq 0\}$ on a separable Banach space X. We note that the idea of applying the set-valued contraction principle due to Covitz and Nadler ([6]) in the space of derivatives of the solutions belongs to Tallos ([10, 12]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main results.

2. Preliminaries

Let denote by I the interval $[0,T]$, $T > 0$ and let X be a real separable Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot,\cdot)$. Denote by $B(X)$ the Banach space of bounded linear operators from X into X.

We recall that a family $\{C(t);\ t \in R\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following conditions are satisfied

(i) $C(0) = I$, where I is the identity operator in X,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)\ \forall t,s \in R$,

(iii) the map $t \to C(t)y$ is strongly continuous $\forall y \in X$.

The strongly continuous sine family $\{S(t);\ t \in R\}$ associated to a strongly continuous cosine family $\{C(t);\ t \in R\}$ is defined by

$$S(t)y \ := \ \int_0^t C(s)yds, \quad y \in X, t \in R.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t);\ t \in R\}$ is defined by

$$Ay \ = \ \left(\frac{d^2}{dt^2}\right) C(t)y|_{t=0}.$$

Fore more details on strongly continuous cosine and sine family of operators we refer to [8, 9, 13].

In what follows A is infinitesimal generator of a cosine family $\{C(t);\ t \in R\}$ and $F(\cdot,\cdot) : I \times X \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values, which define the following Cauchy problem associated to a second-order differential inclusion

$$x'' \in Ax + F(t,x), \quad x(0) = x_0, \quad x'(0) = y_0. \tag{2}$$

A continuous mapping $x(\cdot) \in C(I, X)$ is called a *mild solution* of problem (2) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I, X)$ such that:

$$f(t) \in F(t, x(t)) \qquad a.e.\,(I), \tag{3}$$

$$x(t) \; = \; C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)f(u)du \quad \forall t \in I, \tag{4}$$

i.e., $f(\cdot)$ is a (Bochner) integrable selection of the set-valued map $F(\cdot, x(\cdot))$ and $x(\cdot)$ is the mild solution of the Cauchy problem

$$x'' \; = \; Ax + f(t) \qquad x(0) = x_0, \qquad x'(0) = y_0. \tag{5}$$

We shall call $(x(\cdot), f(\cdot))$ a *trajectory-selection pair* of (2) if $f(\cdot)$ verifies (3) and $x(\cdot)$ is a mild solution of (5).

We shall use the following notations for the solution sets of (2).

$$\begin{aligned} \mathcal{S}(x_0, y_0) = \{&(x(\cdot), f(\cdot)); \\ &(x(\cdot), f(\cdot)) \text{ is a trajectory-selection pair of (2)}\}, \end{aligned} \tag{6}$$

$$\mathcal{S}_1(x_0, y_0) = \{x(\cdot); \quad x(\cdot) \text{ is a mild solution of (2)}\}. \tag{7}$$

In what follows the following conditions are satisfied.

Hypothesis 1. *(i) A is infinitesimal generator of a given strongly continuous bounded cosine family $\{C(t);\ t \in R\}$.*

(ii) $F(\cdot, \cdot) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and for every $x \in X$, $F(\cdot, x)$ is measurable.

(iii) There exists $L(\cdot) \in L^1(I, R_+)$ such that for almost all $t \in I, F(t, \cdot)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \leq L(t)|x - y| \quad \forall\;\; x, y \in X,$$

where $d_H(A, B)$ is the Hausdorff distance between $A, B \subset X$

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\}.$$

(iv) $d(0, F(t, 0)) \leq L(t)$ a.e. (I)

Let $m(t) = \int_0^t L(u)du$ and let $M \geq 0$ be such that $|C(t)| \leq M \;\forall t \in I$. Note that $|S(t)| \leq Mt \;\forall t \in I$.

Given $\alpha \in R$ we consider on $L^1(I, X)$ the following norm

$$|f|_1 = \int_0^T e^{-\alpha m(t)} |f(t)| dt, \qquad f \in L^1(I, X),$$

which is equivalent with the usual norm on $L^1(I,X)$.

Consider the following norm on $C(I,X)\times L^1(I,X)$

$$|(x,f)|_{C\times L} = |x|_C + |f|_1 \quad \forall\,(x,f)\in C(I,X)\times L^1(I,X),$$

where, as usual, $|x|_C = \sup_{t\in I}|x(t)|\ \forall\, x\in C(I,X)$.

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z,d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z. T is said to be a λ-contraction if there exists $0<\lambda<1$ such that:

$$d_H(T(x),T(y)) \le \lambda d(x,y) \quad \forall x,y\in Z$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z\in Z$ such that $z\in T(z)$ ([6]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T. Obviously, $Fix(T)$ is closed.

Proposition 1. *([11]) Let Z be a complete metric space and suppose that T_1,T_2 are λ-contractions with closed values in Z. Then*

$$d_H(Fix(T_1),Fix(T_2)) \ \le\ \frac{1}{1-\lambda}\sup_{z\in Z} d_H(T_1(z),T_2(z)).$$

3.The main results

We are ready now to show that the set of all trajectory-selection pairs of (2) depends Lipschitz-continuously on the initial condition.

Theorem 1. *Let Hypothesis 1 be satisfied and let $\alpha > MT$.*

Then the map $(x_0,y_0)\to \mathcal{S}(x_0,y_0)$ is Lipschitz-continuous on $X\times X$ with nonempty closed values in $C(I,X)\times L^1(I,X)$.

Proof. Let us consider $x_0,y_0\in X$, $f(\cdot)\in L^1(I,X)$ and define the following set valued maps

$$(8)\quad M_{x_0,y_0,f}(t) \ =\ F\left(t, C(t)x_0+S(t)y_0+\int_0^t S(t-u)f(u)du\right), \quad t\ge 0,$$

$$(9)\qquad T_{x_0,y_0}(f)\ =\ \left\{\ \phi(\cdot)\in L^1(I,X);\ \ \phi(t)\in M_{x_0,y_0,f}(t)\ \ a.e.\ (I)\right\}.$$

We shall prove first that $T_{x_0,y_0}(f)$ is nonempty and closed for every $f\in L^1(I,X)$. The fact that that the set valued map $M_{x_0,y_0,f}(\cdot)$ is measurable is well known. For example, the map $t \to C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)f(u)du$ can be approximated by step functions and we can apply Theorem III. 40 in [3]. Since the values of F are closed and X is separable with the measurable selection theorem (Theorem III.6 in [3]) we infer that $M_{x_0,y_0,f}(\cdot)$ admits a measurable selection ϕ. According to Hypothesis 1 one has

$$\begin{aligned} |\phi(t)| &\le d(0, F(t,0)) + d_H(F(t,0), F(t,x(t)) \le L(t)(1+|x(t)|) \\ &\le L(t)\left(1 + M|x_0| + Mt|y_0| + \int_0^t M(t-s)|f(s)|ds\right). \end{aligned}$$

Thus integrating by parts we obtain

$$\begin{aligned} \int_0^T e^{-\alpha m(t)}|\phi(t)|dt \;\le\; \int_0^T e^{-\alpha m(t)} L(t) \Bigg(1 + M|x_0| + Mt|y_0| \\ + \int_0^t M(t-s)|f(s)|ds\Bigg)\, dt \;\le\; \frac{1+M|x_0|}{\alpha} + \frac{MT|y_0|}{\alpha} + \frac{MT|f|_1}{\alpha}. \end{aligned}$$

Hence, if $\phi(\cdot)$ is a measurable selection of $M_{x_0,y_0,f}(\cdot)$, then $\phi(\cdot) \in L^1(I,X)$ and thus $T_{x_0,y_0}(f) \neq \emptyset$.

The set $T_{x_0,y_0}(f)$ is closed. Indeed, if $\phi_n \in T_{x_0,y_0}(f)$ and $|\phi_n - \phi|_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T_{x_0,y_0}(f)$.

The next step of the proof will show that $T_{x_0,y_0}(\cdot)$ is a contraction on $L^1(I,X)$.

Let $f, g \in L^1(I,X)$ be given, $\phi \in T_{x_0,y_0}(f)$ and let $\varepsilon > 0$. Consider the following set valued map

$$\begin{aligned} G(t) \;=\; M_{x_0,y_0,g}(t) \cap \Bigg\{ x \in X; \\ |\phi(t) - x| \le L(t)\left|\int_0^t S(t-s)(f(s)-g(s))ds\right| + \varepsilon \Bigg\}. \end{aligned}$$

Since

$$\begin{aligned} d\left(\phi(t), M_{x_0,y_0,g}(t)\right) &\le d\Bigg(F\left(t, C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)f(u)du\right), \\ &\qquad F\left(t, C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)g(u)du\right)\Bigg) \\ &\le L(t)\left|\int_0^t S(t-u)(f(u)-g(u))du\right| \end{aligned}$$

we deduce that $G(\cdot)$ has nonempty closed values. Moreover, according to Proposition III. 4 in [3], $G(\cdot)$ is measurable. Let $\psi(\cdot)$ be a measurable selection of $G(\cdot)$. It follows that $\psi \in T_{x_0,y_0}(g)$ and

$$\begin{aligned}
|\phi-\psi|_1 &= \int_0^T e^{-\alpha m(t)}|\phi(t)-\psi(t)|dt \\
&\leq \int_0^T e^{-\alpha m(t)}L(t)\left(\int_0^t M(t-s)|f(s)-g(s)|ds\right)dt \\
&\quad + \int_0^T \varepsilon e^{-\alpha m(t)}dt \\
&\leq \frac{MT}{\alpha}|f-g|_1+\varepsilon\int_0^T e^{-\alpha m(t)}dt.
\end{aligned}$$

Since ε was arbitrary, we deduce that

$$d(\phi, T_{x_0,y_0}(g)) \leq \frac{MT}{\alpha}|f-g|_1.$$

Replacing f by g we obtain

$$d(T_{x_0,y_0}(f), T_{x_0,y_0}(g)) \leq \frac{MT}{\alpha}|f-g|_1,$$

hence $T_{x_0,y_0}(\cdot)$ is a contraction on $L^1(I,X)$.

Consequently $T_{x_0,y_0}(\cdot)$ admits a fixed point $f(\cdot) \in L^1(I,X)$. We define $x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)f(u)du$.

We have that $\mathcal{S}(x_0,y_0) \subset C(I,X) \times L^1(I,X)$ is a closed subset. Let $(x_n,f_n) \in \mathcal{S}(x_0,y_0)$, $|(x_n,f_n)-(x,f)|_{C\times L} \to 0$. In particular, $f_n \in Fix(T_{x_0,y_0})$, which is a closed set, and thus $f(\cdot) \in Fix(T_{x_0,y_0})$. We define $y(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-u)f(u)du$ and we prove that $y(\cdot) = x(\cdot)$. One may write

$$\begin{aligned}
|y-x_n|_C &= \sup_{t\in I}|y(t)-x_n(t)| \\
&\leq \sup_{t\in I} M\int_0^t (t-u)|f_n(u)-f(u)|du \leq MTe^{\alpha m(T)}|f_n-f|_1
\end{aligned}$$

and finally we get that $y(\cdot) = x(\cdot)$.

We prove next the following inequality

$$d_H(T_{x_1,y_1}(f), T_{x_2,y_2}(f)) \leq \frac{1}{\alpha}(M|x_1-x_2| + MT|y_1-y_2|) \tag{10}$$

$\forall f \in L^1(I,X)$, $x_1,x_2,y_1,y_2 \in X$. Let us consider the set-valued map

$$\begin{aligned}
G_1(t) = M_{x_1,x_2,f}(t) \cap \{z \in X;& \\
&|\phi(t)-z| \leq L(t)(|C(t)||x_1-x_2| + |S(t)||y_1-y_2|) + \varepsilon\},
\end{aligned}$$

$t \in I$, where $\phi(\cdot)$ is a measurable selection of $M_{x_1,y_1,f}(\cdot)$ and $\varepsilon > 0$.

With the same arguments used for the set valued map $G(\cdot)$, we deduce that $G_1(\cdot)$ is measurable with nonempty closed values. Let $\psi(\cdot)$ be a measurable selection of $G_1(\cdot)$. It follows that $\psi(\cdot) \in T_{x_2,y_2}(f)$ and

$$\begin{aligned}
|\phi - \psi|_1 &= \int_0^T e^{-\alpha m(t)} |\phi(t) - \psi(t)| dt \\
&\leq \int_0^T e^{-\alpha m(t)} L(t) \left(|C(t)|\ |x_1 - x_2| + |S(t)|\ |y_1 - y_2|\right) dt \\
&\quad + \varepsilon \int_0^T e^{-\alpha m(t)} dt \\
&\leq \frac{M}{\alpha} |x_1 - x_2| + \frac{MT}{\alpha} |y_1 - y_2| + \varepsilon \int_0^T e^{-\alpha m(t)} dt.
\end{aligned}$$

Since ε was arbitrary, we deduce that

$$d(\phi, T_{x_2,y_2}(f)) \leq \frac{1}{\alpha} \left(M|x_1 - x_2| + MT|y_1 - y_2|\right).$$

Replacing (x_1, y_1) by (x_2, y_2) we obtain (10).

From (10) and Proposition 1 we obtain

$$d_H(Fix(T_{x_1,y_1}), Fix(T_{x_2,y_2})) \leq \frac{1}{\alpha - MT} \left(M|x_1 - x_2| + MT|y_1 - y_2|\right).$$

Let $x_1, x_2, y_1, y_2 \in X$ and $(x(\cdot), f(\cdot)) \in \mathcal{S}(x_1, y_1)$. In particular, $f(\cdot) \in Fix(T_{x_1,y_1})$ and thus, for every $\varepsilon > 0$ there exists $g(\cdot) \in Fix(T_{x_2,y_2})$ such that

$$|f - g|_1 \leq \frac{1}{\alpha - MT} \left(M|x_1 - x_2| + MT|y_1 - y_2|\right) + \varepsilon. \tag{11}$$

Put $z(t) = C(t)x_2 + S(t)y_2 + \int_0^t S(t-u)g(u)du$. One has

$$\begin{aligned}
|x - z|_C &= \sup_{t \in I} |x(t) - z(t)| \leq M|x_1 - x_2| + MT|y_1 - y_2| \\
&\quad + \sup_{t \in I} \int_0^t M(t-s)|f(s) - g(s)| ds \\
&\quad + M|x_1 - x_2| + MT|y_1 - y_2| + MTe^{\alpha m(t)}|f - g|_1 \\
&\leq \left(1 + \frac{MTe^{\alpha m(t)}}{\alpha - MT}\right) \left(M|x_1 - x_2| + MT|y_1 - y_2|\right) + \frac{MTe^{\alpha m(t)}}{\alpha - MT} \varepsilon.
\end{aligned}$$

If we denote $k = \max\{M + \frac{M^2Te^{\alpha m(t)}}{\alpha - MT}, MT + \frac{M^2T^2e^{\alpha m(t)}}{\alpha - MT}\}$ we deduce first that

$$d((x, f), \mathcal{S}(x_2, y_2)) \leq k[|x_1 - x_2| + |y_1 - y_2|]$$

and by interchanging (x_1, y_1) and (x_2, y_2) we obtain

$$d_H(\mathcal{S}(x_1, y_1), \mathcal{S}(x_2, y_2)) \ \leq \ k[|x_1 - x_2| + |y_1 - y_2|]$$

and the proof is complete. ■

Obviously, from Theorem 1 we also obtain

Corollary 1. *Let Hypothesis 1 be satisfied and let $\alpha > MT$. Then the map $(x_0, y_0) \to \mathcal{S}_1(x_0, y_0)$ is Lipschitz continuous on $X \times X$ with nonempty values in $C(I, X)$.*

In general, under the hypothesis of Theorem 1 the solution set $\mathcal{S}_1(x_0, y_0)$ is not closed in $C(I, X)$. The next result shows that if X is reflexive and the multifunction $F(\cdot, \cdot)$ is convex valued and integrably bounded then $\mathcal{S}_1(x_0, y_0) \subset C(I, X)$ is closed.

Let B be the closed unit ball in X.

Proposition 2. *Assume that X is reflexive, $\alpha > MT$ and let $F(\cdot, \cdot) : I \times X \to \mathcal{P}(X)$ be a convex valued set valued map that satisfies Hypothesis 1. Assume that there exists $k(\cdot) \in L^1(I, X)$ such that for almost all $t \in I$ and for all $x \in X$, $F(t, x) \subset k(t)B$.*

Then for every $x_0, y_0 \in X$, the set $\mathcal{S}_1(x_0, y_0) \subset C(I, X)$ is closed.

Proof. Let $x_n(\cdot) \in \mathcal{S}_1(x_0, y_0)$ such that $|x_n - x|_C \to 0$. There exists $h_n(\cdot) \in L^1(I, X)$ such that $(x_n(\cdot), h_n(\cdot))$ is a trajectory-selection pair of (2) $\forall n \in N$. We define $f_n(t) = e^{-\alpha m(t)} h_n(t)$, $t \in I$.

The set valued map $F(\cdot, \cdot)$ being integrably bounded, we have that $f_n(\cdot)$ is bounded in $L^1(I, X)$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall E \subset I$, $\mu(E) < \delta$ $|\int_E f_n(s) ds| < \varepsilon$ uniformly with respect to n. Moreover, X is reflexive and so by the Dunford-Pettis criterion ([7]), taking a subsequence and keeping the same notations, we may assume that $f_n(\cdot)$ converges weakly in $L^1(I, X)$ to some $f(\cdot) \in L^1(I, X)$.

We recall that for convex subsets of a Banach space the strong closure coincides with the weak closure. We apply this result. Since $f_n(\cdot)$ converges weakly in $L^1(I, X)$ to $f(\cdot) \in L^1(I, X)$ for all $h \geq 0$, $f(\cdot)$ belongs to the weak closure of the convex hull $co\{f_n(\cdot)\}_{n \geq h}$ of the subset $\{f_n(\cdot)\}_{n \geq h}$. It coincides with the strong closure of $co\{f_n(\cdot)\}_{n \geq h}$. Hence there exist $\lambda_i^n > 0$, $i = n, \ldots k(n)$ such that

$$\sum_{i=1}^{k(n)} \lambda_i^n = 1, \qquad g_n(\cdot) \ = \ \sum_{i=n}^{k(n)} \lambda_i^n f_i(\cdot) \in co\{f_n(\cdot)\}_{n \geq h}$$

and such that $g_n(\cdot)$ converges strongly to $f(\cdot)$ in $L^1(I, X)$. Let

$$l_n(\cdot) \ = \ \sum_{i=n}^{k(n)} \lambda_i^n h_i(\cdot).$$

Then there exists a subsequence $g_{n_j}(\cdot)$ that converges to $f(\cdot)$ almost everywhere. In particular, $l_{n_j}(\cdot)$ converges almost everywhere to $l(\cdot) = e^{\alpha m(\cdot)} f(\cdot) \in L^1(I, X)$. Hence using the Lebesque dominated convergence theorem, for every $t \in I$ we obtain

$$\lim_{j\to\infty} \int_0^t S(t-u) l_{n_j}(u) du \;=\; \int_0^t S(t-u) l(u) du.$$

We define

$$y(t) \;=\; C(t)x_0 + S(t)y_0 + \int_0^t S(t-u) l(u) du, \qquad t \in I.$$

and observe that

$$\begin{aligned} |x(t) - y(t)| \;\le\; & |x(\cdot) - x_{n_j}(\cdot)|_C \\ & + \left| \int_0^t S(t-u) l_{n_j}(u) du - \int_0^t S(t-u) l(u) du \right|, \end{aligned}$$

which yields $x(t) = y(t)\ \forall t \in I$.

Let us observe now that for almost every $t \in I$

$$l_{n_j}(t) \in \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} F(t, x_i(t)) \subset F(t, x(t)) + L(t) \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} |x(t) - x_i(t)| B.$$

Since $\lim_{i\to\infty} |x(t) - x_i(t)| = 0$, we deduce that $f(t) \in F(t, x(t))$ *a.e.*(I) and the proof is complete. ■

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