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Ayşegül Erençin and Fatma Taşdelen

ON CERTAIN KANTOROVICH TYPE OPERATORS

Abstract. In the present paper, we introduce a generalization of the Kantorovich type operators $K_n^*(f;x)$ defined in [1]. We give approximation properties of these operators with the help of Bohman-Korovkin Theorem. We also compute rate of convergence by means of modulus of continuity, the elements of local Lipschitz class and Peetre's K-functional.

Key words: Kantorovich type operators, modulus of continuity, local Lipschitz class, Peetre's K-functional.

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1. Introduction

In [7], Lupaş proposed the linear positive operators

$$L_n(f;x) \;=\; 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right), \qquad x \geq 0, \quad n \in \mathbb{N} \tag{1}$$

with the help of the identity

$$\frac{1}{(1-a)^{\alpha}} \;=\; \sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}a^k, \qquad |a| < 1$$

where $(\alpha)_0 = 1$, $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$, $k \geq 1$.

Agratini [1], studied the rate of convergence on a finite interval and established a Voronovskaja type formula for these operators. The author also introduced Kantorovich type generalization of the operators (1) as follows:

$$K_n^*(f;x) \;=\; n2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \qquad x \geq 0, \;\; n \in \mathbb{N}. \tag{2}$$

In [5], we presented the modification of the operators (1)

$$L_n^*(f;x) \;=\; 2^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)_k}{2^k k!}f\left(\frac{k}{b_n}\right), \qquad x \geq 0, \;\; n \in \mathbb{N}$$

where a_n, b_n are increasing and unbounded sequences of positive numbers such that

$$\frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right), \qquad \lim_{n\to\infty} \frac{1}{b_n} = 0$$

and we studied the convergence properties of these operators in weighted spaces of continuous functions on positive semi-axis.

In this paper, we consider the generalization of the Kantorovich type operators $K_n^*(f;x)$ given by (2) as follows:

$$(3) \qquad K_n(f;x) = b_n 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \int_{\frac{k}{b_n}}^{\frac{k+1}{b_n}} f(t)dt, \qquad x \geq 0, \quad n \in \mathbb{N}$$

where $a_n \geq b_n$ are increasing and unbounded sequences of positive numbers such that

$$(4) \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = 1, \qquad \lim_{n\to\infty} \frac{1}{b_n} = 0$$

and f is an integrable function on $[0,\infty)$ and bounded on every compact subinterval of $[0,\infty)$.

2. Main results

In [5], we proved that

$$L_n^*(1;x) = 1, \quad L_n^*(t;x) = \frac{a_n}{b_n}\, x, \quad L_n^*(t^2;x) = \frac{a_n^2}{b_n^2} x^2 + \frac{2a_n}{b_n^2}\, x.$$

By using these relations we can state

Lemma 1. *The operators K_n defined by (3) verify*

$$\begin{aligned} K_n(1;x) &= 1 \\ K_n(t;x) &= \frac{a_n}{b_n}\, x + \frac{1}{2b_n} \\ K_n(t^2;x) &= \frac{a_n^2}{b_n^2}\, x^2 + \frac{3a_n}{b_n^2}\, x + \frac{1}{3b_n^2} \end{aligned}$$

for all $x \geq 0$.

Lemma 2. *The operators K_n defined by (3) verify*

$$K_n((t-x)^2;x) = \left(\frac{a_n}{b_n} - 1\right)^2 x^2 + \left(\frac{3a_n}{b_n^2} - \frac{1}{b_n}\right) x + \frac{1}{3b_n^2}$$

for all $x \geq 0$.

Proof. Since K_n is linear, we can get

$$K_n((t-x)^2;x) \;=\; K_n(t^2;x) - 2xK_n(t;x) + x^2K_n(1;x).$$

We note that throughout this paper $[a,b]$ defines any subinterval of $[0,\infty)$. Now, we can give the following theorem for the convergence of the operators K_n. ■

Theorem 1. *If $f \in C[a,b]$, then the operators $K_n(f;x)$ defined by (3) converges to $f(x)$ uniformly on $[a,b]$.*

Proof. By using Lemma 1 and conditions (4) we can apply the well known Bohman-Korovkin Theorem (see [3], [6]) to obtain the required result. ■

Now, we study rate of convergence for the sequence of linear positive operators K_n with the help of modulus of continuity. Let us recall that if $I \subset \mathbb{R}$ is a given interval and f is a bounded real valued function defined on I the modulus of continuity $\omega(f;\delta)$ of f denoted by

$$\omega(f;\delta) \;=\; \sup\{|f(t)-f(x)|;\; x,t \in I,\; |t-x| < \delta\}$$

for any $\delta > 0$.

Theorem 2. *Let f be an integrable function on $[0,\infty)$ and bounded on every compact subinterval of $[0,\infty)$. Then any $x \geq 0$, the operators K_n defined by (3) satisfies the inequality*

$$|K_n(f;x) - f(x)| \;\leq\; 2\omega(f;\delta_{n,x})$$

where

$$\delta_{n,x} \;=\; \left[\left(\frac{a_n}{b_n}-1\right)^2 x^2 + \left(\frac{3a_n}{b_n^2} - \frac{1}{b_n}\right)x + \frac{1}{3b_n^2}\right]^{\frac{1}{2}}.$$

Proof. From the linearity and monotonicity of the operators K_n we can write

$$(5) \qquad |K_n(f;x) - f(x)| \;\leq\; b_n 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \int_{\frac{k}{b_n}}^{\frac{k+1}{b_n}} |f(t)-f(x)|dt.$$

On the other hand, by using well known properties of the modulus of continuity $\omega(f;\delta)$

$$|f(t)-f(x)| \leq \omega(f;|t-x|)$$

and

$$\omega(f;\lambda\delta) \;\leq\; \left(1+\lambda^2\right)\omega(f;\delta)$$

we have

$$|f(t) - f(x)| \leq \left(\frac{(t-x)^2}{\delta^2} + 1 \right) \omega(f; \delta) \tag{6}$$

for every $\lambda > 0$ and $\delta > 0$. Thus substituting (6) in (5), from Lemma 2 we find

$$\begin{aligned} (7) \quad & |K_n(f;x) - f(x)| \\ & \leq \omega(f;\delta) \left\{ \frac{1}{\delta^2} \left[b_n 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \int_{\frac{k}{b_n}}^{\frac{k+1}{b_n}} (t-x)^2 dt \right] + 1 \right\} \\ & = \omega(f;\delta) \left[\frac{1}{\delta^2} K_n((t-x)^2; x) + 1 \right] \\ & = \omega(f;\delta) \left\{ \frac{1}{\delta^2} \left[\left(\frac{a_n}{b_n} - 1 \right)^2 x^2 + \left(\frac{3a_n}{b_n^2} - \frac{1}{b_n} \right) x + \frac{1}{3b_n^2} \right] + 1 \right\}. \end{aligned}$$

■

Thus if we take $\delta = \delta_{n,x}$ in (7) then we obtain the desired result.

Theorem 3. *If f is differentiable on $[a, b]$ and $f \in C^1[a, b]$, then for each $x \in [a, b]$ the operators K_n defined by (3) satisfies the inequality*

$$|K_n(f;x) - f(x)| \leq |f'(x)| \left[\left(\frac{a_n}{b_n} - 1 \right) x + \frac{1}{2b_n} \right] + 2\delta_{n,x} \omega(f'; \delta_{n,x})$$

where $\delta_{n,x}$ defined as in Theorem 2.

Proof. Since $K_n(1;x) = 1$, from the well known result of O. Shisha and B. Mond (see[8], [9]), it follows that

$$\begin{aligned} (8) \quad |K_n(f;x) - f(x)| \leq \; & |f'(x)| \, |K_n(t;x) - x| + \sqrt{K_n((t-x)^2;x)} \\ & \times \left[1 + \frac{1}{\delta} \sqrt{K_n((t-x)^2;x)} \right] \omega(f';\delta). \end{aligned}$$

In (8), using Lemma 1 and Lemma 2, we can get

$$|K_n(f;x) - f(x)| \leq |f'(x)| \left[\left(\frac{a_n}{b_n} - 1 \right) x + \frac{1}{2b_n} \right] + \delta_{n,x} \left(1 + \frac{1}{\delta} \delta_{n,x} \right) \omega(f';\delta).$$

If we choose $\delta = \delta_{n,x}$ we obtain the required result. ■

Theorem 4. *Let K_n defined by (3). If f is local $Lip(\alpha)$ on $[a, b]$, i.e.,*

$$|f(t) - f(x)| \leq M|t-x|^\alpha$$

for all $(x,t) \in [a,b] \times [0,\infty)$ *with* $M > 0$ *and* $0 < \alpha \leq 1$*, then we have*

$$|K_n(f;x) - f(x)| \;\leq\; M\delta_{n,x}^{\alpha}$$

where $\delta_{n,x}$ *defined as in Theorem 2.*

Proof. Let f be local $Lip(\alpha)$ on $[a,b]$. By linearity and monotonicity of the operators K_n, we can get

$$(9)\qquad \begin{aligned} |K_n(f;x) - f(x)| &\leq b_n 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \int_{\frac{k}{b_n}}^{\frac{k+1}{b_n}} |f(t) - f(x)| dt \\ &\leq M b_n 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \int_{\frac{k}{b_n}}^{\frac{k+1}{b_n}} |t - x|^{\alpha} dt. \end{aligned}$$

Applying the Hölder inequality for space of integrable functions with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$ we write (9) as follows:

$$|K_n(f;x) - f(x)| \;\leq\; M b_n^{\frac{\alpha}{2}} 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \left[\int_{\frac{k}{b_n}}^{\frac{k+1}{b_n}} (t-x)^2 dt \right]^{\frac{\alpha}{2}}.$$

We apply the Hölder inequality for L^p space again with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$ and take into consideration $L_n^*(1;x) = 1$ to obtain

$$|K_n(f;x) - f(x)| \;\leq\; M \left[K_n((t-x)^2;x) \right]^{\frac{\alpha}{2}} \;=\; M\delta_{n,x}^{\alpha}.$$

Thus the proof is completed. ∎

Now we recall the following space and norm: $C^2[a,b]$:= the space of the functions f for which f, f', $f'' \in C[a,b]$ and the norm on the space $C^2[a,b]$ can be defined as

$$||f||_{C^2[a,b]} \;=\; ||f||_{C[a,b]} + ||f'||_{C[a,b]} + ||f''||_{C[a,b]}.$$

We consider the Peetre's K-functional (similarly as in [2], [4])

$$(10)\qquad K(f;\delta_n) \;=\; \inf_{g \in C^2[a,b]} \left\{ \|f - g||_{C[a,b]} + \delta_n \|g||_{C^2[a,b]} \right\}.$$

Theorem 5. *Let* K_n *be given by (3). Then for all* $f \in C[0,b]$ *with fixed* b $(1 \leq b < \infty)$ *we have*

$$||K_n(f;x) - f(x)||_{C[0,b]} \;\leq\; 2K(f;\delta_n)$$

where $K(f;\delta_n)$ *is Peetre's* K*-functional defined by (10) and*

$$\delta_n \;=\; \frac{1}{2} \left[\left(\frac{a_n^2}{2b_n^2} + \frac{a_n}{b_n} - \frac{3}{2} \right) b^2 + \left(\frac{3a_n}{2b_n^2} + \frac{1}{2b_n} \right) b + \frac{1}{6b_n^2} \right].$$

Proof. Assume that $g \in C^2[0,b]$. Thus from the Taylor expansion we can get

$$|K_n(g;x) - g(x)| \leq |g'(x)|\,|K_n((t-x);x)| + \frac{1}{2}\,|g''(x)|\,\left|K_n((t-x)^2;x)\right|. \tag{11}$$

Since

$$\left|K_n((t-x)^2;x)\right| \leq \left|K_n(t^2;x) - x^2\right| + 2x\,|K_n(t;x) - x|$$

by using Lemma 1, we can write inequality (11) as

$$\begin{aligned} |K_n(g;x) - g(x)| \leq\ & |g'(x)| \left[\left(\frac{a_n}{b_n} - 1\right)x + \frac{1}{2b_n}\right] \\ & + |g''(x)| \left[\left(\frac{a_n^2}{2b_n^2} + \frac{a_n}{b_n} - \frac{3}{2}\right)x^2 + \left(\frac{3a_n}{2b_n^2} + \frac{1}{2b_n}\right)x + \frac{1}{6b_n^2}\right] \end{aligned}$$

and consequently

$$\begin{aligned} ||K_n(g;x) - g(x)||_{C[0,b]} \leq\ & \left(||g'||_{C[0,b]} + ||g''||_{C[0,b]}\right) \\ & \times \left[\left(\frac{a_n^2}{2b_n^2} + \frac{a_n}{b_n} - \frac{3}{2}\right)b^2 + \left(\frac{3a_n}{2b_n^2} + \frac{1}{2b_n}\right)b + \frac{1}{6b_n^2}\right] \\ \leq\ & ||g||_{C^2[0,b]} \left[\left(\frac{a_n^2}{2b_n^2} + \frac{a_n}{b_n} - \frac{3}{2}\right)b^2 + \left(\frac{3a_n}{2b_n^2} + \frac{1}{2b_n}\right)b + \frac{1}{6b_n^2}\right]. \end{aligned} \tag{12}$$

On the other hand, since K_n is a linear operator, we have

$$|K_n(f;x) - f(x)| \leq |K_n(f-g;x)| + |f(x) - g(x)| + |K_n(g;x) - g(x)|$$

and so

$$||K_n(f;x) - f(x)||_{C[0,b]} \leq 2||f-g||_{C[0,b]} + ||K_n(g;x) - g(x)||_{C[0,b]}. \tag{13}$$

Combining (13) with (12), we can write

$$\begin{aligned} ||K_n(f;x) - f(x)||_{C[0,b]} \leq 2 \Bigg\{ & ||f-g||_{C[0,b]} + ||g(x)||_{C^2[0,b]} \\ & \times \frac{1}{2}\left[\left(\frac{a_n^2}{2b_n^2} + \frac{a_n}{b_n} - \frac{3}{2}\right)b^2 + \left(\frac{3a_n}{2b_n^2} + \frac{1}{2b_n}\right)b + \frac{1}{6b_n^2}\right]\Bigg\}. \end{aligned} \tag{14}$$

If we take infimum over $g \in C^2[0,b]$ from both sides of (14) by choosing

$$\delta_n = \frac{1}{2}\left[\left(\frac{a_n^2}{2b_n^2} + \frac{a_n}{b_n} - \frac{3}{2}\right)b^2 + \left(\frac{3a_n}{2b_n^2} + \frac{1}{2b_n}\right)b + \frac{1}{6b_n^2}\right].$$

the proof is completed. Note that $\delta_n \to 0$ when $n \to \infty$. ∎

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AYŞEGÜL ERENÇIN
ABANT İZZET BAYSAL UNIVERSITY
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
14280 BOLU, TURKEY
e-mail: erencina@hotmail.com

FATMA TAŞDELEN
ANKARA UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
06100 ANKARA, TURKEY
e-mail: tasdelen@science.ankara.edu.tr

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