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SOLVABILITY CONDITIONS FOR A NONLOCAL BOUNDARY VALUE PROBLEM FOR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. The aim of the paper is to find efficient conditions for the unique solvability of the problem

$$u'(t) \;=\; \ell(u)(t) + q(t), \quad u(a) \;=\; h(u) + c,$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operators, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$.

Key words: linear functional differential equation, nonlocal boundary value problem, existence and uniqueness.

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1. Introduction and notation

On the interval $[a, b]$, we consider the boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \tag{1}$$

$$u(a) = h(u) + c, \tag{2}$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator, $q \in L([a,b];\mathbb{R})$, $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a linear nondecreasing functional (i.e. it maps the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$), and $c \in \mathbb{R}$.

By a solution of the equation (1) we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ satisfying the equatity (1) almost everywhere on the interval $[a, b]$. A solution of the equation (1) satisfying the boundary condition (2) is said to be a solution of the problem (1), (2).

In this paper, the efficient sufficient conditions are given for the unique solvability of the problem (1), (2). It is clear that

$$u(a) = \lambda u(b) + c \tag{3}$$

with $\lambda \geq 0$ is a special case of the boundary condition (2). In papers [6, 7, 8], the problem (1), (3) is studied in detail. The results obtained here can be regarded as an extension of those from [6, 7].

The paper is organized as follows. Main results given in Section 2 are concretized in Section 3 for diferential equation with deviating arguments

$$u'(t) = p(t)u\big(\tau(t)\big) - g(t)u\big(\mu(t)\big) + q(t), \tag{1'}$$

where $p, g \in L([a,b];\mathbb{R}_+)$, $q \in L([a,b];\mathbb{R})$, and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions. The proofs of all statements established in this paper can be found in Section 4.

We will suppose in the sequel that the operator ℓ and the functional h appearing in (1) and (2) satisfy the following assumptions:

(i) If $h(1) = 1$ then the operator ℓ is "nontrivial" in the sense that the condition $\ell(1) \not\equiv 0$ holds.

(ii) $\widetilde{h} \not\equiv 0$, where the functional $\widetilde{h}$ is defined by

$$\widetilde{h}(v) = h(v) - v(a) \quad \text{for} \quad v \in C([a,b];\mathbb{R}).$$

The following notation is used throughout the paper:

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|v\|_C = \max\{|v(t)| : t \in [a,b]\}.$$

$C([a,b];D) = \{v \in C([a,b];\mathbb{R}) : v : [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.

$\widetilde{C}([a,b];D)$, where $D \subseteq \mathbb{R}$, is the set of absolutely continuous functions $v : [a,b] \to D$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|p\|_L = \int_a^b |p(s)|ds.$$

$L([a,b];D) = \{p \in L([a,b];\mathbb{R}) : p : [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

$\mathcal{P}_{ab}$ is the set of operators $\ell \in \mathcal{L}_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

F_{ab} is the set of linear bounded functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$.

PF_{ab} is the set of functionals $h \in F_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.

$C_h([a,b];\mathbb{R}) = \{v \in C([a,b];\mathbb{R}) : v(a) = h(v)\}$, where $h \in F_{ab}$.

In what follows, the equalities and inequalities with measurable functions are undestood to hold almost everywhere.

2. Main results

Recall that, throughout the paper, we suppose that $h \in PF_{ab}$. Introduce the following definition.

Definition 1. *We say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\widetilde{V}_{ab}^{+}(h)$ if every function $v \in \widetilde{C}([a,b];\mathbb{R})$ satisfying*

$$v'(t) \geq \ell(v)(t) \quad \text{for} \quad t \in [a,b] \quad \text{and} \quad v(a) \geq h(v)$$

is nonnegative.

Remark 1. Sufficient conditions guaranteeing the inclusion $\ell \in \widetilde{V}_{ab}^{+}(h)$ are given in [11].

Theorem 1. *Let there exist an operator $\overline{\ell} \in \widetilde{V}_{ab}^{+}(h)$ such that the condition*

$$\ell(v)(t) \operatorname{sgn} v(t) \leq \overline{\ell}(|v|)(t) \quad \text{for} \quad t \in [a,b] \tag{4}$$

holds on the set $C_h([a,b];\mathbb{R})$. Then the problem (1), (2) *has a unique solution.*

Theorem 2. *Let there exist operators $\varphi_0 \in \widetilde{V}_{ab}^{+}(h)$ and $\varphi_1 \in \mathcal{P}_{ab}$ such that the condition*

$$|\ell(v)(t) - \varphi_0(v)(t)| \leq \varphi_1(|v|)(t) \quad \text{for} \quad t \in [a,b] \tag{5}$$

holds on the set $C_h([a,b];\mathbb{R})$. If, moreover,

$$\varphi_0 + \varphi_1 \in \widetilde{V}_{ab}^{+}(h), \tag{6}$$

then the problem (1), (2) *has a unique solution.*

Corollary 1. *Let $h(1) < 1$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. If, moreover, there exist $\varepsilon \in [0, 1/2]$ such that*

$$-\varepsilon\ell_1 \in \widetilde{V}_{ab}^{+}(h), \qquad \ell_0 + (1-2\varepsilon)\ell_1 \in \widetilde{V}_{ab}^{+}(h), \tag{7}$$

then the problem (1), (2) *has a unique solution.*

Remark 2. By a suitable choice of the number ε in Corollary 1 and, by virtue of the results from [11], we can derive several efficient conditions for the solvability of the problem (1), (2).

In particular, for $\varepsilon = \frac{1}{2}$, resp. $\varepsilon = \frac{1}{3}$, the assumption (7) reads as

$$\ell_0 \in \widetilde{V}_{ab}^{+}(h), \qquad -\frac{1}{2}\,\ell_1 \in Vp,$$

resp.

$$\ell_0 + \frac{1}{3}\,\ell_1 \in \widetilde{V}_{ab}^{+}(h), \qquad -\frac{1}{3}\,\ell_1 \in \widetilde{V}_{ab}^{+}(h).$$

Notation 1. *Let $h \in PF_{ab}$. For any $\lambda \geq 0$, we put*

$$h_\lambda(v) = h(v) - \lambda v(b) \quad \text{for} \quad v \in C([a,b];\mathbb{R}). \tag{8}$$

Obviously, $h_0 \in PF_{ab}$. Therefore, we can set

$$\lambda^* = \sup\big\{\lambda \geq 0 : h_\lambda \in PF_{ab}\big\}. \tag{9}$$

It is clear also that $0 \leq \lambda^* \leq h(1)$ and $h_{\lambda^*} \in PF_{ab}$.

Theorem 3. *Let $h(1) < 1$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Let, moreover, there exist a function $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[\,)$ such that*

$$\gamma'(t) \geq \ell_0(\gamma)(t) + \ell_1(1)(t) \quad \text{for} \quad t \in [a,b], \tag{10}$$

$$\gamma(a) > h(\gamma), \tag{11}$$

and

$$\gamma(b) - \gamma(a) < 1 + \lambda^* + 2\sqrt{1 + \lambda^* - h(1)}\,, \tag{12}$$

where the number λ^* is defined by (9). Then the problem (1), (2) has a unique solution.

3. Equation with deviating arguments

In this section, we will give some consequences of the main results for the equation with deviating arguments $(1')$.

Theorem 4. *Let $h(1) < 1$. Assume that the functions p and τ satisfy one of the following items:*

a)

$$\int_a^b p(s)ds < 1 - h(1);$$

b) $h(z_0) > 0$ and

$$\max\left\{\frac{h(z_1) + (1 - h(1))z_1(t)}{h(z_0) + (1 - h(1))z_0(t)} : t \in [a,b]\right\} < 1 - \frac{h(z_0)}{1 - h(1)}\,,$$

where

$$z_0(t) = \int_a^t p(s)ds \quad \text{for} \quad t \in [a,b], \tag{13}$$

(14) $$z_1(t) = \int_a^t p(s) \left(\int_a^{\tau(s)} p(\xi) d\xi \right) ds \quad \text{for} \quad t \in [a, b];$$

c)

(15) $$h(\beta_0) < 1,$$

(16) $$h(\beta_1)\beta_0(b) + \big(1 - h(\beta_0)\big)\beta_1(b) < 1 - h(\beta_0),$$

where

(17) $$\beta_0(t) = \exp\left(\int_a^t p(s)ds \right) \quad \text{for} \quad t \in [a, b],$$

(18)
$$\beta_1(t) = \int_a^t p(s)\sigma(s) \left(\int_s^{\tau(s)} p(\xi)d\xi \right) \exp\left(\int_s^t p(\eta)d\eta \right) ds \quad \text{for} \quad t \in [a, b],$$

and

(19) $$\sigma(t) = \frac{1}{2}\big(1 + \operatorname{sgn}(\tau(t) - t)\big) \quad \text{for} \quad t \in [a, b];$$

d) $\int_a^{\tau^*} p(s)ds \neq 0$ *and*

$$\text{ess sup}\left\{ \int_t^{\tau(t)} p(s)ds : t \in [a, b] \right\} < \eta^*,$$

where $\tau^* = \text{ess sup}\big\{\tau(t) : t \in [a, b]\big\}$,

$$\eta^* = \sup\left\{ \frac{1}{y} \ln \frac{y\beta_0^y(\tau^*)}{\beta_0^y(\tau^*) - \big(1 - h\big(\beta_0^y\big)\big)(1 - h(1))^{-1}} : y > 0,\ h(\beta_0^y) < 1 \right\}$$

and β_0 *given by* (17), *while the functions* g *and* μ *satisfy*

(20) $$\mu(t) \le t \quad \text{for} \quad t \in [a, b]$$

and one of the following items:

A)

$$\int_a^b g(s)ds \le 2;$$

B)

$$\int_a^b g(s) \int_{\mu(s)}^s g(\xi) \exp\left(\frac{1}{2}\int_{\mu(\xi)}^s g(\eta)d\eta\right) d\xi ds \le 4;$$

C) $g \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^t g(s)ds : t \in [a,b]\right\} < 2\omega^*,$$

where

$$\omega^* = \sup\left\{\frac{1}{x}\ln\left(x + \frac{x}{\exp\left(\frac{x}{2}\int_a^b g(s)ds\right) - 1}\right) : x > 0\right\}.$$

Then the problem (1′), (2) *has a unique solution.*

Theorem 5. *Let* $h(1) < 1$*. Assume that* (20) *holds and the functions* g *and* μ *satisfy one of the following items:*

A)

$$\int_a^b g(s)ds \le 3\,;$$

B)

$$\int_a^b g(s) \int_{\mu(s)}^s g(\xi) \exp\left(\frac{1}{3}\int_{\mu(\xi)}^s g(\eta)d\eta\right) d\xi ds \le 9\,;$$

C) $g \not\equiv 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_{\mu(t)}^t g(s)ds : t \in [a,b]\right\} < 3\omega^*\,,$$

where

$$\omega^* = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\left(\frac{x}{3}\int\limits_a^b g(s)ds\right) - 1} \right) : x > 0 \right\}.$$

Let, moreover, either

a) the condition (15) *holds, where*

$$\beta_0(t) = \exp\left(\frac{1}{3} \int\limits_a^t g(s)ds \right) \quad \textit{for} \quad t \in [a,b],$$

or

b) $h(z_0) > 0$ *and*

$$\max\left\{ \frac{h(z_1) + (1-h(1))z_1(t)}{h(z_0) + (1-h(1))z_0(t)} : t \in [a,b] \right\} < 3 - \frac{h(z_0)}{1-h(1)},$$

where

$$z_0(t) = \int\limits_a^t g(s)ds \quad \textit{for} \quad t \in [a,b], \tag{21}$$

$$z_1(t) = \int\limits_a^t g(s) \left(\int\limits_a^{\mu(s)} g(\xi)d\xi \right) ds \quad \textit{for} \quad t \in [a,b], \tag{22}$$

be fulfilled. Then the problem $(1')$, (2) *with* $p \equiv 0$ *has a unique solution.*

Theorem 6. *Let* $h(1) < 1$. *Let, moreover,*

$$\tau(t) \le t \quad \textit{for} \quad t \in [a,b], \tag{23}$$

the condition (15) *hold, and*

$$h(\beta_1)\big(\beta_0(b) - 1\big) + \beta_1(b)\big(1 - h(\beta_0)\big) < \omega\big(1 - h(\beta_0)\big), \tag{24}$$

where the function β_0 *is defined by* (17),

$$\beta_1(t) = \int\limits_a^t g(s) \exp\left(\int\limits_s^t p(\xi)d\xi \right) ds \quad \textit{for} \quad t \in [a,b], \tag{25}$$

$$\omega = 1 + \lambda^* + 2\sqrt{1 + \lambda^* - h(1)}, \tag{26}$$

and the number λ^* *is given by* (9). *Then the problem* $(1')$, (2) *has a unique solution.*

Theorem 7. *Let $h(1) < 1$. Let, moreover, the condition* (15) *be fulfilled and*

$$\frac{1-h(1)}{1+\lambda^*-h(1)}\left(\frac{\beta_0(b)h(\beta_2)}{1-h(\beta_0)}+\beta_2(b)\right) < \omega(1-A), \tag{27}$$

where the functions β_0, β_1, and σ are defined by (17)–(19), *the numbers ω and λ^* are given by* (26) *and* (9), *respectively,*

$$A = \frac{h(\beta_1)}{1-h(\beta_0)}\,\beta_0(b)+\beta_1(b), \tag{28}$$

and

$$\beta_2(t) = \int_a^t p(s)\left(\int_a^{\tau(s)} g(\xi)d\xi\right)\exp\left(\int_s^t p(\eta)d\eta\right)ds + \int_a^t g(s)ds \quad for \quad t\in[a,b]. \tag{29}$$

Then the problem (1′), (2) *has a unique solution.*

4. Proofs

The following statement is well–known from the general theory of boundary value problems for functional differentional equations (see, e.g., [1, 2, 9, 12, 5]).

Lemma 1. *The problem* (1), (2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{1_0}$$

$$u(a) = h(u) \tag{2_0}$$

has only the trivial solution.

Remark 3. Acccording to Definition 1 and Lemma 1, it is clear that the inclusion $\ell\in\widetilde{V}^+_{ab}(h)$ guarantees the unique solvability of the problem (1), (2) for any $q\in L([a,b];\mathbb{R})$ and $c\in\mathbb{R}$.

Proof of Theorem 1. According to Lemma 1, it is sufficient to show that the homogeneous problem (1_0), (2_0) has only the trivial solution.

Let u be a solution of the problem (1_0), (2_0). Then $u \in C_h([a,b];\mathbb{R})$ and, in view of (4), we get

$$|u(t)|' = \ell(u)(t)\operatorname{sgn} u(t) \le \overline{\ell}(|u|)(t) \quad \text{for} \quad t \in [a,b]. \tag{30}$$

On the other hand, the condition (2_0), by virtue of the assumption $h \in PF_{ab}$, yields

$$|u(a)| = |h(u)| \le h(|u|). \tag{31}$$

By virtue of the assumption $\overline{\ell} \in \widetilde{V}_{ab}^{+}(h)$, the conditions (30) and (31) imply

$$|u(t)| \le 0 \quad \text{for} \quad t \in [a,b].$$

Hence, the homogeneous problem (1_0), (2_0) has only the trivial solution. ■

Proof of Theorem 2. According to Lemma 1, it is sufficient to show that the homogeneous problem (1_0), (2_0) has only the trivial solution.

Let u be a solution of the problem (1_0), (2_0). Then $u \in C_h([a,b];\mathbb{R})$ and, in view (5), we get

$$\begin{aligned} u'(t) &= \varphi_0(u)(t) + \ell(u)(t) - \varphi_0(u)(t) \\ &\le \varphi_0(u)(t) + \varphi_1(|u|)(t) \quad \text{for} \quad t \in [a,b], \end{aligned} \tag{32}$$

$$\begin{aligned} u'(t) &= \varphi_0(u)(t) + \ell(u)(t) - \varphi_0(u)(t) \\ &\ge \varphi_0(u)(t) - \varphi_1(|u|)(t) \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{33}$$

According to the assumption $\varphi_0 \in \widetilde{V}_{ab}^{+}(h)$ and Remark , the problem

$$\alpha'(t) = \varphi_0(\alpha)(t) + \varphi_1(|u|)(t), \tag{34}$$

$$\alpha(a) = h(\alpha) \tag{35}$$

has a unique solution α. Moreover, since $\varphi_1 \in \mathcal{P}_{ab}$, (34) and (35) imply

$$\alpha(t) \ge 0 \quad \text{for} \quad t \in [a,b]. \tag{36}$$

It follows from (32)–(34) that

$$\begin{aligned} (u-\alpha)'(t) &\le \varphi_0(u-\alpha)(t) \quad \text{for} \quad t \in [a,b], \\ (u+\alpha)'(t) &\ge \varphi_0(u+\alpha)(t) \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{37}$$

On the other hand, the conditions (2_0) and (35) yield

$$(u-\alpha)(a) = h(u-\alpha), \qquad (u+\alpha)(a) = h(u+\alpha). \tag{38}$$

By virtue of the assumption $\varphi_0 \in \widetilde{V}_{ab}^{+}(h)$, (37) and (38) imply

$$|u(t)| \leq \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{39}$$

Consequently, in view of the assumption $\varphi_1 \in \mathcal{P}_{ab}$, we get from (34) the inequality

$$\alpha'(t) \leq (\varphi_0 + \varphi_1)(\alpha)(t) \quad \text{for} \quad t \in [a,b],$$

which, together with (6), (35), and (36), yields $\alpha \equiv 0$. Hence, (39) yields $u \equiv 0$, i.e., the homogeneous problem (1_0), (2_0) has only the trivial solution. ■

Proof of Corollary 1. The validity of corollary immediatelly follows from Theorem 2 with $\varphi_0 = -\varepsilon\,\ell_1$ and $\varphi_1 = \ell_0 + (1-\varepsilon)\,\ell_1$. ■

To prove Theorem 3, we need the following lemma established in [11, Theorem 2.1].

Lemma 2. *Let $\ell \in \mathcal{P}_{ab}$ and let $h \in PF_{ab}$ be such that $h(1) < 1$. Then $\ell \in \widetilde{V}_{ab}^{+}(h)$ if and only if there exists a function $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[\,)$ satisfying*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b], \qquad \gamma(a) > h(\gamma).$$

Proof of Theorem 3. According to Lemma 1, it is sufficient to show that the homogeneous problem (1_0), (2_0) has only the trivial solution.

Suppose that, on the contrary, the problem (1_0), (2_0) possesses a nontrivial solution u. According to Lemma 2, the conditions (10), (11), and the assumption $\ell_1 \in \mathcal{P}_{ab}$, it is clear that

$$\ell_0 \in \widetilde{V}_{ab}^{+}(h). \tag{40}$$

Therefore, by virtue of the assumption $\ell_1 \in \mathcal{P}_{ab}$, it follows easily from Definition 1 that u changes its sign. Put

$$M = \max\big\{u(t) : t \in [a,b]\big\}, \qquad m = -\min\big\{u(t) : t \in [a,b]\big\}, \tag{41}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{42}$$

Obviously,

$$M > 0, \qquad m > 0 \tag{43}$$

and without loss of generality we can assume that $t_m < t_M$.

From (1_0), (2_0), (10), and (11), by virtue of (41), (43), and the assumption $\ell_1 \in \mathcal{P}_{ab}$, we get

$$\begin{aligned}(M\gamma(t) + u(t))' &\geq \ell_0(M\gamma + u)(t) + \ell_1(M - u)(t)\\ &\geq \ell_0(M\gamma + u)(t) \quad \text{for} \quad t \in [a,b],\end{aligned} \tag{44}$$

$$M\gamma(a) + u(a) > h(M\gamma + u), \tag{45}$$

and

$$\begin{aligned}(m\gamma(t) - u(t))' &\geq \ell_0(m\gamma - u)(t) + \ell_1(m + u)(t) \\ &\geq \ell_0(m\gamma - u)(t) \quad \text{for} \quad t \in [a, b],\end{aligned} \tag{46}$$

$$m\gamma(a) - u(a) \ > \ h(m\gamma - u). \tag{47}$$

Hence, according to (40), the previous inequalities yield

$$M\gamma(t) + u(t) \geq 0, \qquad m\gamma(t) - u(t) \geq 0 \quad \text{for} \quad t \in [a, b].$$

Consequently, by virtue of the assumption $\ell_0 \in \mathcal{P}_{ab}$, it follows from (44) and (46) that

$$-u'(t) \leq M\gamma'(t), \qquad u'(t) \ \leq \ m\gamma'(t) \quad \text{for} \quad t \in [a, b]. \tag{48}$$

The integration of the second inequality in (48) from t_m to t_M, in view of (42) and (43), implies

$$M + m \leq m\big(\gamma(t_M) - \gamma(t_m)\big),$$

i.e.,

$$0 < M \leq m\big(\gamma(t_M) - \gamma(t_m) - 1\big). \tag{49}$$

On the other hand, the integrations of the first inequality in (48) from a to t_m and from t_M to b, in view of (42) and (43), yield

$$u(a) + m \leq M\big(\gamma(t_m) - \gamma(a)\big), \qquad M - u(b) \leq M\big(\gamma(b) - \gamma(t_M)\big). \tag{50}$$

Further, the condition (2_0), on account of (8), (41) and the condition $h_{\lambda^*} \in PF_{ab}$, results in

$$u(a) - \lambda^* u(b) = h_{\lambda^*}(u) \ \geq \ -m h_{\lambda^*}(1) = m\big(\lambda^* - h(1)\big). \tag{51}$$

It is also clear that $\lambda^* < 1$ because we suppose that $h(1) < 1$. Therefore, from (50) and (51) we get

$$\begin{aligned}m\big(1 + \lambda^* - h(1)\big) + \lambda^* M \ &\leq \ u(a) + m + \lambda^*\big(M - u(b)\big) \\ &\leq \ M\Big(\gamma(t_m) - \gamma(a) + \lambda^*\big(\gamma(b) - \gamma(t_M)\big)\Big).\end{aligned}$$

Hence, in view of (43) and the condition $\lambda^* < 1$,

$$0 < m\big(1 + \lambda^* - h(1)\big) \leq M\big(\gamma(t_m) - \gamma(a) + \gamma(b) - \gamma(t_M) - \lambda^*\big). \tag{52}$$

From (49) and (52) we get

$$\gamma(b)-\gamma(a)>1+\lambda^* \tag{53}$$

and

$$\begin{aligned} 0 &< 1+\lambda^*-h(1) \\ &\leq \Big(\gamma(t_M)-\gamma(t_m)-1\Big)\Big(\gamma(t_m)-\gamma(a)+\gamma(b)-\gamma(t_M)-\lambda^*\Big). \end{aligned} \tag{54}$$

Finally, in view of the inequality $4xy \leq (x+y)^2$, (54) implies

$$1+\lambda^*-h(1) \leq \frac{1}{4}\Big(\gamma(b)-\gamma(a)-1-\lambda^*\Big)^2,$$

which, on account of (53), contradicts (12).

The contradiction obtained proves that the homogeneous problem (1_0), (2_0) has only the trivial solution. ■

Proof of Theorem 4. Let the operators $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ be defined by

$$\ell_0(v)(t) \overset{\text{def}}{=} p(t)v\big(\tau(t)\big), \qquad \ell_1(v)(t) \overset{\text{def}}{=} g(t)v\big(\mu(t)\big) \quad \text{for} \quad t \in [a,b]. \tag{55}$$

According to the statements from [11], each of the items a)–d) guarantees the inclusion

$$\ell_0 \in \widetilde{V}^+_{ab}(h).$$

On the other hand, each of the items A)–C) implies

$$-\frac{1}{2}\,\ell_1 \in Vp.$$

Therefore, the assumptions of Corollary 1 are satisfied with $\varepsilon = \frac{1}{2}$. ■

Proof of Theorem 5. Let the operator $\ell_1 \in \mathcal{P}_{ab}$ be defined by (55). According to the assertions from [11], each of the items a) and b) guarantees the inclusion

$$\frac{1}{3}\,\ell_1 \in \widetilde{V}^+_{ab}(h).$$

On the other hand, each of the items A)–C) implies

$$-\frac{1}{3}\,\ell_1 \in \widetilde{V}^+_{ab}(h).$$

Therefore, the assumptions of Corollary 1 are satisfied with $\varepsilon = \frac{1}{3}$ and $\ell_0 \equiv 0$. ■

Proof of Theorem 6. Let the operators $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ be defined by (55). According to (24), there exists $\varepsilon > 0$ such that

$$\frac{h(\beta_1)+\varepsilon}{1-h(\beta_0)}\big(\beta_0(b)-1\big)+\beta_1(b)<\omega. \tag{56}$$

Put

$$\gamma(t)=\frac{h(\beta_1)+\varepsilon}{1-h(\beta_0)}\,\beta_0(t)+\beta_1(t) \quad \text{for} \quad t\in[a,b],$$

where the functions β_0 and β_1 are given by (17) and (25), respectively. It is not difficult to verify that

$$\gamma'(t)=p(t)\gamma(t)+g(t) \quad \text{for} \quad t\in[a,b], \qquad \gamma(a)=h(\gamma)+\varepsilon, \tag{57}$$

and $\gamma(t)\geq 0$ for $t\in[a,b]$. Consequently, $\gamma'(t)\geq 0$ for $t\in[a,b]$ and thus (57) implies $\gamma(t)>0$ for $t\in[a,b]$. Further, by virtue of (23), we get

$$p(t)\gamma\big(\tau(t)\big)\leq p(t)\gamma(t) \quad \text{for} \quad t\in[a,b]. \tag{58}$$

Hence, on account of (26) and (55)–(58), the function γ satisfies the inequalities (10)–(12).

Therefore, the assumptions of Theorem 3 are fulfilled. ■

Proof of Theorem 7. Let the operators $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ be defined by (55). According to (27), there exists $\varepsilon > 0$ such that

$$\frac{1-h(1)}{1+\lambda^*-h(1)}\left(\frac{\beta_0(b)\big(h(\beta_2)+\varepsilon\big)}{1-h(\beta_0)}+\beta_2(b)\right)\leq \omega(1-A). \tag{59}$$

From (27) we get $A<1$, which, by virtue of (15), (28), and Theorem 4.2 in [11], guarantees the inclusion (40). Thus, in view of Remark 3, the problem

$$\gamma'(t)=p(t)\gamma\big(\tau(t)\big)+g(t), \tag{60}$$

$$\gamma(a)=h(\gamma)+\varepsilon \tag{61}$$

has a unique solution γ. It is clear that the conditions (10) and (11) are fulfilled. On account of (40), it follows from Definition 1 that $\gamma(t)\geq 0$ for $t\in[a,b]$. Hence, (60) yields

$$0\leq\gamma(a)\leq\gamma(t)\leq\gamma(b) \quad \text{for} \quad t\in[a,b]. \tag{62}$$

Further, the condition (61), on account of the assumption $h\in PF_{ab}$, implies $\gamma(a)\geq\varepsilon>0$ and thus

$$\gamma(t)>0 \quad \text{for} \quad t\in[a,b].$$

On the other hand, it easily follows from (60) that γ satisfies

$$\gamma'(t) = p(t)\gamma(t) + p(t)\int_t^{\tau(t)} p(s)\gamma\big(\tau(s)\big)ds + p(t)\int_t^{\tau(t)} g(s)ds + g(t) \quad \text{for} \quad t \in [a,b].$$

Hence, the Cauchy formula, in view of the notation (17) and (29), implies

$$\gamma(t) = \gamma(a)\beta_0(t) + \int_a^t p(s)\left(\int_s^{\tau(s)} p(\xi)\gamma\big(\tau(\xi)\big)d\xi\right)\exp\left(\int_s^t p(\eta)d\eta\right)ds + \beta_2(t)$$

for $t \in [a,b]$. Whence, in view of (25) and (62), we get

$$\gamma(t) \le \gamma(a)\beta_0(t) + \gamma(b)\beta_1(t) + \beta_2(t) \quad \text{for} \quad t \in [a,b]. \tag{63}$$

Taking now into account (63) and the assumption $h \in PF_{ab}$, the condition (61) yields

$$\gamma(a) \le \gamma(a)h(\beta_0) + \gamma(b)h(\beta_1) + h(\beta_2) + \varepsilon. \tag{64}$$

Thus, from (63) and (64) we get

$$\gamma(b) \le A\gamma(b) + \frac{h(\beta_2) + \varepsilon}{1 - h(\beta_0)}\,\beta_0(b) + \beta_2(b). \tag{65}$$

On the other hand, the condition (61), by virtue of (62) and the assumption $h_{\lambda^*} \in PF_{ab}$, implies

$$\begin{aligned}\gamma(a) &= \lambda^*\gamma(b) + h_{\lambda^*}(\gamma) + \varepsilon > \lambda^*\gamma(b) + \gamma(a)h_{\lambda^*}(1)\\ &= \lambda^*\gamma(b) + \gamma(a)\big(h(1) - \lambda^*\big).\end{aligned}$$

Whence we get

$$\gamma(b) - \gamma(a) < \frac{1 - h(1)}{1 + \lambda^* - h(1)}\,\gamma(b). \tag{66}$$

Finally, it is clear that the conditions (59), (65), and (66) guarantee the inequality (12).

Therefore, the assumptions of Theorem 3 are satisfied. ■

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