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MINIMAL STRUCTURES, m-OPEN MULTIFUNCTIONS IN THE SENSE OF KURATOWSKI AND BITOPOLOGICAL SPACES

Abstract. By using m-open multifunctions from a topological space into an m-space, we establish the unified theory for several weak forms of open multifunctions in the sense of Kuratowski between bitopological spaces.

Key words: m-structure, m-open set, (i, j)-K-m-open multifunction, bitopological space, multifunction.

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1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of open functions and open multifunctions. By using these sets, several authors introduced and studied various types of modifications of open functions and open multifunctions in topological spaces and bitopological spaces. Maheshwari and Prasad [20] and Bose [5] introduced the concepts of semi-open sets and semi-open functions in bitopological spaces. Jelić [11], [13], Kar and Bhattacharyya [14] and Khedr et al. [15] introduced and studied the concepts of preopen sets and preopen functions in bitopological spaces. The notions of α-open sets and α-open functions in bitopological spaces were studied in [12], [24] and [16]. Some forms of open multifunctions between topological and bitopological spaces are studied in [4], [6] and [7]. Recently, in [30] and [31] the present authors introduced the notions of minimal structures, m-spaces and m-continuity.

In the present paper, we introduce the notion of a K-m-open multifunctions from a topological space into an m-space and establish the unified theory for several weak forms of open multifunctions in the sense of Kuratowski between bitopological spaces. We obtain some characterizations of K-m-open multifunctions and characterize the set of all points at which a multifunction is not K-m-open. In the last part, some new modifications of open multifunctions between bitopological spaces are introduced and investigated.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 1. *Let (X, τ) be a topological space. A subset A of X is said to be* α-open *[25] (resp.* semi-open *[18],* preopen *[22],* β-open *[1],* b-open *[3] or* γ-open *[10]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)), A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$).*

The family of all semi-open (resp. preopen, α-open, β-open, b-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$).

Definition 2. *The complement of a semi-open (resp. preopen, α-open, β-open , b-open) set is said to be* semi-closed *[8] (resp.* preclosed *[9],* α-closed *[23],* β-closed *[1],* b-closed *[3]).*

Definition 3. *The intersection of all semi-closed (resp. preclosed, α-closed, β-closed, b-closed) sets of X containing A is called the* semi-closure *[8] (resp.* preclosure *[9],* α-closure *[23],* β-closure *[2],* b-closure *[3]) of A and is denoted by $s\mathrm{Cl}(A)$ (resp. $p\mathrm{Cl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$, $b\mathrm{Cl}(A)$).*

Definition 4. *The union of all semi-open (resp. preopen, α-open, β-open, b-open) sets of X contained in A is called the* semi-interior *(resp.* preinterior*,* α-interior*,* β-interior*,* b-interior*) of A and is denoted by $s\mathrm{Int}(A)$ (resp. $p\mathrm{Int}(A)$, $\alpha\mathrm{Int}(A)$, ${}_{\beta}\mathrm{Int}(A)$, $b\mathrm{Int}(A)$).*

Throughout the present paper, (X, τ) and (Y, σ) (briefly X and Y) always denote topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \quad \text{and} \quad F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 5. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be* K-open *[17] at a point $y \in Y$ if for each open set U of X such that $y \in f(U)$, there exists an open set V of Y such that $y \in V \subset f(U)$. If f is K-open at each point $y \in Y$, then f is said to be K-open.*

Remark 1. A function $f : (X, \tau) \to (Y, \sigma)$ is K-open if and only if $f(U)$ is open for each open set U of X.

Definition 6. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be* semi-open *[26] (resp.* preopen *[22],* α-open *[23],* β-open *[1]) if $f(U)$ is semi-open (resp. preopen, α-open, β-open) for each open set U of X.*

Definition 7. *A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be* open *[4] (resp.* semi-open *[29],* preopen *[7],* α-open *[6],* β-open*) if $F(U)$ is open (resp. semi-open, preopen, α-open, β-open) for each open set U of X.*

3. Minimal structures and K-m-open multifunctions

Definition 8. *A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a* minimal structure *(or briefly* m-structure*) [30], [31] on X if $\emptyset \in m_X$ and $X \in m_X$.*

By (X, m_X) (or briefly (X, m)), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Definition 9. *Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the* m_X-closure *of A and the* m_X-interior *of A are defined in [21] as follows:*
(1) $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 2. Let (X,τ) be a topological space and A be a subset of X. If $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\text{BO}(X)$), then we have
(a) $m_X\text{-Cl}(A) = \text{Cl(A)}$ (resp. $\text{sCl}(A)$, $\text{pCl}(A)$, $\alpha\text{Cl}(A$, ${}_\beta\text{Cl}(A)$, $\text{bCl}(A)$),
(b) $m_X\text{-Int}(A) = \text{Int(A)}$ (resp. $\text{sInt}(A)$, $\text{pInt}(A)$, $\alpha\text{Int}(A$, ${}_\beta\text{Int}(A)$, $\text{bInt}(A)$).

Lemma 1 (Maki et al. [21]). *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*
(1) $m_X\text{-Cl}(X - A) = X - m_X\text{-Int}(A)$ *and*
$m_X\text{-Int}(X - A) = X - m_X\text{-Cl}(A)$,
(2) *If* $(X - A) \in m_X$, *then* $m_X\text{-Cl}(A) = A$ *and if* $A \in m_X$, *then*
$m_X\text{-Int}(A) = A$,
(3) $m_X\text{-Cl}(\emptyset) = \emptyset, m_X\text{-Cl}(X) = X$, $m_X\text{-Int}(\emptyset) = \emptyset$ *and* $m_X\text{-Int}(X) = X$,
(4) *If* $A \subset B$, *then* $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$ *and* $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$,
(5) $A \subset m_X\text{-Cl}(A)$ *and* $m_X\text{-Int}(A) \subset A$,
(6) $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ *and*
$m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$.

Lemma 2 (Popa and Noiri [30]). *Let (X, m_X) be an m-space and A a subset of X. Then $x \in m_X\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 10. *A minimal structure m_X on a nonempty set X is said to have property $\mathcal{B}$ [21] if the union of any family of subsets belonging to m_X belongs to m_X.*

Lemma 3 (Popa and Noiri [32]). *Let (X, m_X) be an m-space and m_X satisfy property $\mathcal{B}$. Then for a subset A of X, the following properties hold:*
(1) *$A \in m_X$ if and only if m_X-$\mathrm{Int}(A) = A$,*
(2) *A is m_X-closed if and only if m_X-$\mathrm{Cl}(A) = A$,*
(3) *m_X-$\mathrm{Int}(A) \in m_X$ and m_X-$\mathrm{Cl}(A)$ is m_X-closed.*

Remark 3. Let (X, τ) be a topological space and $m_X = \mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$), then m_X satisfies property $\mathcal{B}$.

Definition 11. *Let (Y, m_Y) be an m-space. A multifunction $F : (X, \tau) \to (Y, m_Y)$ is said to be K-m-open at $y \in Y$ if for each open set U of X such that $y \in F(U)$, there exists $V \in m_Y$ such that $y \in V \subset F(U)$. If F is K-m-open at each point $y \in Y$, then F is said to be K-m-open.*

Theorem 1. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*
(1) *F is K-m-open at $y \in Y$;*
(2) *For each open set U of X such that $y \in F(U)$, $y \in m_Y$-$\mathrm{Int}(F(U))$;*
(3) *If $A \in \mathcal{P}(X)$ and $y \in F(\mathrm{Int}(A))$, then $y \in m_Y$-$\mathrm{Int}(F(A))$;*
(4) *If $B \in \mathcal{P}(Y)$ and $y \in F(\mathrm{Int}(F^+(B)))$, then $y \in m_Y$-$\mathrm{Int}(B)$.*

Proof. (1) $\Rightarrow$ (2): Let U be any open set of X such that $y \in F(U)$. Then there exists $V \in m_Y$ such that $y \in V \subset F(U)$; hence $y \in m_Y$-$\mathrm{Int}(F(U))$.

(2) $\Rightarrow$ (3): Let $A \in \mathcal{P}(X)$. Then $\mathrm{Int}(A)$ is open in X. Since $y \in F(\mathrm{Int}(A))$, by (2) $y \in m_Y$-$\mathrm{Int}(F(\mathrm{Int}(A))) \subset m_Y$-$\mathrm{Int}(F(A))$.

(3) $\Rightarrow$ (4): Let $B \in \mathcal{P}(Y)$ and $y \in F(\mathrm{Int}(F^+(B)))$. Then, $y \in m_Y$-$\mathrm{Int}(F(F^+(B))) \subset m_Y$-$\mathrm{Int}(B)$.

(4) $\Rightarrow$ (1): Let U be any open set of X such that $y \in F(U)$. Since $U \subset F^+(F(U))$, $U \subset \mathrm{Int}(F^+(F(U)))$ and $F(U) \subset F(\mathrm{Int}(F^+(F(U))))$. Therefore, by (4) $y \in m_Y$-$\mathrm{Int}(F(U))$ and hence there exists $V \in m_Y$ such that $y \in V \subset F(U)$. This shows that F is K-m-open at y. ■

Theorem 2. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*
(1) *F is K-m-open;*
(2) *$F(U) = m_Y$-$\mathrm{Int}(F(U))$ for every open set U of X;*
(3) *$F(\mathrm{Int}(A)) \subset m_Y$-$\mathrm{Int}(F(A))$ for any subset A of X;*
(4) *$\mathrm{Int}(F^+(B)) \subset F^+(m_Y$-$\mathrm{Int}(B))$ for any subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let U be any open set of X and $y \in F(U)$. Since F is K-m-open, there exists $V \in m_Y$ such that $y \in V \subset F(U)$. Therefore,

$y \in m_Y$-Int$(F(U))$. Hence $F(U) \subset m_Y$-Int$(F(U))$. By Lemma 1, $F(U) = m_Y$-Int$(F(U))$.

(2) $\Rightarrow$ (3): Let A be any subset of X and $x \in \mathrm{Int}(A)$. Then, by (2) $F(x) \subset F(\mathrm{Int}(A)) \subset m_Y$-Int$(F(\mathrm{Int}(A)))$. Therefore, we obtain $F(\mathrm{Int}(A)) \subset m_Y$-Int$(F(A))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By (3), we have $F(\mathrm{Int}(F^+(B))) \subset m_Y$-Int$(F(F^+(B))) \subset m_Y$-Int$(B)$. Hence, $\mathrm{Int}(F^+(B)) \subset F^+(m_Y$-Int$(B))$.

(4) $\Rightarrow$ (1): Let $y \in Y$ and U be any open set of X such that $y \in F(U)$. Since $U \subset F^+(F(U))$, by (4) $U \subset \mathrm{Int}(F^+(F(U))) \subset F^+(m_Y$-Int$(F(U)))$. Therefore, $F(U) \subset m_Y$-Int$(F(U))$ and $y \in m_Y$-Int$(F(U))$. Hence there exists $V \in m_Y$ such that $y \in V \subset F(U)$. This shows that F is K-m-open. ■

Remark 4. (a) Let m_Y have property $\mathcal{B}$. Then, it follows from Lemma 3 and Theorem 2 that F is K-m-open if and only if $F(U)$ is m_Y-open for each open set U of X.

(b) If $F : (X, \tau) \to (Y, \sigma)$ is a multifunction and $m_Y = \sigma$ (resp. SO(Y), PO(Y), $\alpha(Y), \beta(Y)$), then by (a) we obtain Definition 7.

(c) If $f : (X, \tau) \to (Y, \sigma)$ is a function and $m_Y = \sigma$ (resp. SO(Y), PO(Y), $\alpha(Y), \beta(Y)$), then by (a) we obtain Definition 6.

For a multifunction $F : (X, \tau) \to (Y, m_Y)$, we denote

$$D^0(F) = \{y \in Y : \ F \ \ is \ \ not \ \ K\text{-}m\text{-open} \ \ at \ \ y\}.$$

Theorem 3. *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, the following properties hold:*

$$\begin{aligned} D^0(F) &= \cup_{U \in \tau}\{F(U) - m_Y\text{-Int}(F(U))\} \\ &= \cup_{A \in \mathcal{P}(\mathcal{X})}\{F(\mathrm{Int}(A)) - m_Y\text{-Int}(F(A))\} \\ &= \cup_{B \in \mathcal{P}(\mathcal{Y})}\{F(\mathrm{Int}(F^+(B))) - (m_Y\text{-Int}(B))\}. \end{aligned}$$

Proof. Let $y \in D^0(F)$. Then, by Theorem 1, there exists an open set U_0 of X such that $y \in F(U_0)$ and $y \notin m_Y$-Int$(F(U_0)))$. Hence $y \in F(U_0) \cap (X - m_Y$-Int$(F(U_0))) = F(U_0) - m_Y$-Int$(F(U_0)) \subset \cup_{U \in \tau}\{F(U) - m_Y$-Int$(F(U))\}$.

Conversely, let $y \in \cup_{U \in \tau}\{F(U) - m_Y$-Int$(F(U))\}$. Then there exists $U_0 \in \tau$ such that $y \in F(U_0) - m_Y$-Int$(F(U_0))$. Therefore, by Theorem 1 F is not K-m-open at y.

The other equations are silimarly proved. ■

4. Minimal structures and bitopological spaces

Throughout the present paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) denote bitopological spaces. For a subset A of X, the closure of A and the interior of A

with respect to τ_i are denoted by $i\mathrm{Cl}(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1, 2$. First, we shall recall some definitions of weak forms of open sets in a bitopological space.

Definition 12. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be*

(1) (i,j)-semi-open *[20] if $A \subset j\mathrm{Cl}(i\mathrm{Int}(A))$, where $i \neq j,\ i,j = 1,2$,*

(2) (i,j)-preopen *[11] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(A))$, where $i \neq j,\ i,j = 1,2$,*

(3) (i,j)-α-open *[12] if $A \subset i\mathrm{Int}(j\mathrm{Cl}(i\mathrm{Int}(A)))$, where $i \neq j,\ i,j = 1,2$,*

(4) (i,j)-semi-preopen *[15] if there exists an (i,j)-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j,\ i,j = 1,2$.*

The family of (i,j)-semi-open (resp. (i,j)-preopen, (i,j)-α-open, (i,j)-semi-preopen) sets of (X, τ_1, τ_2) is denoted by $(i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$).

Remark 5. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then $(i,j)\mathrm{SO}(X)$, $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$ and $(i,j)\mathrm{SPO}(X)$ are all m-structures on X. Hence, if $m_{ij} = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$), then we have

(1) m_{ij}-$\mathrm{Cl}(A) = (i,j)$-$\mathrm{sCl}(A)$ [20] (resp. (i,j)-$\mathrm{pCl}(A)$ [15], (i,j)-$\alpha\mathrm{Cl}(A)$ [24], (i,j)-$\mathrm{spCl}(A)$ [15]),

(2) m_{ij}-$\mathrm{Int}(A) = (i,j)$-$\mathrm{sInt}(A)$ (resp. (i,j)-$\mathrm{pInt}(A)$, (i,j)-$\alpha\mathrm{Int}(A)$, (i,j)-$\mathrm{spInt}(A)$).

Remark 6. Let (X, τ_1, τ_2) be a bitopological space.

(a) Let $m_{ij} = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\alpha(X)$). Then, by Lemma 1 we obtain the result established in Theorem 13 of [20] and Theorem 1.13 of [19] (resp. Theorem 3.6 of [24]).

(b) Let $m_{ij} = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$). Then, by Lemma 2 we obtain the result established in Theorem 1.15 of [19] (resp. Theorem 3.5 of [15], Theorem 3.5 of [24], Theorem 3.6 of [15]).

Remark 7. Let (X, τ_1, τ_2) be a bitopological space.

(a) It follows from Theorem 2 of [20] (resp. Theorem 4.2 of [14] or Theorem 3.2 of [15], Theorem 3.2 of [24], Theorem 3.2 of [15]) that $(i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$) is an m-structure on X satisfying property $\mathcal{B}$.

(b) Let $m_{ij} = (i,j)\mathrm{SO}(X)$ (resp. $(i,j)\mathrm{PO}(X)$, $(i,j)\alpha(X)$, $(i,j)\mathrm{SPO}(X)$). Then, by Lemma 3 we obtain the result established in Theorem 1.13 of [19] (resp. Theorem 3.5 of [15], Theorem 3.6 of [24], Theorem 3.6 of [15]).

5. K-m-open multifunctions in bitopological spaces

Definition 13. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* (i, j)-semi-open *[5] (resp.* (i, j)-preopen *[14],* (i, j)-α-open *[16],* (i, j)-semi-preopen*) if for each τ_i-open set U of X, $f(U)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-semi-preopen) in Y.*

Definition 14. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* (i, j)-almost open (*or* (i, j)-preopen) *[7] if for each $U \in \tau_i$, $F(U)$ is (i, j)-preopen.*

Remark 8. (a) By Remark 7(a), $(i, j)\mathrm{SO}(Y)$, $(i, j)\mathrm{PO}(Y)$, $(i, j)\alpha(Y)$ and $(i, j)\mathrm{SPO}(Y)$ are all m-structures on Y satisfying property $(\mathcal{B})$. Therefore, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-semi-open (resp. (i, j)-preopen, (i, j)-α-open, (i, j)-semi-preopen) if and only if $f : (X, \tau_i) \to (Y, m_{ij})$ is m-open, where $m_{ij} = (i, j)\mathrm{SO}(Y)$ (resp. $(i, j)\mathrm{PO}(Y)$, $(i, j)\alpha(Y)$, $(i, j)\mathrm{SPO}(Y)$).

(b) A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-preopen if and only if $F : (X, \tau_i) \to (Y, m_{ij})$ is m-open, where $m_{ij} = (i, j)\mathrm{PO}(Y)$.

Definition 15. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y determined by σ_1 and σ_2. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* (i, j)-K-m-open *at $y \in Y$ if $F : (X, \tau_i) \to (Y, m_{ij})$ is K-m-open at $y \in Y$. F is said to be* (i, j)-K-m-open *if it is (i, j)-K-m-open at each point of Y.*

Remark 9. (a) A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-*K-m-open* at $y \in Y$ if for each τ_i-open set U such that $y \in F(U)$, there exists $V \in m_{ij}$ such that $y \in V \subset F(U)$.

(b) Let $m_{ij} = m(\sigma_1, \sigma_2)$ have property $\mathcal{B}$. Then it follows from Remark 4 that a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-K-m-open if and only if $F(U)$ is m_{ij}-open for every τ_i-open set U of X.

(c) If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a function and $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction, where m_{ij} has property $\mathcal{B}$, then by Definition 15 we obtain Definitions 13 and 14.

By Definition 15 and Theorems 1-3, we obtain the following theorems.

Theorem 4. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y determined by σ_1 and σ_2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-K-m-open at $y \in Y$;*

(2) *For each τ_i-open set U of X such that $y \in F(U)$, $y \in m_{ij}$-$\mathrm{Int}(F(U))$;*

(3) *If $A \in \mathcal{P}(X)$ and $y \in F(\mathrm{Int}(A))$, then $y \in m_{ij}$-$\mathrm{Int}(F(A))$;*

(4) *If $B \in \mathcal{P}(Y)$ and $y \in F(\mathrm{Int}(F^+(B)))$, then $y \in m_{ij}$-$\mathrm{Int}(B)$.*

Theorem 5. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y determined by σ_1 and σ_2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is (i, j)-K-m-open;*
(2) *$F(U) = m_{ij}\text{-Int}(F(U))$ for every τ_i-open set U of X;*
(3) *$F(\text{Int}(A)) \subset m_{ij}\text{-Int}(F(A))$ for any subset A of X;*
(4) *$\text{Int}(F^+(B)) \subset F^+(m_{ij}\text{-Int}(B))$ for any subset B of Y.*

For a function $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y determined by σ_1 and σ_2, we denote

$$D^0_{ij}(F) = \{y \in Y : F \text{ is not } (i,j)\text{-}K\text{-}m\text{-open at } y\},$$

then by Definition 15 and Theorem 3 we obtain the following theorem:

Theorem 6. *Let (Y, σ_1, σ_2) be a bitopological space and $m_{ij} = m(\sigma_1, \sigma_2)$ an m-structure on Y determined by σ_1 and σ_2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties hold:*

$$\begin{aligned} D^0_{ij}(F) &= \cup_{U \in \tau}\{F(U) - m_{ij}\text{-Int}(F(U))\} \\ &= \cup_{A \in \mathcal{P}(\mathcal{X})}\{F(i\text{Int}(A)) - m_{ij}\text{-Int}(F(A))\} \\ &= \cup_{B \in \mathcal{P}(\mathcal{Y})}\{F(i\text{Int}(F^+(B))) - (m_{ij}\text{-Int}(B))\}. \end{aligned}$$

6. New forms of modifications of open multifunctions

There are many modifications of open sets in topological spaces. In order to define some new modifications of open sets in a bitopological space, let recall θ-open sets and δ-open sets due to Veličko [33]. Let (X, τ) be a topological space. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of a subset A of X if $\text{Cl}(V) \cap A \neq \emptyset$ (resp. $\text{Int}(\text{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\text{Cl}_\theta(A)$ (resp. $\text{Cl}_\delta(A)$). If $A = \text{Cl}_\theta(A)$ (resp. $A = \text{Cl}_\delta(A)$), then A is said to be *θ-closed* (resp. *δ-closed*) [33]. The complement of a θ-closed (resp. δ-closed) set is said to be *θ-open* (resp. *δ-open*). The union of all θ-open (resp. δ-open) sets contained in A is called the *θ-interior* (resp. *δ-interior*) of A and is denoted by $\text{Int}_\theta(A)$ (resp. $\text{Int}_\delta(A)$).

Definition 16. *A subset A of a bitopological space (Y, σ_1, σ_2) is said to be*

(1) (i,j)-δ-semi-open *[27] if $A \subset j\text{Cl}(i\text{Int}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,*
(2) (i,j)-δ-preopen *[28] if $A \subset i\text{Int}(j\text{Cl}_\delta(A))$, where $i \neq j$, $i, j = 1, 2$,*
(3) (i,j)-δ-semi-preopen (*simply* (i,j)-δ-sp-open) *if there exists an (i,j)-δ-preopen set U such that $U \subset A \subset j\text{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.*

Definition 17. *A subset A of a bitopological space (Y, σ_1, σ_2) is said to be*

(1) (i,j)-θ-semi-open *if $A \subset j\mathrm{Cl}(i\mathrm{Int}_\theta(A))$, where $i \neq j$, $i,j = 1,2$,*
(2) (i,j)-θ-preopen *if $A \subset i\mathrm{Int}(j\mathrm{Cl}_\theta(A))$, where $i \neq j$, $i,j = 1,2$,*
(3) (i,j)-θ-semi-preopen (*simply* (i,j)-θ-sp-open) *if there exists an (i,j)-θ-preopen set U such that $U \subset A \subset j\mathrm{Cl}(U)$, where $i \neq j$, $i,j = 1,2$.*

Let (Y, σ_1, σ_2) be a bitopological space. The family of (i,j)-δ-semi-open (resp. (i,j)-δ-preopen, (i,j)-δ-*sp*-open, (i,j)-θ-semi-open, (i,j)-θ-preopen, (i,j)-θ-*sp*-open) sets of (Y, σ_1, σ_2) is denoted by $(i,j)\delta\mathrm{SO}(Y)$ (resp. $(i,j)\delta \mathrm{PO}(Y)$, $(i,j)\delta\mathrm{SPO}(Y)$, $(i,j)\theta\mathrm{SO}(Y)$, $(i,j)\theta\mathrm{PO}(Y)$, $(i,j)\theta\mathrm{SPO}(Y)$).

Remark 10. Let (Y, σ_1, σ_2) be a bitopological space. The family $(i,j)\delta\mathrm{SO}(Y)$, $(i,j)\delta\mathrm{PO}(Y)$, $(i,j)\delta\mathrm{SPO}(Y)$, $(i,j)\theta\mathrm{SO}(Y)$, $(i,j)\theta\mathrm{PO}(Y)$ and $(i,j)\theta\mathrm{SPO}(Y)$ are all m-structures with property $\mathcal{B}$.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, we can define many new types of (i,j)-K-m-open multifunctions. For example, in case $m_{ij} = (i,j)\delta\mathrm{SO}(Y)$ (resp. $(i,j)\delta\mathrm{PO}(Y)$, $(i,j)\delta\mathrm{SPO}(Y)$, $(i,j)\theta\mathrm{SO}(Y)$, $(i,j)\theta\mathrm{PO}(Y)$, $(i,j)\theta\mathrm{SPO}(Y)$) we can define new types of (i,j)-K-m-open multifunctions as follows:

Definition 18. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* (i,j)-δ-semi-open (*resp.* (i,j)-δ-preopen, (i,j)-δ-sp-open) *if $F : (X, \tau_i) \to (Y, m_{ij})$ is (i,j)-K-m-open and $m_{ij} = (i,j)\delta\mathrm{SO}(Y)$ (resp. $(i,j)\delta\mathrm{PO}(Y)$, $(i,j)\delta\mathrm{SPO}(Y)$).*

Definition 19. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* (i,j)-θ-semi-open (*resp.* (i,j)-θ-preopen, (i,j)-θ-sp-open) *if $F : (X, \tau_i) \to (Y, m_{ij})$ is (i,j)-K-m-open and $m_{ij} = (i,j)\theta\mathrm{SO}(Y)$ (resp. $(i,j)\theta\mathrm{PO}(Y)$, $(i,j)\theta\mathrm{SPO}(Y)$).*

Definition 20. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be* (i,j)-semi-open (*resp.* (i,j)-α-open, (i,j)-semi-preopen) *if $F : (X, \tau_i) \to (Y, m_{ij})$ is (i,j)-K-m-open and $m_{ij} = (i,j)\mathrm{SO}(Y)$ (resp. $(i,j)\alpha(Y)$, $(i,j)\mathrm{SPO}(Y)$), equivalently if for each τ_i-open set U of X, $F(U)$ is (i,j)-semi-open (resp. (i,j)-α-open, (i,j)-semi-preopen) in Y.*

Conclusion. We can apply the characterizations established in Sections 5 to the multifunctions defined in Definitions 18, 19 and 20 and also to multifunctions defined by using any m-structure $m_{ij} = m(\sigma_1, \sigma_2)$ determined by σ_1 and σ_2 in a bitopological space (Y, σ_1, σ_2).

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