F A S C I C U L I M A T H E M A T I C I

Nr 41 2009

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ON SOME CLASSES OF DIFFERENCE DOUBLE SEQUENCE SPACES

Abstract. In this article we introduce the difference double sequence spaces ${}_2\ell_\infty(\Delta, q)$, ${}_2c(\Delta, q)$, ${}_2c_0(\Delta, q)$, ${}_2c^B(\Delta, q)$, ${}_2c_0^B(\Delta, q)$, ${}_2c^R(\Delta, q)$ and ${}_2c_0^R(\Delta, q)$ defined over a seminormed space (X, q), seminormed by q. We examine some topological and algebraic properties of these spaces like symmetricity, solidness, monotonocity, convergence free, nowhere densenes etc. We prove some inclusion results too.

Key words: difference sequence, completeness, solid space, symmetric space, convergence free.

AMS Mathematics Subject Classification: 40A05, 40B05, 40D05.

1. Introduction

Throughout w, c, c_0 and ℓ_∞ denote the classes of *all, convergent, null* and *bounded* scalar valued single sequences respectively.

Some initial works on double sequence spaces is found in Bromwich [2]. Later on the classes of double sequences were investigated by Hardy [3], Moricz [6], Moricz and Rhoades [7], Basarir and Solancan [1], Tripathy [9], Turkmenoglu [12] and many others.

Let (X, q) be a seminormed space, seminormed by q. Throughout the article ${}_2w(X)$, ${}_2\ell_\infty(X)$, ${}_2c(X)$, ${}_2c^R(X)$, ${}_2c^B(X)$, ${}_2c_0(X)$, ${}_2c_0^R(X)$, ${}_2c_0^B(X)$ denote the spaces of *all, bounded, convergent in Pringsheim's sense, regularly convergent, bounded convergent in Pringsheim's sense, null in Pringsheims sense, regularly null* and *bounded null in Pringsheim's sense* double sequence spaces respectively defined over (X, q). For $X = C$, the field of complex numbers these represent the corresponding scalar valued sequence spaces.

Throughout a double sequence will be denoted as $A = < a_{nk} >$ i.e. a double infinite array of elements a_{nk}, for $n, k \in N$.

A double sequence $< a_{nk} >$ is said to converge in *Pringsheim's sense* if

$$\lim_{n,k\to\infty} a_{nk} = L \quad \text{exists},$$

where n and k tend to infinity independent of each other.

The notion of regular convergence for double sequences was introduced by Hardy [3].

A double sequence $< a_{nk} >$ is said to *converge regularly* if it converges in the pringsheim's sense and the following limits hold:

$$\lim_{n\to\infty} a_{nk} = L_k, \quad \text{exist for each} \quad k \in N.$$

and

$$\lim_{k\to\infty} a_{nk} = M_n, \quad \text{exist for each} \quad n \in N.$$

When $L = L_k = M_n = \theta$, for all $n, k \in N$, we say that $< a_{nk} >$ is regularly null.

Hence the definition is equivalent to the following single statement:

$$\lim_{\max\{n,k\}\to\infty} a_{nk} = \theta.$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [5] as follows:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$.

The above spaces are Banach spaces normed by

$$||(x_k)|| = |x_1| + \sup_{k\geq 1} |\Delta x_k|$$

Later on the notion was further investigated by Tripathy [13] and many others.

2. Definitions and preliminaries

Definition 1. *A double sequence space E is said to be solid (or normal) if $< \alpha_{nk} a_{nk} > \in E$ whenever $< a_{nk} > \in E$ for all double sequences $< \alpha_{nk} >$ of scalars with $|\alpha_{nk}| \leq 1$ for all $n, k \in N$.*

Definition 2. *Let $K = \{(n_i, k_j) : i, j \in N;\ n_1 < n_2 < \ldots.$ and $k_1 < k_2 < \ldots.\} \subseteq N \times N$ and E be a double sequence space. A K-step space of E is a sequence space*

$$\lambda_K^E = \{< a_{n_i k_i} > \in {}_2w \ : < a_{nk} > \in E\}.$$

A canonical pre-image of a sequence $< a_{n_i k_i} > \in E$ is a sequence $< b_{nk} > \in E$ defined as follows:

$$b_{nk} = \begin{cases} a_{nk} & \text{if} \quad (n,k) \in K, \\ \theta & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

Definition 3. *A double sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Remark 1. From the above notions, it follows that "If a sequence space E solid then E is monotone".

Definition 4. *A double sequence space E is said to be symmetric if $< a_{nk} > \in E$ implies $< a_{\pi(n)\pi(k)} > \in E$, where π is a permutation of N.*

Definition 5. *A double sequence space E is said to be convergence free if $< b_{nk} > \in E$ whenever $< a_{nk} > \in E$ and $b_{nk} = \theta$, whenever $a_{nk} = \theta$, where θ is the zero element of X.*

We introduce the following difference double sequence spaces defined over the seminormed space (X, q).

$$Z(\Delta, q) = \{< a_{nk} > \in {}_2w(q) \, : < \Delta a_{nk} > \in Z(q)\},$$

where $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ and $\Delta a_{nk} = a_{nk} - a_{n+1,k} - a_{n,k+1} + a_{n+1,k+1}$ for all $n, k \in N$.

3. Main results

The proof of the following two results are routine works.

Theorem 1. *The classes $Z(\Delta, q)$ where $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ are linear spaces.*

Theorem 2. *The sequence spaces $Z(\Delta, q)$ where $Z = {}_2\ell_\infty$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ are seminormed spaces, seminormed by*

$$f(< a_{nk} >) = \sup_n q(a_{n1}) + \sup_k q(a_{1k}) + \sup_{n,k} q(\Delta a_{nk})$$

Remark 2. *Theorem 2 holds good if we consider the function:*

$$g(< a_{nk} >) = \sup_{n,k} q(\Delta a_{nk}),$$

instead of f.

Theorem 3. *Let (X, q) be a complete seminormed space, then $Z(\Delta, q)$ for $Z = {}_2\ell_\infty$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$, ${}_2c_0^B$ are complete.*

Proof. We establish the result for the space ${}_2\ell_\infty(\Delta, q)$.

Let $(A^i) = (< a^i_{nk} >)$ be a Cauchy sequence in ${}_2\ell_\infty(\Delta, q)$. Thus for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$f(A^i - A^j) < \varepsilon, \quad \text{for all} \quad i, j \geq n_0.$$

Then

(1) $\sup_n q(a^i_{n1} - a^j_{n1}) + \sup_k q(a^i_{1k} - a^j_{1k}) + \sup_{n,k} q(\Delta(< a^i_{nk} - a^j_{nk} >)) < \varepsilon,$

for all $i, j \geq n_0$.

$\Rightarrow < a^i_{n1} >$ is a Cauchy sequences in X, for each $n \in N$.

$\Rightarrow < a^i_{n1} >$ converges in X for each $n \in N$.

(2) Let $\lim_{i\to\infty} a^i_{n1} = a_{n1}$, for each $n \in N$.

(3) Similarly $\lim_{i\to\infty} a^i_{1k} = a_{1k}$, for each $k \in N$.

(4) and $\lim_{i\to\infty} \Delta a^i_{nk} = x_{nk}$, for all $n, k \in N$.

From (2), (3) and (4) we have $\lim_{i\to\infty} a^i_{nk} = a_{nk} \in X$, for each $n, k \in N$.

From (1) we have for all $i, j \geq n_0$,

$q(a^i_{n1} - a^j_{n1}) < \varepsilon$ for each $n \in N$; $\quad q(a^i_{1k} - a^j_{1k}) < \varepsilon$, for each $k \in N$, and $q(\Delta(< a^i_{nk} - a^j_{nk} >)) < \varepsilon$, for all $n, k \in N$.

$\Rightarrow \lim_{j\to\infty} q(a^i_{n1} - a^j_{n1}) < \varepsilon$ for each $n \in N$; $\quad \lim_{j\to\infty} q(a^i_{1k} - a^j_{1k}) < \varepsilon$, for each $k \in N$,

and $\lim_{j\to\infty} q(\Delta a^i_{nk} - \Delta a^j_{nk}) < \varepsilon$, for all $n, k \in N$.

$\Rightarrow q(a^i_{n1} - a_{n1}) < \varepsilon$ for each $n \in N$; $\;q(a^i_{1k} - a_{1k}) < \varepsilon$, for each $k \in N$, and $q(\Delta a^i_{nk} - \Delta a_{nk}) < \varepsilon$, for all $n, k \in N$.

Since the right hand side is free from n and k so we have for all $i \geq n_0$,

$\sup_n q(a^i_{n1} - a_{n1}) < \varepsilon$; $\;\sup_k q(a^i_{1k} - a_{1k}) < \varepsilon$ and $\sup_{n,k} q(\Delta a^i_{nk} - \Delta a_{nk}) < \varepsilon$

$$\Rightarrow f(A^i - A) = \sup_n q(a^i_{n1} - a_{n1}) + \sup_k q(a^i_{1k} - a_{1k}) + \sup_{n,k} q(\Delta a^i_{nk} - \Delta a_{nk}) < 3\varepsilon$$

Hence $A^i - A \in {}_2\ell_\infty(\Delta, q)$, for all $i \geq n_0$.

Since ${}_2\ell_\infty(\Delta, q)$ is linear space, so we have

$$A = A^i - (A^i - A) \in {}_2\ell_\infty(\Delta, q), \quad \text{for all} \quad i \geq n_0.$$

Hence ${}_2\ell_\infty(\Delta, q)$ is complete. ∎

Proposition 1. *The spaces $Z(\Delta, q)$ where $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c^R_0$, ${}_2c^B$, ${}_2c^B_0$ are not symmetric.*

Proof. The result follows from the following examples.

Example 1. Let $X = \ell_\infty$; $q(x) = \sup\limits_i |x^i|$, for $x = (x^i) \in \ell_\infty$ and consider the sequence $< a_{nk} >$ defined by

$$a_{nk} = \begin{cases} e, & \text{for } n = 1 \text{ and all } k \in N, \\ \theta, & \text{otherwise.} \end{cases}$$

Then $< a_{nk} > \in Z(\Delta, q)$ for $Z = {}_2c,\ {}_2c_0,\ {}_2c^R,\ {}_2c_0^R,\ {}_2c^B,\ {}_2c_0^B$. Consider the rearranged sequence $< b_{nk} >$ of $< a_{nk} >$ defined by

$$b_{nk} = \begin{cases} e, & \text{for } n = k, \\ \theta, & \text{otherwise.} \end{cases}$$

Then $< b_{nk} > \notin {}_2c(\Delta, q)$. Hence the spaces $Z(\Delta, q)$ for $Z = {}_2c,\ {}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ are not symmetric.

Example 2. Let $X = C$, the field of complex numbers, $q(x) = |x|$ and consider the sequence $< a_{nk} >$ defined by

$$a_{nk} = \begin{cases} n, & \text{for } \ k = 1 \text{ and all } n \in N, \\ 0, & \text{otherwise.} \end{cases}$$

Then $< a_{nk} > \in {}_2\ell_\infty(\Delta)$. Let $< b_{nk} >$ be a rearrangement of $< a_{nk} >$ defined by

$$b_{nk} = \begin{cases} i, & \text{for } n = k, \text{ and } n = i^2,\ i \in N, \\ \theta, & \text{otherwise.} \end{cases}$$

■

Then $< b_{nk} > \notin {}_2\ell_\infty(\Delta)$. Hence ${}_2\ell_\infty(\Delta)$ is not symmetric.

Proposition 2. *$Z(q) \subset Z_0(\Delta, q)$, for $Z = {}_2c$, ${}_2c^R$ and ${}_2c^B$ and the inclusions are strict.*

Proof. Let $< a_{nk} > \in {}_2c(q)$. Then for a given $\varepsilon > 0$ there exists $n_0,\ k_0 \in N$ such that

$$q(a_{nk} - L) < \frac{\varepsilon}{4} \quad \text{for all } n \geq n_0,\ k \geq k_0. \tag{5}$$

Hence for all $n \geq n_0$, $k \geq k_0$,

$$\begin{aligned} q(\Delta a_{nk}) &\leq q(a_{nk} - L) + q(a_{n+1,k} - L) + q(a_{n,k+1} - L) \\ &\quad + q(a_{n+1,k+1} - L) \\ &< \varepsilon, \text{ by (5).} \end{aligned}$$

Thus $< a_{nk} > \in {}_2c_0(\Delta, q)$. The other cases can be proved similarly. ■

The inclusions are strict follows from the following example:

Example 3. Let $X = C$, and consider the sequence $< a_{nk} >$ be defined by

$$a_{nk} = n + k - 1, \quad \text{for all} \quad n, k \in N.$$

Then $\Delta a_{nk} = 0$, for all $n, k \in N$. Hence $< a_{nk} > \in {}_2c_0^R(\Delta, q) \subset {}_2c_0(\Delta, q)$ but $< a_{nk} > \notin {}_2c(q)$.

Theorem 4. *The spaces $Z(\Delta, q)$ for $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ are not monotone and as such are not solid.*

Proof. The spaces are not monotone follows from the following examples. Since the spaces are not monotone, are not solid is clear from the Remark 1. ∎

Example 4. Let $X = C$, and consider the sequence $< a_{nk} >$ defined by

$$a_{nk} = 1, \quad \text{for all} \ \ n, k \in N.$$

Consider the sequence $< b_{nk} >$ in the pre-image space defined by

$$b_{nk} = \begin{cases} a_{nk}, & \text{for} \quad n = k = i^2, \ \ i \in N, \\ 0, & \text{otherwise.} \end{cases}$$

Then $< a_{nk} > \in Z(\Delta, q)$ for $Z = {}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ but $< b_{nk} > \notin {}_2c(\Delta)$. Hence $Z(\Delta)$ for $Z = {}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ are not monotone.

Example 5. For the space ${}_2\ell_\infty(\Delta, q)$, let $X = C$ and $q(x) = |x|$. Consider the sequence $< a_{nk} >$ defined by

$$a_{nk} = n + k, \quad \text{for all} \ \ n, k \in N.$$

Consider the sequence $< b_{nk} >$ in the pre-image space, defined as in Example 4. Then $< a_{nk} > \in {}_2\ell_\infty(\Delta, q)$, but $< b_{nk} > \notin {}_2\ell_\infty(\Delta, q)$. Thus ${}_2\ell_\infty(\Delta, q)$ is not monotone.

The proof of the following result is a routine verification.

Proposition 3. (*a*) *$Z(q) \subset Z(\Delta, q)$ where $Z = {}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$, and the inclusions are strict.*

(*b*) *$Z(\Delta, q) \subset {}_2\ell_\infty(\Delta, q)$ for $Z = {}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$, and the inclusions are strict.*

The following result follows from Proposition 3 and Theorem 3.

Proposition 4. *The spaces $Z(\Delta, q)$, for $Z = {}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ are nowhere dense subsets of ${}_2\ell_\infty(\Delta, q)$.*

Remark 3. If we consider a normed linear space $(X, ||.||)$ then the spaces $Z(X, ||.||)$ for $Z = {}_2\ell_\infty$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ will be normed linear spaces, normed by

$$||A||_\Delta = \sup_k ||a_{1k}|| + \sup_n ||a_{n1}|| + \sup_{n,k} ||\Delta a_{nk}||.$$

Remark 4. When $(X, ||.||)$ will be a Banach space, the spaces $Z(X, ||.||)$ for $Z = {}_2\ell_\infty$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ will be Banach spaces, normed by $||A||_\Delta$.

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Received on 19.03.2008 and, in revised form, on 11.06.2008.