F A S C I C U L I M A T H E M A T I C I

Nr 42 2009

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MORE ON ALMOST CONTRA λ-CONTINUOUS FUNCTIONS

Abstract. In 1996, Dontchev [14] introduced and investigated a new notion of non-continuity called contra-continuity. Recently, Baker et al. [6] offered a new generalization of contra-continuous functions via λ-closed sets, called almost contra λ-continuous functions. It is the objective of this paper to further study some more properties of such functions.

Key words: topological spaces, λ-open sets, λ-closed sets, almost contra λ-continuous functions, λR-closed graph.

AMS Mathematics Subject Classification: 54B05, 54C08.

1. Introduction and preliminaries

In 1986, Maki [25] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel(= saturated set), i.e. to the intersection of all open supersets of A. Arenas et al. [3] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. Quite recently, Caldas et al. ([7], [11]) introduced the notion of λ-closure of a set by utilizing the notion of λ-open sets defined in [3]. In [14], Dontchev introduced and studied a new notion of non-continuity called contra-continuity. It is the aim of this paper to continue our work ([6], [9], [8]) and present some more properties of almost contra λ-continuity which is a generalization of contra-continuity. Moreover, we present some of the basic properties and preservation theorems of almost contra λ-continuous functions. Furthermore, we investigate the relationships between almost contra λ-continuous functions and functions with λR-closed graph.

Throughout this paper, by (X,τ) and (Y,σ) (or X and Y) we always mean topological spaces. Let A be a subset of X. We denote the interior, the closure and the complement of a set A by $Int(A)$, $Cl(A)$ and $X\backslash A$ or A^c, respectively. A subset A of X is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A subset A of a space X is called preopen [24] (resp. semi-open [23], β-open [1](also called semipreopen [2]) if $A \subset Int(Cl(A))$ (resp. $A \subset Cl(Int(A))$, $A \subset$

$Cl(Int(Cl(A)))$). The complement of a preopen (resp. semi-open, β-open) set is said to be preclosed (resp. semi-closed, β-closed). The collection of all regular closed (resp. semi-open) subsets of X will be denoted by $RC(X)$(resp. $SO(X)$). We set $RC(X,x) = \{V \in RC(X) : x \in V\}$ (resp. $SO(X,x) = \{V \in SO(X) : x \in V\}$). A subset A of (X,τ) is called λ-*closed* [3] if $A = L \cap D$, where L is a Λ-set and D is a closed set. The complement of a λ-closed set is called λ-*open*. We denote the collection of all λ-open sets (resp. λ-closed sets) by $\lambda O(X,\tau)$ (resp. $\lambda C(X,\tau)$). We set $\lambda O(X,x) = \{U : x \in U \in \lambda O(X,\tau)\}$ and $\lambda C(X,x) = \{U : x \in U \in \lambda C(X,\tau)\}$. A point x in a topological space (X,τ) is called a λ-*cluster point* of A [7] if $A \cap U \neq \emptyset$ for every λ-open set U of X containing x. The set of all λ-cluster points is called the λ-*closure* of A and is denoted by $Cl_\lambda(A)$ ([3], [7]).
A point $x \in X$ is said to be a λ-interior point of A if there exists a λ-open set U containing x such that $U \subset A$. The set of all λ-interior points of A is said to be λ-interior of A and is denoted by $Int_\lambda(A)$.

Lemma 1 ([3], [7]). *Let A, B and A_i ($i \in I$) be subsets of a topological space (X,τ). The following properties hold:*

(1) *If A_i is λ-closed for each $i \in I$, then $\cap_{i\in I} A_i$ is λ-closed.*
(2) *If A_i is λ-open for each $i \in I$, then $\cup_{i\in I} A_i$ is λ-open.*
(3) *A is λ-closed if and only if $A = Cl_\lambda(A)$.*
(4) *A is λ-open if and only if $A = Int_\lambda(A)$.*
(5) *$Cl_\lambda(A) = \cap\{F \in \lambda C(X,\tau) : A \subset F\}$.*
(6) *$A \subset Cl_\lambda(A)$.*
(7) *If $A \subset B$, then $Cl_\lambda(A) \subset Cl_\lambda(B)$.*
(8) *$Cl_\lambda(A)$ is λ-closed.*

Definition 1. *A function $f : X \to Y$ is said to be:*

(1) *λ-continuous [3]. If $f^{-1}(V)$ is λ-closed for every closed set V in Y, equivalently if the inverse image of every open set V in Y is λ-open in X.*

(2) *almost λ-continuous [21] if $f^{-1}(V)$ is λ-closed in X for every regular closed set V in Y.*

(3) *almost contra pre-continuous ([16], [27]) if $f^{-1}(V)$ is preclosed in X for every regular open set V in Y.*

(4) *almost contra β-continuous [5] if $f^{-1}(V)$ is β-closed in X for every regular open set V in Y.*

(5) *almost contra λ-continuous if $f^{-1}(V)$ is λ-closed in X for each regular open set V of Y.*

Definition 2. *Let A be a subset of a space (X,τ). The set $\bigcap\{U \in RO(X) : A \subset U\}$ is called the r-kernel of A [17] and is denoted by $rker(A)$.*

Lemma 2 (Ekici [17]). *The following properties hold for the subsets A, B of a space X :*

(1) *$x \in rker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in RC(X, x)$.*
(2) *$A \subset rker(A)$ and $A = rker(A)$ if A is regular open in X.*
(3) *If $A \subset B$, then $rker(A) \subset rker(B)$.*

Theorem 1 ([6]). *Let $f : X \to Y$ be a function from a topological space X into a topological space Y. The following statements are equivalent:*
(1) *f is almost contra λ-continuous;*
(2) *The inverse image of each regular closed set in Y is λ-open in X;*
(3) *For each point x in X and each regular closed set V in Y containing $f(x)$, there is a λ-open set U in X containing x such that $f(U) \subset V$;*
(4) *For each point x in X and each semiopen set V in Y containing $f(x)$, there is a λ-open set U in X containing x such that $f(U) \subset Cl(V)$;*
(5) *$f(Cl_\lambda(A)) \subset rker(f(A))$ for every subset A of X;*
(6) *$Cl_\lambda(f^{-1}(B)) \subset f^{-1}(rker(B))$ for every subset B of Y.*

2. Some more properties

Recall that a topological space (X, τ) is said to be:
(i) λ-T_1 [10] if for any distinct pair of points x and y in X, there exist $U \in \lambda O(X)$ containing x but not y and $V \in \lambda O(X)$ containing y but not x.
(ii) λ-T_2 [10] if for any distinct pair of points x and y in X, there exist $U \in \lambda O(X, x)$ and $V \in \lambda O(X, y)$ such that $U \cap V = \emptyset$.
(iii) Weakly Hausdorff [30] (briefly weak-T_2) if every point of X is an intersection of regular closed sets of X.
(iv) s-Urysohn [4] if for each pair of distinct points x and y in X, there exist $U \in SO(X, x)$ and $V \in SO(X, x)$ such that $Cl(U) \cap Cl(V) = \emptyset$.

Remark 1. Observe that T_0, λ-T_1 and λ-T_2 are equivalent [18] and s-Urysohn $\Rightarrow$ weak-T_2 $\Rightarrow$ T_1 $\Rightarrow$ T_0.

Theorem 2. *If X is a topological space and for each pair of distinct points x_1 and x_2 in X, there exists a map f of X into a Urysohn topological space Y such that $f(x_1) \neq f(x_2)$ and f is almost contra λ-continuous at x_1 and x_2 , then X is T_0.*

Proof. Let x_1 and x_2 be any distinct points in X. Then by hypothesis, there is a Urysohn space Y and a function $f : X \to Y$ which satisfies the conditions of the theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since Y is Urysohn , there exist open sets U_{y_1} and U_{y_2} of y_1 and y_2, respectively, in Y such that $Cl(U_{y_1}) \cap Cl(U_{y_2}) = \emptyset$. Since f is almost contra λ-continuous at x_i, there exists a λ-open set W_{x_i} of x_i in X such that $f(W_{x_i}) \subset Cl(U_{y_i})$ for $i = 1, 2$. Hence we get $W_{x_1} \cap W_{x_2} = \emptyset$ since $Cl(U_{y_1}) \cap Cl(U_{y_2}) = \emptyset$. Hence X is λ-T_2 and therefore by Remark 1, X is T_0. ■

Corollary 1. *If f is an almost contra λ-continuous injection of a topological space X into a Urysohn space Y, then X is T_0.*

Proof. For each pair of distinct points x_1 and x_2 in X, f is an almost contra λ-continuous function of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$ since f is injective. Hence by Theorem 2, X is T_0. ■

Theorem 3. *If f is an almost contra λ-continuous injection of a topological space X into a weakly Hausdorff space Y, then X is T_0.*

Proof. Since Y is weakly Hausdorff and f is injective, for any distinct points x_1 and x_2 of X, there exist $V_1, V_2 \in RC(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \notin V_1$, $f(x_2) \in V_2$ and $f(x_1) \notin V_2$. Since f is almost contra λ-continuous, by Theorem 2 $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are λ-open sets and $x_1 \in f^{-1}(V_1)$, $x_2 \notin f^{-1}(V_1)$, $x_2 \in f^{-1}(V_2)$, $x_1 \notin f^{-1}(V_2)$. Then, there exists $U_1, U_2 \in \lambda O(X)$ such that $x_1 \in U_1 \subset f^{-1}(V_1)$, $x_2 \notin U_1$, $x_2 \in U_2 \subset f^{-1}(V_2)$ and $x_1 \notin U_2$. Thus X is T_0. ■

Corollary 2. *If f is an almost contra λ-continuous injection of a topological space X into a s-Urysohn space Y, then X is T_0.*

Recall that a topological space is called a λ-space [3] if the union of any two λ-closed sets is a λ-closed set. Observe that if $f, g : X \to Y$ are almost contra λ-continuous functions, X is a λ-space and Y is s-Urysohn, then it is obvious that $E = \{x \in X \mid f(x) = g(x)\}$ is λ-closed in X.

We say that the product space $X = X_1 \times ... \times X_n$ has Property P_Λ if A_i is a λ-open set in a topological space X_i, for $i = 1, 2, ...n$, then $A_1 \times ... \times A_n$ is also λ-open in the product space $X = X_1 \times ... \times X_n$.

Theorem 4. *Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be two functions, where*
(1) *$X = X_1 \times X_2$ has the Property P_Λ.*
(2) *Y is a Urysohn space.*
(3) *f_1 and f_2 are almost contra λ-continuous .*
Then $\{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}$ is λ-closed in the product space $X = X_1 \times X_2$.

Proof. Let A denote the set $\{(x_1, x_2) : f_1(x_1) = f_2(x_2)\}$. In order to show that A is λ-closed, we show that $(X_1 \times X_2)\backslash A$ is λ-open. Let $(x_1, x_2) \notin A$. Then $f_1(x_1) \neq f_2(x_2)$. Since Y is Urysohn, there exist open sets V_1 and V_2 of $f_1(x_1)$ and $f_2(x_2)$, respectively, such that $Cl(V_1) \cap Cl(V_2) = \emptyset$. Since f_i $(i = 1, 2)$ is almost contra λ-continuous and $Cl(V_i)$ is regular closed, then $f_i^{-1}(Cl(V_i))$ is a λ-open set containing x_i in X_i $(i = 1, 2)$. Hence by (1), $f_1^{-1}(Cl(V_1)) \times f_2^{-1}(Cl(V_2))$ is λ-open. Furthermore $(x_1, x_2) \in f_1^{-1}(Cl(V_1)) \times f_2^{-1}(Cl(V_2)) \subset (X_1 \times X_2)\backslash A$. It follows that $(X_1 \times X_2)\backslash A$ is λ-open. Thus A is λ-closed in the product space $X = X_1 \times X_2$. ■

Corollary 3. *Assume that the product space $X \times X$ has the Property P_Λ. If $f : X \to Y$ is almost contra λ-continuous and Y is a Urysohn space. Then $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is λ -closed in the product space $X \times X$.*

Recall that a topological space X is called a $T_{\frac{1}{2}}$-space ([15], [22]) if every singleton is open or closed.

Lemma 3. *Let (X, τ) be a $T_{\frac{1}{2}}$-space and $f : X \to Y$. If f is almost contra-β-continuous or almost contra-pre-continuous then f is almost contra-λ-continuous.*

Proof. It follows directly from Theorem 2.6 of [3]. ■

Remark 2. Observe that a topological space (X, τ) in which every two non-void λ-closed subsets of (X, τ) intersect is indiscrete. It is obvious that if a topological space X is indiscrete and $f : X \to Y$ is a surjective almost contra λ-continuous function, then Y is hyperconnected. Recall that a topological space is hyperconnected if every open set is dense. To see this, suppose that Y is not hyperconnected. This implies that there exists an open set V such that $Cl(V) \neq Y$. Thus, there exist disjoint regular open sets D and E in Y, i.e, $Int(Cl(V))$ and $Y \backslash Cl(V)$. Since f is a surjective almost contra λ-continuous function, we have $A = f^{-1}(D)$ and $B = f^{-1}(E)$ such that A and B are disjoint non-empty λ-closed subsets of X. By hypothesis, X is indiscrete and this implies that $A \cap B \neq \emptyset$. But this is a contradiction. Hence Y is hyperconnected.

Theorem 5. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then f is almost contra λ-continuous if g is almost contra λ-continuous.*

Proof. Let $x \in X$ and V be a regular open subset of Y containing $f(x)$. Then we have $X \times V$ is a regular open. Since g is almost contra λ-continuous, $g^{-1}(X \times V) = f^{-1}(V)$ is λ-closed. Hence f is almost contra λ-continuous. ■

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 3. *A function $f : X \to Y$ has a λ-closed graph if for each $(x, y) \in (X \times Y) - G(f)$, there exists $U \in \lambda O(X, x)$ and an open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 4. *The graph, $G(f)$ of a function $f : X \to Y$ is λ-closed if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exists $U \in \lambda O(X, x)$ and an open set V of Y containing y such that $f(U) \cap V = \emptyset$.*

Theorem 6. *If $f : X \to Y$ is a function with λ-closed graph, then for each $x \in X$, $f(x) = \cap\{Cl(f(U)) : U \in \lambda O(X, x)\}$.*

Proof. Suppose the theorem is false. Then there exists a $y \neq f(x)$ such that $y \in \cap\{Cl(f(U)) : U \in \lambda O(X, x)\}$. This implies that $y \in Cl(f(U))$, for every $U \in \lambda O(X, x)$. So $V \cap f(U) \neq \emptyset$, for every $V \in O(Y, y)$ which contradicts the hypothesis that f is a function with λ-closed graph. Hence the theorem. ■

Theorem 7. *If $f : X \to Y$ is almost contra λ-continuous and Y is Haudorff, then $G(f)$ is λ-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exists disjoint open sets V and W of Y such that $y \in V$ and $f(x) \in W$. Then $f(x) \notin Y - Cl(W)$. Since $Y - Cl(W)$ is a regular open set containg V, it follows that $f(x) \notin \mathrm{rKer}(V)$ and hence $x \notin f^{-1}(\mathrm{rKer}(V))$. Then by Theorem 1(6) $x \notin \mathrm{Cl}_\lambda(f^{-1}(V))$. Therefore we have $(x, y) \in (X - \mathrm{Cl}_\lambda(f^{-1}(V))) \times V \subset (X \times Y) - G(f)$, which proves that $G(f)$ is λ-closed. ■

Theorem 8. *Let $f : X \to Y$ have a λ-closed graph.*
(1) *If f is injective, then X is T_0.*
(2) *If f is surjective, then Y is T_1.*

Proof. (1) Let x_1 and x_2 be two points in X. Then $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since f has a λ-closed graph, there exist $U \in \lambda O(X, x_1)$ and an open set V of Y containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Then $U \cap f^{-1}(V) = \emptyset$. Since $x_2 \in f^{-1}(V)$, $x_2 \notin U$. Therefore U is a λ-open set containing x_1 but not x_2, which proves that X is λ-T_1 and hence by Remark 1 that X is T_0.

(2) Let y_1 and y_2 be two points in Y. Since Y is surjective, there exists $x \in X$ such that $f(x) = y_1$. Then $(x, y_2) \in (X \times Y) - G(f)$. Since f has a λ-closed graph, there exist $U \in \lambda O(X, x)$ and an open set V of Y containing y_2 such that $f(U) \cap V = \emptyset$. Since $y_1 = f(x)$ and $x \in U$, $y_1 \in f(U)$. Therefore $y_1 \notin V$, which proves that Y is T_1. ■

It is clear that if $f : X \to Y$ has a λ-closed graph and X is a λ-space , then $f^{-1}(K)$ is λ-closed for every compact subset K of Y.

3. λR-closed graphs

Definition 4. *A function $f : X \to Y$ has a λR-closed graph if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in \lambda O(X, x)$ and $V \in RC(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.*

Remark 3. The above definition is equivalent with the statement that a function $f : X \to Y$ has a λR-closed graph if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \lambda O(X, x)$ and $V \in SO(Y, y)$ such that $(U \times Cl(V)) \cap G(f) = \emptyset$.

Lemma 5. *A graph $G(f)$ of a function $f : X \to Y$ is λR-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \lambda O(X)$ containing x and $V \in RC(Y)$ containing y such that $f(U) \cap V = \emptyset$.*

Remark 4. Observe that a graph $G(f)$ of a function $f : X \to Y$ is λR-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in \lambda O(X)$ containing x and $V \in SO(Y)$ containing y such that $f(U) \cap Cl(V) = \emptyset$.

Theorem 9. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is λ-continuous;*
(2) *for each $x \in X$ and each $V \in O(Y, f(x))$, there exists $U \in \lambda O(X, x)$ such that $f(U) \subset V$.*

Proof. Straightforward. ■

Remark 5. Examples 3.4 and 3.5 in [6] show that λ-continuity and almost contra λ-continuity are, in general, independent.

Theorem 10. *If $f : X \to Y$ is λ-continuous and Y is Hausdorff, then $G(f)$ is λR-closed.*

Proof. Let $(x, y) \in X \times Y\backslash G(f)$. Since Y is Hausdorff, then there exists a set $V \in O(Y, y)$ such that $f(x) \notin Cl(V)$. Now $Y\backslash Cl(V) \in O(Y, f(x))$. Therefore, by the λ-continuity of f there exists $U \in \lambda O(X, x)$ such that $f(U) \subset Y\backslash Cl(V)$. Consequently, $f(U) \cap Cl(V) = \emptyset$ where $Cl(V)$ is a regular closed set since V is open. By Lemma 5, $G(f)$ is λR-closed. ■

Theorem 11. *Let $f : X \to Y$ has a λ R-closed graph.*
(1) *If f is injective, then X is T_0.*
(2) *If f is surjective, then Y is weakly-T_2.*

Proof. (1) Suppose that x and y are any two distinct points of X. We have $(x, f(y)) \in X \times Y\backslash G(f)$. Since f has a λR-closed graph, then there exist a λ-open neighborhood U of x and a regular closed set F of Y containing $f(y)$ such that $f(U) \cap F = \emptyset$. Hence $U \cap f^{-1}(F) = \emptyset$. This means that we have $y \notin U$. Thus X is T_0.

(2) Let y_1 and y_2 be any distinct points of Y. Since f is surjective, then $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in X \times Y\backslash G(f)$. Since f has a λR-closed graph, then there exist a λ-open neighborhood U of x and a regular closed set F of Y containing y_2 such that $f(U) \cap F = \emptyset$. This means that $y_1 \notin F$. It follows that Y is weakly-T_2. ■

Theorem 12. *If $f : X \to Y$ is almost contra λ-continuous and Y is Urysohn, then $G(f)$ is λR-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$, then $y \neq f(x)$. Since Y is Urysohn there exist open sets V and W in Y such that $y \in V$, $f(x) \in W$ with $Cl(V) \cap Cl(W) = \emptyset$. Since f is almost contra λ-continuous, by Theorem 1(3) and since $Cl(W)$ is regular closed containing $f(x)$ there exists $U \in \lambda O(X, x)$ such that $f(U) \subset Cl(W)$. Therefore, we obtain $f(U) \cap Cl(V) = \emptyset$. By definition $G(f)$ is λR-closed in $X \times Y$. ∎

Theorem 13. *If $f : X \to Y$ is almost contra λ-continuous and Y is s-Urysohn, then $G(f)$ is λR-closed in $X \times Y$.*

Definition 5. *A subset A of a space X is said to be S-closed relative to X [26] if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of A by semi-open sets of X, there exists a finite subset ∇_0 of ∇ such that $A \subset \bigcup\{Cl(V_\alpha) \mid \alpha \in \nabla_0\}$. A space X is said to be S-closed [32] if X is S-closed relative to X.*

It should be noted that if a function $f : X \to Y$ has a λR-closed graph and X is λ-space, then $f^{-1}(K)$ is λ-closed in X for every subset K which is S-closed relative to Y.

Definition 6. *A topological space X is said to be:*

(1) *strongly λS-closed if every λ-closed cover of X has a finite subcover. (resp. $A \subset X$ is strongly λS-closed if the subspace A is strongly λS-closed).*

(2) *nearly-compact [28] if every regular open cover of X has a finite subcover.*

Theorem 14. *If $f : X \to Y$ is an almost contra λ-continuous surjection and X is strongly λS-closed, then Y is nearly compact.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be a regular open cover of Y. Since f is almost contra λ-continuous, we have that $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of X by λ-closed sets. Since X is strongly λS-closed, there exists a finite subset I_0 of I such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective $Y = \bigcup\{V_\alpha : \alpha \in I_0\}$ and therefore Y is nearly compact. ∎

Definition 7. *A topological space X is said to be almost-regular [29] if for each regular closed set F of X and each point $x \in X\backslash F$, there exist disjoint open sets U and V such that $F \subset V$ and $x \in U$.*

Theorem 15. *If a function $f : X \to Y$ is almost contra λ-continuous and Y is almost-regular, then f is almost λ-continuous.*

Proof. Let x be an arbitrary point of X and V an open set of Y containing $f(x)$. Since Y is almost-regular, by Theorem 3.2 of [29] there exists a regular open set W in Y containing $f(x)$ such that $Cl(W) \subset Int(Cl(V))$. Since f is almost contra λ-continuous, and $Cl(W)$ is regular closed in Y, by Theorem 1(3) there exists $U \in \lambda O(X,x)$ such that $f(U) \subset Cl(W)$. Then $f(U) \subset Cl(W) \subset Int(Cl(V))$. Hence, f is almost λ-continuous. ■

Recall that Caldas et al. [7] introduced the notion of λ-frontier of A, denoted by $Fr_\lambda(A)$, as $Fr_\lambda(A) = Cl_\lambda(A) \backslash Int_\lambda(A)$, equivalently $Fr_\lambda(A) = Cl_\lambda(A) \cap Cl_\lambda(X \backslash A)$.

Theorem 16. *The set of points $x \in X$ at which $f : (X,\tau) \to (Y,\sigma)$ is not almost contra λ-continuous is identical with the union of the λ-frontiers of the inverse images of regular closed sets of Y containing $f(x)$.*

Proof. Necessity. Suppose that f is not almost contra λ-continuous at a point x of X. Then there exists a regular closed set $F \subset Y$ containing $f(x)$ such that $f(U)$ is not a subset of F for every $U \in \lambda O(X,x)$. Hence we have $U \cap (X \backslash f^{-1}(F)) \neq \emptyset$ for every $U \in \lambda O(X,x)$. It follows that $x \in Cl_\lambda(X \backslash f^{-1}(F))$. We also have $x \in f^{-1}(F) \subset Cl_\lambda(f^{-1}(F))$. This means that $x \in Fr_\lambda(f^{-1}(F))$.

Sufficiency. Suppose that $x \in Fr_\lambda(f^{-1}(F))$ for some $F \in RC(Y, f(x))$ Now, we assume that f is almost contra λ-continuous at $x \in X$. Then there exists $U \in \lambda O(X,x)$ such that $f(U) \subset F$. Therefore, we have $x \in U \subset f^{-1}(F)$ and hence $x \in Int_\lambda(f^{-1}(F)) \subset X \backslash Fr_\lambda(f^{-1}(F))$. This is a contradiction. This means that f is not almost contra λ-continuous. ■

Definition 8. *A space (X,τ) is called λ-compact ([7], [8]) (also called λO-compact [19]) if every cover of X by λ-open sets has a finite subcover.*

Definition 9. *A space X is said to be*

(1) *S-Lindelöf [12] if every cover of X by regular closed sets has a countable subcover,*

(2) *countably S-closed [1] if every countable cover of X by regular closed sets has a finite subcover,*

(3) *mildly compact [31] if every clopen cover of X has a finite subcover.*

Theorem 17. *Let $f : (X,\tau) \to (Y,\sigma)$ be an almost contra λ-continuous surjection.*

(1) *If X is λO-compact, then Y is S-closed.*

(2) *If X is S-Lindelöf, then Y is S-Lindelöf.*

(3) *If X is countably λO-compact, then Y is countably S-closed.*

Proof. We prove only (1) since the proofs of (2) and (3) are analogous.

Suppose that $\{V_\alpha \mid \alpha \in \nabla\}$ be any regular closed cover of Y. Since f is almost contra λ-continuous, then $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a λ-open cover of X. Thus, there exists a finite subset ∇_0 of ∇ such that $X = \bigcup\{f^{-1}(V_\alpha) \mid \alpha \in$

$\nabla_0\}$. We have $Y = \bigcup\{V_\alpha \mid \alpha \in \nabla_0\}$ and this shows that Y is S-closed [[20], Theorem 3.2]. ■

Acknowledgement. S.P. Moshokoa acknowledges the support by the South African National Research Foundation under Grant number 2053847.

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Received on 11.06.2008 and, in revised form, on 14.07.2009.