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OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR DIFFERENCE EQUATIONS

Abstract. We shall establish some new criteria for the oscillation of third order nonlinear difference equations of the form

$$\Delta^2\left(a(n)(\Delta(x(n))^{\alpha}\right) + q(n)f(x[g(n)]) = 0$$

and

$$\Delta^2\left(a(n)(\Delta x(n))^{\alpha}\right) = q(n)f(x[g(n)]) + p(n)h(x[\sigma(n)])$$

when $\sum^{\infty} a^{-1/\alpha}(n) < \infty$.

Key words: oscillation, nonoscillation, comparison, first and second order.

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1. Introduction

This paper is concerned with the oscillatory behavior of the third order nonlinear difference equations

$$\Delta^2\left(a(n)(\Delta(x(n))^{\alpha}\right) + q(n)f(x[g(n)]) \;=\; 0 \tag{1}$$

and

$$\Delta^2\left(a(n)(\Delta x(n))^{\alpha}\right) = q(n)f(x[g(n)]) + p(n)h(x[\sigma(n)]), \tag{2}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}$, n_0 is a nonnegative integer, and Δ is the forward difference operator, $\Delta x(n) = x(n+1) - x(n)$ and $\{a(n)\}$, $\{g(n)\}$, $\{p(n)\}$, $\{q(n)\}$ and $\{\sigma(n)\}$ are sequences of real numbers.

The following conditions are always assumed to hold:

(*i*) α is the ratio of two positive odd integers,

(*ii*) $a(n) > 0$ for $n \in \mathbb{N}(n_0)$ and

$$\sum_{k=n_0}^{\infty} a^{-1/\alpha}(k) < \infty, \tag{3}$$

(iii) $p(n)$ and $q(n) \geq 0$ for $n \in \mathbb{N}(n_0)$,

(iv) $g,\ \sigma : \mathbb{N}(n_0) \to \mathbf{Z}$ is such that $g(n) < n,\ \sigma(n) > n,\ \Delta g(n) \geq 0$ and $\Delta\sigma(n) \geq 0$ for $n \in \mathbb{N}(n_0)$ and $\lim_{n\to\infty} g(n) = \infty$,

(v) $f,\ h \in C^1(\mathbb{R}, \mathbb{R}),\ xf(x) > 0,\ xh(x) > 0,\ f'(x) \geq 0$ and $h'(x) \geq 0$ for $x \neq 0$,

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0, \tag{4}$$

and

$$-h(-xy) \geq h(xy) \geq h(x)h(y) \quad \text{for} \quad xy > 0. \tag{5}$$

By a solution of equation (1)-(2) we mean a real sequence $\{x(n)\}$ defined on $\mathbb{N}(n_0)$, which satisfies equation (1)-(2) A nontrivial solution of equation (1)-(2) is said to be nonoscillatory if it is either eventually positive or eventually negative and it is oscillatory otherwise. Equation (1)-(2) is said to be oscillatory if all its solutions are oscillatory.

The problem of determining the nonoscillation and oscillation of all solutions of difference equations has been a very active area of research in the last three decades. In the second order case oscillation theories for differential and difference equations are well established, see [1-4] and for some higher order cases we refer the reader to [3,5-12]. It seems not much is known regarding the oscillation of equations (1) and (2) particularly, when condition (3) holds. Therefore, the purpose of this paper is to establish some new criteria for the oscillation of all solutions of equations (1) and (2). We note that the obtained results include the previous results for equations (1) and (2) when $\sum^{\infty} a^{-1/\alpha}(s) = \infty$. Also, the results of this paper not only extend the known results, but also improve and unify these criteria.

2. Oscillation criteria for equation (1)

In this section we shall investigate the oscillatory behavior of all solutions of equation (1). In what follows for $n,\ n_1 \in \mathbb{N}(n_0)$, we let

$$A[n, n_1] = \sum_{k=n_1}^{n-1} \left(\frac{k}{a(k)}\right)^{1/\alpha} \quad \text{and} \quad A(n) = \sum_{k=n}^{\infty} a^{-1/\alpha}(k).$$

Theorem 1. *Let conditions (i) – (v), (1.3) and (1.4) hold, and assume that there exists a nondecreasing sequence* $\{\eta(n)\}$,

$$\eta : \mathbb{N}(n_0) \to \mathbf{Z} \quad \text{such that} \quad g(n) < \eta(n) < n \quad \text{for} \quad n \in \mathbb{N}(n_0). \tag{6}$$

If both first order difference equations

$$\Delta z(n) + cq(n)f(A[g(n), n_1])f\left(z^{1/\alpha}[g(n)]\right) = 0 \tag{7}$$

for any constant c, $0 < c < 1$ and all $n_1 \geq n_0$, and

$$\Delta w(n) + q(n) f\left(A\left(g\left(n\right)\right)\right) f\left(\left[\eta(n) - g(n)\right]^{1/\alpha}\right) f\left(w^{1/\alpha}[\eta(n)]\right) = 0 \tag{8}$$

are oscillatory, and

$$\sum_{k=n_1}^{\infty} \left(\frac{1}{a(k)} \sum_{u=n_1}^{k-1} \sum_{v=n_1}^{u-1} q(v) f(A[g(v)]) \right)^{1/\alpha} = \infty, \tag{9}$$

for $n_1 \geq n_0$, then equation (1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1), say, $x(n) > 0$ and $x[g(n)] > 0$ for $n \geq n_0 \geq 0$. From equation (1), we have $\Delta^2\left(a(n)(\Delta x(n))^{\alpha}\right) \leq 0$ for $n \geq n_0$. Thus, we conclude that $\Delta\left(a(n)(\Delta x(n))^{\alpha}\right)$ and $\Delta x(n)$ are of one sign for $n \geq n_1 \geq n_0$. Now, there are four possibilities to consider:

(*I*) $\Delta\left(a(n)(\Delta x(n))^{\alpha}\right) > 0$ and $\Delta x(n) > 0$ for $n \geq n_1$,
(*II*) $\Delta\left(a(n)(\Delta x(n))^{\alpha}\right) > 0$ and $\Delta x(n) < 0$ for $n \geq n_1$,
(*III*) $\Delta\left(a(n)(\Delta x(n))^{\alpha}\right) < 0$ and $\Delta x(n) < 0$ for $n \geq n_1$,
(*IV*) $\Delta\left(a(n)(\Delta x(n))^{\alpha}\right) < 0$ and $\Delta x(n) > 0$ for $n \geq n_1$.

The Case (*IV*) cannot hold. In fact, if we let $y(n) = a(n)(\Delta x(n))^{\alpha}$, then we find that $\Delta^2 y(n) < 0$ and $\Delta y(n) < 0$ for $n \geq n_1$ and hence $\lim_{n\to\infty} y(n) = -\infty$, which contradicts the positivity of $y(n)$.

Case (*I*). There exist an $n_2 \in \mathbb{N}(n_0)$, $n_2 \geq n_1$ and a constant b, $0 < b < 1$ such that

$$y(n) \;\geq\; bn\Delta y(n) \quad \text{for} \quad n \geq n_2,$$

or

$$\Delta x(n) \;\geq\; b^{1/\alpha} \left(\frac{n}{a(n)}\right)^{1/\alpha} (\Delta y(n))^{1/\alpha} \quad \text{for} \quad n \geq n_2, \tag{10}$$

where $y(n) = a(n)(\Delta x(n))^{\alpha}$, $n \geq n_2$.

Summing (10) from n_2 to $n-1$, we get

$$x(n) \geq b^{1/\alpha} A[n, n_2] z^{1/\alpha}(n) \quad \text{for} \quad n \geq n_2,$$

where $z(n) = \Delta y(n)$, $n \geq n_2$.

There exists an $n_3 \in \mathbb{N}(n_0)$, $n_3 \geq n_2$ such that

$$x[g(n)] \geq b^{1/\alpha} A[g(n), n_2] z^{1/\alpha}[g(n)] \quad \text{for} \quad n \geq n_3. \tag{11}$$

Using (4) and (11) in equation (1), we find

$$\begin{aligned} -\Delta z(n) &= q(n) f(x[g(n)]) \\ &\geq f(b^{1/\alpha}) q(n) f(A[g(n), n_2]) f\left(z^{1/\alpha}[g(n)]\right) \quad \text{for} \quad n \geq n_3. \end{aligned} \tag{12}$$

Summing (12) from n to $u \geq n \geq n_3$ and letting $u \to \infty$, we have

$$z(n) \geq f(b^{1/\alpha}) \sum_{k=n}^{\infty} q(k) f(A[g(k), n_2]) f\left(z^{1/\alpha}[g(k)]\right).$$

The sequence $\{z(n)\}$ is strictly decreasing for $n \geq n_3$. Hence, by the analog of Theorem 1 in [14] (also, see [1]), we conclude that there exists a positive solution $\{z(n)\}$ of equation (2.2) with $\lim_{n\to\infty} z(n) = 0$, which is a contradiction.

Case (II). For $n \geq s \geq n_1$, we obtain

$$a(s)\,(-\Delta x(s))^{\alpha} \geq a(n)\,(-\Delta x(n))^{\alpha}$$

or

$$x(n) \geq [a(n)\,(-\Delta x(n))^{\alpha}]^{1/\alpha} \sum_{k=n}^{\infty} a^{-1/\alpha}(k).$$

Replacing n by $g(n)$, we find

$$\begin{aligned}(13)\quad x[g(n)] &\geq [a(g(n))\,(-\Delta x(g(n)))^{\alpha}]^{1/\alpha} A(g(n)) \quad \text{for} \quad n \geq n_2 \geq n_1 \\ &=: y^{1/\alpha}[g(n)] A(g(n)) \quad \text{for} \quad n \geq n_2,\end{aligned}$$

where $y(n) = -a(n)(\Delta x(n))^{\alpha}$ for $n \geq n_2$.

Using (4) and (13) in equation (1), we have

$$(14) \qquad \Delta^2 y(n) \geq q(n) f(A(g(n))) f\left(y^{1/\alpha}[g(n)]\right) \quad \text{for} \quad n \geq n_2.$$

Clearly, $y(n) > 0$ and $\Delta y(n) < 0$ for $n \geq n_2$.

For $n \geq s \geq n_2$, we have

$$y(s) \geq (n-s)(-\Delta y(n)).$$

Replacing s and n by $g(n)$ and $\eta(n)$ respectively, we obtain

$$(15) \qquad y[g(n)] \geq (\eta(n) - g(n)) w[\eta(n)] \quad \text{for} \quad n \geq n_3 \geq n_2,$$

where $w(n) = -\Delta y(n)$, $n \geq n_3$.

Using (15) and (4) in (14), we have

$$\Delta w(n) + q(n) f(A(g(n))) f\left([\eta(n) - g(n)]^{1/\alpha}\right) f\left(w^{1/\alpha}[\eta(n)]\right) \leq 0 \text{ for } n \geq n_3.$$

The rest of the proof is similar to that of Case (I) above and hence omitted.

Case (III). For $s \geq n \geq n_1$, we have

$$a(s)(-\Delta x(s))^{\alpha} \geq a(n)(-\Delta x(n))^{\alpha},$$

or

$$-\Delta x(s) \;\geq\; -\,(a^{1/\alpha}(n)\Delta x(n))(a^{-1/\alpha}(s)). \tag{16}$$

Summing (16) from n to $u \geq n$ and letting $u \to \infty$, we get

$$\begin{aligned} x(n) \;&\geq\; -(a^{1/\alpha}(n)\Delta x(n)) \sum_{k=n}^{\infty} a^{-1/\alpha}(k) \\ &=\; -(a^{1/\alpha}(n)\Delta x(n))A\,(n)\,. \end{aligned} \tag{17}$$

Clearly,

$$a^{1/\alpha}(n)(-\Delta x(n)) \geq a^{1/\alpha}(n_1)(-\Delta x(n_1)) = \bar{b} > 0, \tag{18}$$

where $\bar{b}$ is a constant. Combining (17) and (18), we find

$$x(n) \geq \bar{b}A(n) \quad \text{for} \quad n \geq n_1.$$

There exists an $n_2 \geq n_1$ such that

$$x[g(n)] \;\geq\; \bar{b}A(g(n)) \quad \text{for} \quad n \geq n_2. \tag{19}$$

Using (4) and (19), we obtain

$$-\Delta^2\left(a(n)(\Delta x(n))^{\alpha}\right) = q(n)f(x[g(n)]) \geq f(\bar{b})q(n)f(A(g(n))). \tag{20}$$

Summing (20) twice from n_2 to $n-1$ one can easily find

$$-a(n)(\Delta x(n))^{\alpha} \geq f(\bar{b}) \sum_{u=n_2}^{n-1} \sum_{k=n_2}^{u-1} q(k)f(A(g(k))),$$

or

$$-\Delta x(n) \geq (f(\bar{b}))^{1/\alpha} \left(\frac{1}{a(n)} \sum_{u=n_2}^{n-1} \sum_{k=n_2}^{u-1} q(k)f(A(g(k))) \right)^{1/\alpha}. \tag{21}$$

Summing (21) from n_2 to $n-1$, we obtain

$$\begin{aligned} &\infty \,>\, x(n_2) \geq x(n_2) - x(n) \\ &\quad\geq\; (f(\bar{b}))^{1/\alpha} \sum_{\ell=n_2}^{n-1} \left(\frac{1}{a(\ell)} \sum_{u=n_2}^{\ell-1} \sum_{k=n_2}^{u-1} q(k)f(A(g(k))) \right)^{1/\alpha} \to \infty \quad \text{as} \quad n \to \infty, \end{aligned}$$

which is a contradiction. This completes the proof. ■

We can combine equations (7) and (8) in one by letting

$$Q(n) = \min\{cq(n)f(A[g(n),n_1]), \quad q(n)f(A(g(n)))f\left([\eta(n)-g(n)]^{1/\alpha}\right)\}, \tag{22}$$

for any constant c, $0<c<1$ and all $n \geq n_1$.

In this case, one can easily replace equations (7) and (8) by the equation

$$\Delta y(n) + Q(n)f\left(y^{1/\alpha}[\eta(n)]\right) = 0. \tag{23}$$

From the proof of Theorem 1 it is clear that if the condition

$$\sum_{k=n_0}^{\infty} a^{-1/\alpha}(k) = \infty \tag{24}$$

holds, then Cases (I) and (II) hold and Cases (III) and (IV) are disregarded. Thus, we have the following result.

Theorem 2. *Let conditions* (i) – (v)*, (4) and (24) hold and assume that there exists a nondecreasing sequence* $\{\eta(n)\}$*,* $\eta : I\!N(n_0) \to \mathbf{Z}$ *such that (6) holds. If equation (23) is oscillatory, then equation (1) is oscillatory.*

The following result is immediate.

Corollary 1. *Let conditions* (i)–(v)*, (3) and (4) hold, and assume that there exists a nondecreasing sequence* $\{\eta(n)\}$ *such that condition (6) holds. Equation (1) is oscillatory if one of the following conditions holds:*

(I_1) $\dfrac{f(u^{1/\alpha})}{u} \geq k_1$ *for* $u \neq 0$ *and some* $k_1 > 0$

$$\limsup_{n\to\infty} \sum_{k=\eta(n)}^{n-1} Q(k) > \frac{1}{k_1},$$

(I_2) $\displaystyle\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$*, and*

$$\sum_{k=n_0}^{\infty} Q(k) = \infty.$$

3. Oscillation criteria for equation (2)

The main goal of this section is to establish criteria for the oscillation of equation (2) of mixed nonlinearities and arguments.

Theorem 3. *Let conditions (i)–(v) and (3) – (5) hold and assume that there exist nondecreasing sequences* $\{\eta(n)\}$, $\{\rho(n)\}$ *and* $\{\theta(n)\}$,

$$(25)\qquad \eta, \rho, \theta : \mathbb{N}(n_0) \to \mathbf{Z} \quad \text{such that} \quad g(n) < \eta(n) < n-1 \quad \text{and} \quad \sigma(n) > \rho(n) > \theta(n) > n \quad \text{for} \quad n \in \mathbb{N}(n_0).$$

If the difference equations

$$(26)\qquad \Delta y(n) - p(n) h\left(\sum_{k=\rho(n)}^{\sigma(n)-1} a^{-1/\alpha}(k)\right) \times h\left([\rho(n)-\theta(n)]^{1/\alpha}\right) h\left(y^{1/\alpha}[\theta(n)]\right) = 0$$

$$(27)\qquad \Delta z(n) + q(n) f\left(\sum_{k=n_0}^{g(n)-1} a^{-1/\alpha}(k)\right) \times f\left([\eta(n)-g(n)]^{1/\alpha}\right) f\left(z^{1/\alpha}[\eta(n)]\right) = 0$$

and

$$(28)\qquad \Delta^2 w(n) + q(n) f(A(g(n))) f(w[g(n)]) = 0$$

are oscillatory, where $A(n)$ *is as in Section 2, then equation (2) is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (2), say, $x(n) > 0$, $x[g(n)] > 0$ and $x[\sigma(n)] > 0$ for $n \geq n_0 \geq 0$. Since $\Delta^2 (a(n)(\Delta x(n))^\alpha) \geq 0$ for $n \geq n_0$, there exists an $n_1 \in \mathbb{N}(n_0)$ such that $\Delta (a(n)(\Delta x(n))^\alpha)$ and $\Delta x(n)$ are of one sign for $n \geq n_1$. Now, as in Theorem 1 there are four possibilities to consider:

(*I*) $\Delta (a(n)(\Delta x(n))^\alpha) > 0$ and $\Delta x(n) > 0$ for $n \geq n_1$,

(*II*) $\Delta (a(n)(\Delta x(n))^\alpha) < 0$ and $\Delta x(n) > 0$ for $n \geq n_1$,

(*III*) $\Delta (a(n)(\Delta x(n))^\alpha) < 0$ and $\Delta x(n) < 0$ for $n \geq n_1$,

(*IV*) $\Delta (a(n)(\Delta x(n))^\alpha) > 0$ and $\Delta x(n) < 0$ for $n \geq n_1$.

The Case (*IV*) cannot hold. In fact, if we let $y(n) = a(n)(\Delta x(n))^\alpha$ for $n \geq n_1$, then $\Delta^2 y(n) > 0$ and $\Delta y(n) > 0$ for $n \geq n_1$ and hence $\lim_{n\to\infty} y(n) = \infty$, which contradicts the negativity of $\Delta x(n)$. Now, we consider:

Case (*I*). For $n \geq s \geq n_1$, we have

$$a(n)(\Delta x(n))^\alpha \geq a(s)(\Delta x(s))^\alpha,$$

or

$$x(n) \geq [a(s)(\Delta x(s))^{\alpha}]^{1/\alpha} \sum_{k=s}^{n-1} a^{-1/\alpha}(k).$$

Replacing n and s by $\sigma(n)$ and $\rho(n)$ respectively, we find

$$\begin{aligned} x[\sigma(n)] &\geq [a(\rho(n))(\Delta x(\rho(n)))^{\alpha}]^{1/\alpha} \sum_{k=\rho(n)}^{\sigma(n)-1} a^{-1/\alpha}(k) \\ &:= y^{1/\alpha}[\rho(n)] \sum_{k=\rho(n)}^{\sigma(n)-1} a^{-1/\alpha}(k) \quad \text{for} \quad n \geq n_2 \geq n_1, \end{aligned} \tag{29}$$

where $y(n) = a(n)(\Delta x(n))^{\alpha}$ for $n \geq n_2$.

Using (29) and (5) in equation (2), we get

$$\begin{aligned} \Delta^2 y(n) &\geq p(n)h(x[\sigma(n)]) \\ &\geq p(n)h\left(\sum_{k=\rho(n)}^{\sigma(n)-1} a^{-1/\alpha}(k)\right) h\left(y^{1/\alpha}[\rho(n)]\right) \quad \text{for} \quad n \geq n_2. \end{aligned} \tag{30}$$

For $n \geq s \geq n_2$, we have

$$y(n) \geq (n-s)\Delta y(s)$$

or

$$y(n)^{\ 1/\alpha} \geq (n-s)^{1/\alpha} (\Delta y(s))^{1/\alpha}.$$

Replacing n and s by $\rho(n)$ and $\theta(n)$ respectively, we get

$$y^{1/\alpha}[\rho(n)] \geq (\rho(n) - \theta(n))^{1/\alpha} z^{1/\alpha}[\theta(n)] \quad \text{for} \quad n \geq n_3 \in \mathbb{N}(n_0), \tag{31}$$

where $z(n) = \Delta y(n)$ for $n \geq n_3$.

Using (31) and (5) in (30), we obtain

$$\begin{aligned} \Delta z(n) &\geq p(n)h\left(\sum_{k=\rho(n)}^{\sigma(n)-1} a^{-1/\alpha}(k)\right) \\ &\quad \times\ h\left([\rho(n) - \theta(n)]^{1/\alpha}\right) h\left(z^{1/\alpha}[\theta(n)]\right) \quad \text{for} \quad n \geq n_3. \end{aligned}$$

Now by a known result in [1,3,13], we arrive at the desired contradiction.

Case (II). For $n \geq n_1$, we have

$$x(n) = x(n_1) + \sum_{k=n_1}^{n-1} \Delta x(k) \geq \left(\sum_{k=n_1}^{n-1} a^{-1/\alpha}(k)\right) y^{1/\alpha}(n),$$

where $y(n) = a(n)(\Delta x(n))^{\alpha}$, $n \geq n_1$. Next, there exists an $n_2 \geq n_1$ such that

$$x[g(n)] \geq \left(\sum_{k=n_1}^{g(n)-1} a^{-1/\alpha}(k) \right) y^{1/\alpha}[g(n)] \quad \text{for} \quad n \geq n_2. \tag{32}$$

Using (32) and (4) in equation (2), we get

$$\Delta^2 y(n) \;\geq\; q(n) f \left(\sum_{k=n_1}^{g(n)-1} a^{-1/\alpha}(k) \right) f \left(y^{1/\alpha}[g(n)] \right) \quad \text{for} \quad n \geq n_2. \tag{33}$$

Clearly, $\Delta y(n) < 0$ for $n \geq n_2$. Thus, for $n \geq s \geq n_2$, we find

$$y(s) \geq (n - s)(-\Delta y(n)).$$

Replacing s and n by $g(n)$ and $\eta(n)$ respectively, we have

$$y[g(n)] \geq (\eta(n) - g(n)) z[\eta(n)] \quad \text{for} \quad n \geq n_3 \geq n_2, \tag{34}$$

where $z(n) = -\Delta y(n)$, $n \geq n_3$.

Using (34) and (4) in (33), we have

$$\Delta z(n) + q(n) f \left(\sum_{k=n_1}^{g(n)-1} a^{-1/\alpha}(k) \right) \times f \left([\eta(n) - g(n)]^{1/\alpha} \right) f \left(z^{1/\alpha}[\eta(n)] \right) \leq 0 \quad \text{for} \quad n \geq n_3.$$

The rest of the proof is similar to that of Theorem 1–Case (I) and hence omitted.

Case (III). As in the proof of Theorem 1–Case (III), we obtain (17). There exists $n_2 \in \mathbb{N}(n_0)$, $n_2 \geq n_1$ such that

$$x[g(n)] \geq A(g(n)) w^{1/\alpha}[g(n)] \quad \text{for} \quad n \geq n_2, \tag{35}$$

where $w(n) = -a(n)(\Delta x(n))^{\alpha}$ for $n \geq n_2$.

Using (35) and (4) in equation (2), we get

$$\Delta^2 w(n) + q(n) f(A(g(n)) f \left(w^{1/\alpha}[g(n)] \right) \leq 0 \quad \text{for} \quad n \geq n_2. \tag{36}$$

By a known result in [1], we arrive at the desired contradiction. This completes the proof. ■

From the proof of Theorem 3, we see that Case (III) is disregarded if condition (24) holds. Thus, one can easily obtain

Theorem 4. *Let conditions (i) – (v) and (4), (5) and (24) hold and assume that there exist nondecreasing sequences $\{\eta(n)\}$, $\{\rho(n)\}$ and $\{\theta(n)\}$ such that (25) holds. If the equations (26) and (27) are oscillatory, then equation (2) is oscillatory.*

Also, from the proof of Theorem 3–Case (III), we obtain the inequality (36). Now, it is easy to see that there exist a constant b, $0 < b < 1$ and an $n_3 \in \mathbb{N}(n_0)$, $n_3 \geq n_2$ such that

$$w[g(n)] \geq bg(n)\Delta w[g(n)] \quad \text{for} \quad n \geq n_3. \tag{37}$$

Using (37) and (4) in (36), we have

$$\Delta v(n) + f(b^{1/\alpha})q(n)f(g^{1/\alpha}(n))f(A(g(n)))f\left(v^{1/\alpha}[g(n)]\right) \leq 0, \quad n \geq n_3,$$

where $v\,(n) = \Delta w(n)$ for $n \geq n_3$.

Now, one may replace equation (28) by

$$\Delta v(n) + cq(n)f(g^{1/\alpha}(n))f(A(g(n)))f\left(v^{1/\alpha}[g(n)]\right) = 0 \tag{38}$$

for any constant c, $0 < c < 1$.

Once again, we may combine equations (27) and (38) in one by letting

$$\tilde{Q}(n) = \min\left\{q(n)f\left(\sum_{k=n_0}^{g(n)-1} a^{-1/\alpha}(k)\right) f\left([\eta(n) - g(n)]^{1/\alpha}\right)\right. \\ \left. \times\, cq(n)f(g^{1/\alpha}(n))f(A(g(n)))\right\} \tag{39}$$

for $n \geq n_0$ and any constant c, $0 < c < 1$.

Now, equations (27) and (38) are replaced by

$$\Delta y(n) + \tilde{Q}(n)f\left(y^{1/\alpha}[\eta(n)]\right) = 0. \tag{40}$$

Thus, Theorem 3 can be restated as follows:

Theorem 3′. *Let conditions (i)–(v) and (3) – (5) hold and assume that there exist nondecreasing sequences $\{\eta(n)\}$, $\{\rho(n)\}$ and $\{\theta(n)\}$ such that (25) holds. If the equations (26) and (40) are oscillatory, then equation (2) is oscillatory.*

The following result is immediate.

Corollary 2. *Let conditions (i) – (v) and (3) – (5) hold and assume that there exist nondecreasing sequences $\{\eta(n)\}$, $\{\rho(n)\}$ and $\{\theta(n)\}$ such that (25) holds. Equation (2) is oscillatory if one of the following conditions holds:*

(II_1) $\dfrac{f(u^{1/\alpha})}{u} \geq k_1$ *and* $\dfrac{h(u^{1/\alpha})}{u} \geq h_1$ *for* $u \neq 0$ *and some* $k_1,\ h_1 > 0$

$$\limsup_{n\to\infty} \sum_{k=n}^{\theta(n)-1} p(k) h\left(\sum_{s=\rho(k)}^{\sigma(k)-1} a^{-1/\alpha}(s)\right) h\left([\rho(k)-\theta(k)]^{1/\alpha}\right) > \frac{1}{h_1}$$

and

$$\limsup_{n\to\infty} \sum_{k=\eta(n)}^{n-1} \tilde{Q}(k) > \frac{1}{k_1},$$

where $\tilde{Q}$ *is as in (39),*

(II_2) $\displaystyle\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$ *and* $\displaystyle\int^{\pm\infty} \frac{du}{h(u^{1/\alpha})} < \infty,$

$$\sum^{\infty} p(n) h\left(\sum_{k=\rho(n)}^{\sigma(n)-1} a^{-1/\alpha}(k)\right) h\left([\rho(n)-\theta(n)]^{1/\alpha}\right) = \infty$$

and

$$\sum^{\infty} \tilde{Q}(n) = \infty.$$

4. Some general remarks

1. Conditions (4) and (5) can be discarded if we let $f(x) = x^{\beta}$ and $h(x) = x^{\gamma}$, where β and γ are ratios of positive odd integers. The details are left to the reader.

2. By applying many other known results oscillation criteria for first order equations, one can easily drawn many oscillation results similar to those in Corollaries 1 and 2 obtained from Theorems 1 and 3 respectively. The details are left to the reader, see [1, 13].

3. The results of this paper are extendable to neutral equations of the form

$$\Delta^2 \left(a(n)(\Delta(x(n)+c(n)x[\tau(n)]))^{\alpha}\right) + q(n) f(x[g(n)]) = 0$$

and

$$\Delta^2 \left(a(n)(\Delta(x(n)+c(n)x[\tau(n)]))^{\alpha}\right) = q(n) f(x[g(n)]) + p(n) h(x[\sigma(n)]),$$

where $\{c(n)\}$ and $\{\tau(n)\}$ are sequences of real numbers and $\lim_{n\to\infty} \tau(n) = \infty$. The details are left to the reader. We also note that we may extend our results to third order dynamic equations of the form

$$\left(a(n)(x^{\Delta}(n))^{\alpha}\right)^{\Delta\Delta} + q(n) f(x[g(n)]) = 0.$$

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