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$(\epsilon,\,\delta)$ CONTRACTIVE CONDITION AND COMMON FIXED POINTS

Abstract. In the present paper we prove a common fixed point theorem (Theorem 1) for four mappings under the $(\epsilon,\,\delta)$ contractive condition, however, without either imposing any additional restriction on δ or assuming the ϕ-contractive condition together with. While proving the theorem, neither the completeness of the metric space is assumed nor any of the mappings is required to be continuous. Thus we also provide one more answer to the problem of Rhoades [24] which ensures the existence of common fixed point, however, does not force the maps to be continuous at the common fixed point. Theorem 2 generalizes further the result obtained in Theorem 1.

Key words: common fixed point, compatible maps, noncompatible maps, weak compatible maps of type (A), contractive condition and reciprocal continuity.

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1. Introduction

For a selfmapping f of a metric space $(X,\,d)$ the most general type of contractive condition is either a Banach type contractive condition

$$(1)\qquad d(fx,fy) \;\le\; k\max\{d(x,y),d(x,fx),d(y,dy),\frac{[d(x,fy)+d(y,fx)]}{2}\},\quad 0\le k<1,$$

or a Meir-Keeler type $(\epsilon,\,\delta)$ contractive condition given $\epsilon>0$ there exists a $\delta>0$ such that

$$(2)\quad \epsilon \;\le\; \max\{d(x,y),d(x,fx),d(y,dy),\frac{[d(x,fy)+d(y,fx)]}{2}\}<\epsilon+\delta \;\Rightarrow\; d(fx,fy)<\epsilon.$$

or a ϕ-contractive condition of the form

$$(3)\quad d(fx,fy)\le\phi(\max\{d(x,y),d(x,fx),d(y,dy),\frac{[d(x,fy)+d(y,fx)]}{2}\})$$

where $R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.

In the more general setting pertaining to common fixed points of four mappings, say A, B, S and T of a metric space (X, d) the conditions (2) and (3) respectively assume the form given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned}(4)\qquad \epsilon &\leq \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \\ &\qquad \frac{[d(Ax,Ty)+d(By,Sx)]}{2}\} < \epsilon + \delta \\ &\Rightarrow d(Ax,By) < \epsilon,\end{aligned}$$

and

$$\begin{aligned}(5)\qquad d(Ax,By) &\leq \phi(\max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \\ &\qquad \frac{[d(Ax,Ty)+d(By,Sx)]}{2}\})\end{aligned}$$

where $R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$. In some results the contractive condition (4) has been replaced by a slightly weaker contractive condition of the form given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned}(6)\qquad \epsilon &< \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \\ &\qquad \frac{[d(Ax,Ty)+d(By,Sx)]}{2}\} < \epsilon + \delta \\ &\Rightarrow d(Ax,By) \leq \epsilon\end{aligned}$$

Jachymski [5] has shown that contractive condition (4) implies (6) but (6) does not imply (4).

In the setting of common fixed point theorems Meir-Keeler type (ϵ, δ) contractive condition alone is not sufficient to guarantee the existence of fixed point. While assuming the (ϵ, δ) contractive condition the existence of fixed point is ensured either by imposing some additional restriction on δ or by assuming some additional condition besides the (ϵ, δ) contractive condition or by imposing strong conditions on the continuity of mappings. Following conditions are known to be assumed in proving a fixed point theorem:

(I) (ϵ, δ) contractive condition is taken and δ is assumed nondecreasing (e.g. Pant ([12], [13]))

(II) (ϵ, δ) contractive condition is taken and δ is assumed lower semicontinuous (e.g. Jungck [7], Jungck *et al* [6])

(III) (ϵ, δ) contractive condition is assumed and relatively stronger continuity conditions are used (e.g. Maiti and Pal [11], Park and Bae [21], Park and Rhoades [20])

(IV) Both the (ϵ, δ) contractive condition and the ϕ-contractive condition are assumed simultaneously, however, without imposing any additional restriction either on ϕ or on δ (e.g. Pant and Pant [14], Pant *et al* [15]).

It has been shown by Jachymski ([4], Proposition 4.2) that the Meir-Keeler type (ϵ, δ) contractive condition implies an analogous ϕ-contractive condition if δ is assumed nondecreasing. Pant *et al* [15] has proved that the (ϵ, δ) contractive condition implies an analogous ϕ-contractive condition if δ is lower semicontinuous. A slightly different version of this result has been proved by Jachymski [4] also. Thus any of the assumption (I) or (II) above implies the assumption (IV), but not conversely (e.g. see Pant and Pant [14], Pant *et al* [15]).

It may be observed that the Meir-Keeler type (ϵ, δ) contractive condition alongwith weaker continuity conditions does not ensure the existence of fixed point. An example to this effect is given by Pant (Example 1, [18]) illustrating that the continuity of one of the mappings is not sufficient to ensure the existence of a common fixed point under an (ϵ, δ) contractive condition.

Adopting the approach (IV) above, Pant and Pant [14] and Pant *et al* [15] have proved common fixed point theorems, where a ϕ-contractive condition is assumed together with the (ϵ, δ) contractive condition, however, without imposing any additional restriction either on δ or on ϕ. Such contractive conditions when taken together are implied by (I) and (II) above but not conversely. As a special case of common fixed point theorems proved under assumption (IV), we can relax the continuity conditions of the mappings in as much as the mappings involved become discontinuous at their common fixed points (see e.g. in Pant *et al* [15]).

In view of this, it is inferred that the most generalized known common fixed point theorem have been obtained under assumption (IV), i.e., by assuming the ϕ-contractive condition together with the (ϵ, δ) contractive condition, however, without either imposing any additional restriction on ϕ or on δ or assuming much stronger continuity conditions on the mappings involved. These theorems can be generalized further if we replace the ϕ-contractive condition with a plane contractive condition or with Lipschitz type analogue of the plane contractive condition. Following this approach, in the present paper we prove a common fixed point theorem (Theorem 1) for four mappings under the (ϵ, δ) contractive condition, however, without either imposing any additional restriction on δ or assuming the ϕ-contractive condition together with it. Further, while proving the theorem; neither the continuity of any of the mapping is required nor the metric space is assumed to be complete. Our theorem (Theorem 1), thus generalize the known results proved under the assumptions I-IV above. Theorem 2 generalizes further the result obtained in Theorem 1. To prove the desired results in the following lines we use the notions of Weak Compatible maps of type (A), Noncompatible maps and Reciprocally Continuous maps.

We begin with the following preliminaries:

Definition 1. *Two selfmaps f and g of a metric space X are called compatible (see Jungck [7]) if* $\lim_n d(fgx_n, gfx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_n fx_n = \lim_n gx_n = t$ *for some t in X.*

It is clear from the above definition that f and g will be noncompatible, if there exists at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either non-zero or non-existent.

Definition 2. *Two selfmappings f and g of a metric space X are called weakly commuting if* $d(fgx, gfx) \leq d(fx, gx)$ *for all x in X. The mappings f and g are said to be weakly commuting at a point z in X if* $d(fgz, gfz) \leq d(fz, gz)$.

Definition 3. *The mappings A and S from a metric space (X, d) into itself are said to be compatible of type (A) (see [3], [22] and [23]) if*

$$\lim_n d(ASx_n, SSx_n) = 0 \quad \textit{and} \quad \lim_n d(SAx_n, AAx_n) = 0,$$

whenever $\{x_n\}$ *is a sequence in X such that*

$$\lim_n Sx_n = \lim_n Ax_n = t \quad \textit{for some} \quad t \in X.$$

Definition 4. *The mappings A and S from a metric space (X, d) into itself are said to be weak compatible of type (A) (see [22] and [23]) if*

$$\lim_n d(ASx_n, SSx_n) \leq \lim_n d(SAx_n, SSx_n)$$

and

$$\lim_n d(SAx_n, AAx_n) \leq \lim_n d(ASx_n, AAx_n),$$

whenever $\{x_n\}$ *is a sequence in X such that* $\lim_n Sx_n = \lim_n Ax_n = t$ *for some t in X.*

Definition 5. *Two selfmappings f and g of a metric space (X, d) are called reciprocally continuous (see [14]) if* $\lim_n fgx_n = ft$ *and* $\lim_n gfx_n = gt$ *whenever* $\{x_n\}$ *is a sequence such that* $\lim_n fx_n = \lim_n gx_n = t$ *for some t in X.*

2. Main results

We define the following for Theorem 1:

$$M(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{[d(Ax,Ty)+d(By,Sx)]}{2}\}$$

Theorem 1. *Let (A,S) and (B,T) be weak compatible mappings of type (A) from a metric space (X,d) into itself such that $AX \subset TX$, $BX \subset SX$ and*

(i) given $\epsilon > 0$, there exists a $\delta > 0$ such that,

$$\epsilon < M(x,y) < \epsilon + \delta \Rightarrow d(Ax,By) \leq \epsilon$$

$(i*)$ $d(Ax,By) < M(x,y)$, whenever the right hand side is positive,

(ii) $d(Ax,By) \leq k\ d(Sx,Ty)$, $k \geq 0$,

then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}.$$

This can be done by virtue of (i). We claim that $\{y_n\}$ is a Cauchy sequence. Two cases arise. Either $y_n = y_{n+1}$ for some n or $y_n \neq y_{n+1}$ for each n. If $y_n = y_{n+1}$ for some n then as shown by Rhoades *et al* [24] we have $y_n = y_{n+k}$ for each $k \geq 1$. For instance, suppose that $y_{2m} = y_{2m+1}$ then $y_{2m+1} = y_{2m+2}$. Otherwise, using (ii) we get

$$\begin{aligned} d(y_{2m+1}, y_{2m+2}) &= d(Ax_{2m+2}, Bx_{2m+1}) = M(x_{2m+2}, x_{2m+1}) \\ &= d(y_{2m+1}, y_{2m+2}), \end{aligned}$$

a contradiction. Hence, $y_{2m+1} = y_{2m+2}$. Similarly, $y_{2m+1} = y_{2m+2}$ implies that $y_{2m+2} = y_{2m+3}$. Proceeding in this manner, it follows that $y_{2m} = y_{2m+k}$ for each $k \geq 1$ and so $\{y_n\}$ is a Cauchy sequence.

Let us, therefore, consider the case when $y_n \neq y_{n+1}$ for each n. Using (ii), we get $d(y_{2n}, y_{2n+1}) < d(y_{2n-1}, y_{2n})$. Similarly, $d(y_{2n-1}, y_{2n}) < d(y_{2n-2}, y_{2n-1})$ and so on. Thus, $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and, therefore, tends to a limit $r \geq 0$. If possible suppose $r > 0$. Then given $\delta > 0$ there exists a positive number N such that for each $n \geq N$ we have

$$r < d(y_{2n}, y_{2n+1}) = M(x_{2n+2}, x_{2n+1}) < r + \delta. \tag{7}$$

Selecting δ in (7) in accordance with (ii), for each $n \geq N$ we get $d(y_{2n+2}, y_{2n+1}) = d(Ax_{2n+2}, Bx_{2n+1}) < r$. This, in turn, gives $d(y_{2n+3}, y_{2n+2}) < d(y_{2n+1}, y_{2n+2}) < r$, contradicting (7). Hence $\lim_n d(y_n, y_{n+1}) = 0$.

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists an $\epsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\epsilon$. Select δ in (ii) so that $0 < \delta \leq \epsilon$. Since $\lim_n d(y_n, y_{n+1}) = 0$, there exists an integer N such that $d(y_n, y_{n+1}) < \delta/6$ whenever $n \geq N$.

Let $n_i \geq N$. Then, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \epsilon + (\delta/3)$. If not, then

$$\begin{aligned} d(y_{n_i}, y_{n_{i+1}}) &\leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) \\ &< \epsilon + (\frac{\delta}{3}) + (\frac{\delta}{6}) < 2\epsilon, \end{aligned}$$

a contradiction. Without loss of generality, we can assume n_i to be odd. Let m_i be the smallest even integer such that $d(y_{n_i}, y_{m_i}) \geq \epsilon + (\frac{\delta}{3})$. Then $d(y_{n_i}, y_{m_i-2}) < \epsilon + (\frac{\delta}{3})$ and

$$\begin{aligned} \epsilon + (\frac{\delta}{3}) \ &\leq \ d(y_{n_i}, y_{m_i}) + d(y_{n_i}, y_{m_i-2}) \\ &\qquad + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_m) \\ &< \ \epsilon + (\frac{\delta}{3}) + (\frac{\delta}{6}) + (\frac{\delta}{6}) = \epsilon + 2(\frac{\delta}{3}) \end{aligned} \tag{8}$$

Also,

$$\begin{aligned} d(y_{n_i}, y_{m_i}) &\leq M(x_{n_{i+1}}, x_{m_{i+1}}) \\ &< \epsilon + 2(\frac{\delta}{3}) + (\frac{\delta}{6}) < \epsilon + \delta, \end{aligned}$$

that is,

$$\epsilon + (\frac{\delta}{3}) \leq M(x_{n_{i+1}}, x_{m_{i+1}}) < \epsilon + \delta.$$

In view of (ii), this yields $d(y_{n_{i+1}}, y_{m_{i+1}}) < \epsilon$. But then

$$\begin{aligned} d(y_{n_i}, y_{m_i}) &\leq d(y_{n_i}, y_{n_i+1}) + d(y_{n_i+1}, y_{m_i+1}) + d(y_{m_i+1}, y_{m_i}) \\ &< (\frac{\delta}{6}) + \epsilon + (\frac{\delta}{6}) = \epsilon + (\frac{\delta}{3}), \end{aligned}$$

which contradicts (8). Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $y_n \to z$. Also

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z. \tag{9}$$

Now, since the pair (A, S) is weak compatible of type (A), we have, $d(ASx_n, SSx_n) \leq d(SAx_n, SSx_n)$ and $d(SAx_n, AAx_n) \leq d(ASx_n, AAx_n)$ which on letting $n \to \infty$ yields $ASx_n = SAx_n$ and $SAx_n = AAx_n$, that is, $Az = Sz$. Since $AX \subset TX$, there exists a w in X such that $Az = Tw$. If $Bw \neq Tw$, using (i) we get

$$\begin{aligned} d(Az, Bw) &< \max\ d(Sz, Tw), d(Az, Sz), d(Bw, Tw), \frac{[d(Az, Tw) + d(Bw, Sz)]}{2} \\ &= \max(d(Bw, Tw), \frac{[d(Az, Tw) + d(Bw, Sz)]}{2}) \\ &= d(Bw, Tw) = d(Bw, Az), \end{aligned}$$

a contradiction. Hence $Bw = Tw$ and $Sz = Az = Tw = Bw$. Since pair (A, S) is weak compatible of type (A),

$$d(ASz, SSz) \leq d(SAz, SSz) \ \text{ and } \ d(SAz, AAz) \leq d(ASz, AAz),$$

which implies that $ASz = SSz$ and $SAz = AAz$. Thus $AAz = ASz = SAz = SSz$. Similarly, since pair (B, T) is also weak compatible of type (A) it can be shown that $BBw = BTw = TBw = TTw$. If $Az \neq AAz$, using (i) we get

$$\begin{aligned} d(Ax, By) = d(AAz, Bw) &< \max\{d(SAz, Tw), d(AAz, SAz), d(Bw, Tw), \\ &\quad \frac{[d(AAz, Tw) + d(Bw, SAz)]}{2}\} \\ &= d(SAz, Tw) = d(AAz, Bw) \end{aligned}$$

a contradiction. Hence $Az = AAz = SAz$ and Az is a common fixed point of A and S. Similarly, $Bw (= Az)$ is a common fixed point of B and T. Uniqueness of the common fixed point follows from (i). Moreover, the proof follows on similar lines when A or B is assumed continuous since $AX \subset TX$ and $BX \subset SX$. This establishes the theorem. ■

Example 1. Let $X = [2, 20)$ and d be the usual metric on X. Define A, B, S and $T : X \to X$ by

$$\begin{gathered} Ax = 2 \ \text{ if } \ x = 2 \ \text{ or } \ > 5, \qquad Ax = 4 \text{ if } 2 < x \leq 5 \\ Bx = 2 \ \text{ if } \ x = 2 \ \text{ or } \ > 5, \qquad Bx = 6 \text{ if } 2 < x \leq 5 \\ S2 = 2, \qquad Sx = 8 \text{ if } 2 < x \leq 5 \qquad Sx = \tfrac{(x+1)}{3} \text{ if } x > 5 \\ T2 = 2, \qquad Tx = x + 9 \text{ if } 2 < x \leq 5, \qquad Tx = \tfrac{(x+5)}{5} \text{ if } x > 5. \end{gathered}$$

Then A, B, S and T satisfy the conditions of the above theorem and have a unique common fixed point $x = 2$. It can be verified in this example that A, B, S and T satisfy the condition (ii) and $\delta(\epsilon) = 7 - \epsilon$ if $\epsilon \leq 2$ and $\delta(\epsilon) = 13 - \epsilon$

if $\epsilon > 2$. It may also be observed that A, B, S and T satisfy the condition (iii) of the above theorem. Further, (A, S) and (B, T) are pairs of weakly compatible mapping of type (A). To see this let us consider a decreasing sequence $\{x_n = 5 + \frac{1}{n} : n > 0\}$, then $Ax_n = 2$, $Sx_n \to 2$ and $d(ASx_n, SSx_n) \leq d(SAx_n, SSx_n)$, $d(SAx_n, AAx_n) \leq d(ASx_n, AAx_n)$. Also, $Bx_n = 2$, $Tx_n \to 2$ and $d(BTx_n, TTx_n) \leq d(TBx_n, TTx_n)$, $d(TBx_n, BBx_n) \leq d(BTx_n, BBx_n)$. It may also be observed that none of the mappings A, B, S or T is continuous and also the metric space is not complete.

Remark 1. Above theorem has been proved under (ϵ, δ) contractive condition without imposing any additional restriction on δ or assuming the ϕ-contractive condition together with. The completeness of the metric space is not necessary and none of the mapping assumed is continuous. Our theorem, thus, generalizes several results including that of Jachymski ([4], Theorem 3.3), Pant ([12], [13], [18]), Pant and Pant [14], Pant *et al* ([15], [19]), Singh and Kasahara [26], Boyd and Wong [1], Maiti and Pal [11] and Park and Rhoades [20].

Remark 2. Despite the Lemma 2.2 of Jachymski [4], an (ϵ, δ) contractive condition does not imply the existence of a common fixed point unless some additional condition is imposed on δ. For example Jungck [7] and Jungck *et al* [6] assume δ to be lower semicontinuous. On the other hand, Pant ([12], [13]) assumes δ to be nondecreasing. However, we have not imposed any additional condition on δ.

We now generalize the above theorem and get the following theorem.

Theorem 2. *Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings from a metric space (X, d) into itself such that $AX \subset TX$, $BX \subset SX$ and*

$$(i)\quad d(Ax, By) < \max\{d(Sx, Ty), k\frac{[d(Ax, Sx) + d(By, Ty)]}{2}, \frac{[d(Ax, Ty) + d(By, Sx)]}{2}\}, \quad 0 < k \leq 2,$$

$$(ii)\quad d(Ax, By) \leq k\ d(Sx, Ty), \qquad k \geq 0.$$

Suppose that one of the pairs (A, S) or (B, T) be noncompatible and other compatible. If mappings in the compatible pair are reciprocally continuous then A, B, S and T have a unique common fixed point.

Proof. Let B and T be noncompatible maps. Then there exists a sequence $\{x_n\}$ in X such that $Bx_n \to t$ and $Tx_n \to t$ for some t in X but $\lim_n d(BTx_n, TBx_n)$ is either nonzero or non existent. Since $BX \subset SX$, for each x_n there exists y_n in X such that $Bx_n = Sy_n$. Thus $Bx_n \to t, Tx_n \to t$

and $Sy_n \to t$. We claim that $Ay_n \to t$. If not, there exists a subsequence $\{Ay_m\}$ of $\{Ay_n\}$, a positive integer M and a number $r > 0$ such that for each $m \geq M$ we have $d(Ay_m, t) \geq r$, $d(Ay_m, Bx_m) \geq r$ and

$$\begin{aligned} d(Ay_m, Bx_m) &< \max\{d(Sy_m, Tx_m), \frac{[d(Ay_m, Sy_m) + d(Bx_m, Tx_m)]}{2}, \\ &\qquad k\ \frac{[d(Ay_m, Tx_m) + d(Bx_m, Sy_m)]}{2}\} \\ &< d(Ay_m, Sy_m) = d(Ay_m, Bx_m), \end{aligned}$$

a contradiction. Hence $Ay_n \to t$.

Suppose that A and S are reciprocally continuous. Then since $Ax_n \to t$, $Sx_n \to t$, by virtue of reciprocal continuity of A and S we get $ASx_n \to At$ and $SAx_n \to St$. On the other hand compatibility of A and S implies that $\lim_n d(ASx_n, SAx_n) = 0$, that is, $At = St$. Since $AX \subset TX$, there exists w in X such that $At = Tw$. We show that $St = At = Tw = Bw$. If $At \neq Bw$

$$d(At, Bw) \leq k\ d(St, Tw) = 0,$$

a contradiction. Hence $St = At = Tw = Bw$. Since compatible maps commute at their coincidence points, we get $ASt = SAt$ and $BTw = TBw$. Moreover, $AAt = ASt = SAt = SSt$ and $BBw = BTw = TBw = TTw$. We claim that At is common fixed point of A and S. If not, by (ii) we get,

$$\begin{aligned} d(St, SSt) &= d(At, AAt) = d(AAt, Bw) \\ &< \max\{d(SAt, Tw), k\ \frac{[d(AAt, SAt) + d(Bw, Tw)]}{2}, \\ &\qquad \frac{[d(AAt, Tw) + d(Bw, SAt)]}{2}\} \\ &= d(AAt, At), \end{aligned}$$

a contradiction. Hence $At = AAt = SAt$ and At is a common fixed point of A and S. Similarly $At = Bw$ is a common fixed point of B and T. The proof is similar when B and T are assumed reciprocally continuous. This completes the proof of the theorem. ∎

We now give and example to illustrate the above theorem.

Example 2. Let $X = [2, 20)$ and d be the usual metric on X. Define A, B, S and $T : X \to X$ by

$$A2 = 2, \qquad Ax = 3 \text{ if } x > 2$$
$$Bx = 2 \text{ if } x = 2 \text{ or } > 5, \qquad Bx = 6 \text{ if } 2 < x \leq 5$$
$$S2 = 2, \qquad Sx = 6 \text{ if } x > 2$$
$$T2 = 2, \qquad Tx = 4 \text{ if } 2 < x \leq 5, \qquad Tx = \tfrac{(x+5)}{5} \text{ if } x > 5.$$

Then A, B, S and T satisfy the conditions of the above theorem and have a unique common fixed point $x = 2$. Here $AX = \{2, 3\}$, $TX = [2, 5)$, $BX = \{2, 6\}$, $SX = \{2, 6\}$, thus $AX \subset TX$, $BX \subset SX$. Condition (i) holds good for $k = 2$. It may be seen that the above example satisfies the condition (ii) of Theorem 2. It may also be observed here that if $x > 2$, it is not possible that $Ax_n \to t$, $Sx_n \to t$. However, at $x = 2$, $Ax_n \to 2$, $Sx_n \to 2$ implies that $t = 2$. Thus $\{x_n\}$ is a sequence consisting of only one term, that is, 2. Hence, $Sx_n = 2 = Ax_n$. Also, $SAx_n = ASx_n$ which implies that $d(ASx_n, SAx_n) = 0$. Thus (A, S) are compatible and reciprocally continuous. Further, let $\{x_n\}$ is a decreasing sequence such that $\lim_n x_n = 5$. Then $Bx_n = 2$, $Tx_n \to 2$, $TBx_n = 2$ and $BTx_n = 6$, that is, $\lim_n d(BTx_n, TBx_n) = 4$. Therefore, B and T are noncompatible.

Remark 3. It is known since the paper of Kannan [8] in 1968 that there exist maps that have a discontinuity in their domain but which have fixed points. In 1988, Rhoades [24] posed an open problem - "Whether there exists a contractive definition which is strong enough to generate a fixed point, but which does not force the map to be continuous at the fixed point." The problem had remained open for more than one decade. Pant ([16], [17]) and Pant *et al* ([15], [19]) have provided some solutions to this problem. In the above theorems we have provided one more answer to this problem. It may be observed that in the Examples above none of the mapping is continuous at their common fixed point.

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