F A S C I C U L I M A T H E M A T I C I

Nr 42 2009

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ON A COMMON FIXED POINT OF TWO RANDOM OPERATORS USING RANDOM MANN ITERATION SCHEME

Abstract. In the present note, it is proved that if a random Mann iteration scheme defined by two operators is convergent under some contractive inequality the limit point is a common fixed point of each of two random operators in a Banach space.

Key words: Mann iteration, fixed point, measurable mappings, Banach space.

AMS Mathematics Subject Classification: 47H10, 47H40.

1. Introduction and preliminaries

In paper [8], Kasahara had shown that if an iterated sequence defined by using a continuous linear mapping is convergent under certain assumptions, then the limit point is a common fixed point of each of two non-linear mappings. Ganguly [6] arrived at the same conclusion under the same contractive definition by taking the sequence of Mann iterates [9]. In this paper, our attempt is to place the random version of Ganguly's result. The study of random fixed points has been an active area of contemporary research in Mathematics. Random iteration scheme has been elaborately discussed by Choudhury ([1], [2], [3], [4]). Looking to the immense applications of iterative algorithms in signal processing and image reconstruction, it is essential to venture upon random iteration.

We first review the following concepts, which are essential for our study in this paper.

Throughout this paper, (Ω, Σ) denotes a measurable space and X stands for a separable Banach space. C is a non-empty subset of X.

A mapping $f : \Omega \to C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of X.

A mapping $F : \Omega \times C \to C$ is said to be a random operator, if $F(., x) : \Omega \to C$ is measurable for every $x \in C$.

A measurable mapping $g : \Omega \to C$ is said to be a random fixed point of the random operator $F : \Omega \times C \to C$, if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

A random operator $F : \Omega \times C \to C$ is said to be continuous if, for fixed $t \in \Omega$, $F(t, .) : C \to C$ is continuous.

Definition 1 (**Random Mann iteration Scheme**). *Let $S, T : \Omega \times C \to C$ be two operators on a non-empty convex subset C of a separable Banach space X. Then the sequence $\{x_n\}$ of random Mann iterates associated with S or T is defined as follows:*

$$\text{Let } x_0 : \Omega \to C \text{ be any given measurable mapping.} \tag{1}$$

$$x_{n+1}(t) = (1 - c_n)x_n(t) + c_n S(t, x_n(t)) \quad \text{for } n > 0, \quad t \in \Omega \tag{2}$$

or

$$x_{n+1}(t) = (1 - c_n)x_n(t) + c_n T(t, x_n(t)) \quad \text{for } n > 0, \quad t \in \Omega \tag{3}$$

where c_n satisfies:

$$c_0 = 1 \quad \text{for } n = 0, \tag{4}$$

$$0 < c_n \leq 1 \quad \text{for } n > 0, \tag{5}$$

$$\lim_n c_n = h > 0. \tag{6}$$

Since C is convex it follows from the above construction that x_n is a mapping from Ω to C for all $n = 0, 1, 2, \ldots$

2. Main result

Theorem 1. *Let $S, T : \Omega \times C \to C$, where C is a nonempty closed convex subset of a separable Banach space X, be two continuous random operators which satisfy the following inequality: for all $x, y \in C$ and $t \in \Omega$*

$$\begin{aligned} \|S(t,x) - T(t,y)\| \leq \alpha \max\{&\beta\|x - y\|, \|x - S(t,x)\|, \|y - T(t,y)\|, \\ &\|x - T(t,y)\|, \|y - S(t,x)\|\} \end{aligned} \tag{7}$$

where $\alpha, \beta \geq 0$, $0 \leq \alpha < 1$.

If the sequence $\{x_n\}$ of random Mann iterates associated with S or T satisfying (1)-(6) converges, then it converges to a common random fixed point of both S and T.

Proof. We may assume that the sequence $\{x_n\}$ defined by (2) is pointwise convergent, that is, for all $t \in \Omega$,

$$x(t) = \lim_{n\to\infty} x_n(t). \tag{8}$$

Since X is a separable Banach space, for any continuous random operator $A : \Omega \times C \to C$ and any measurable mapping $f : \Omega \to C$, the mapping $x(t) = A(t, f(t))$ is a measurable mapping [7].

Since $x(t)$ is measurable and C is convex, it follows that $\{x_n\}$ constructed in the random iteration from (2)-(6) is a sequence of measurable mappings. Hence, $x : \Omega \to C$ being limit of measurable mapping sequence is also measurable. For $t \in \Omega$, from (2) it follows that

$$\begin{aligned}
\|x(t) - T(t,x(t))\| &\le \|x(t) - x_{n+1}(t)\| + \|x_{n+1}(t) - T(t,x(t))\| \\
&\le \|x(t) - x_{n+1}(t)\| + (1-c_n)\|x_n(t) - T(t,x(t))\| \\
&\quad + c_n\|S(t,x_n(t)) - T(t,x(t))\| \\
&\le \|x(t) - x_{n+1}(t)\| + \|(1-c_n)x_n(t) + c_n S(t,x(t)) - T(t,x(t))\| \\
&\le \|x(t) - x_{n+1}(t)\| + (1-c_n)\|x_n(t) - T(t,x(t))\| \\
&\quad + c_n\|S(t,x_n(t)) - T(t,x(t))\| \\
&\le \|x(t) - x_{n+1}(t)\| + (1-h)\|x_n(t) - T(t,x(t))\| \\
&\quad + c_n\,\alpha\,\max\{\beta\|x_n(t) - x(t)\|, \|x_n(t) - S(t,x_n(t))\|, \\
&\qquad \|x(t) - T(t,x(t))\|, \|x_n(t) - T(t,x(t))\|, \|x(t) - S(t,x_n(t))\|\},
\end{aligned} \tag{9}$$

by equations (6) and (7).

Now,

$$c_n(S(t,x_n(t)) - x_n(t)) = c_n\,S(t,x_n(t)) - c_n x_n(t) = x_{n+1}(t) - x_n(t)$$

by (2), so that

$$\|S(t,x_n(t)) - x_n(t)\| \le \frac{1}{c_n}\|x_{n+1}(t) - x_n(t)\|.$$

This shows that for $t \in \Omega$, $S(t,x_n(t)) - x_n(t) \to 0$ and so $S(t,x_n(t)) \to x(t)$ as $n \to \infty$, as S is a continuous random operator and x is a measurable mapping. Consequently from (9) on taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}
\|x(t) - T(t,x(t))\| &\le (1-h)\|x(t) - T(t,x(t))\| \\
&\quad + c_n\alpha\,\max\{0,0,\|x(t) - T(t,x(t))\|, \|x(t) - T(t,x(t))\|, 0\} \\
&\le (1-h+h\alpha)\|x(t) - T(t,x(t))\|,
\end{aligned}$$

which implies $T(t,x(t)) = x(t)$ for all $t \in \Omega$, as T is a continuous random operator and x is measurable. Therefore,

$$\begin{aligned}\|S(t,x(t)) - x(t)\| &= \|S(t,x(t)) - T(t,x(t))\| \\ &\le \alpha \max\{\, \beta\,\|x(t)-x(t)\|, \|x(t)-S(t,x(t))\|, \\ &\qquad \|x(t)-T(t,x(t))\|, \|x(t)-T(t,x(t))\|, \|x(t)-S(t,x(t))\|\} \\ &\le \alpha \max\{0, \|x(t)-S(t,x(t))\|, 0, 0, \|x(t)-S(t,x(t))\|\}.\end{aligned}$$

Since $\alpha < 1$, it follows that for all $t \in \Omega$, and x is measurable, $S(t,x(t)) = x(t)$, which proves the theorem. ■

Remark. In the deterministic case:
1. Our theorem gives the result of Ganguly [6].
2. For $S = T$ with $c_n = t$, $0 < t < 1$, our result gives Theorem 3 of Ciric [5].
3. It extends Theorem 1 of Rhodes [9].

Open problem. The random Mann iteration will not in general be convergent for an arbitrary closed convex set. It will be interesting to find the type of subsets C for which the theorem is valid. Further in that case measurable perturbative effects on the random iteration may be investigated.

Acknowledgement. We are grateful to Prof. B.S. Choudhury for making his papers available to us. Also we are thankful to Prof. A.K. Ganguly for his valuable advice for the preparation of this paper.

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Received on 11.01.2008 and, in revised form, on 03.03.2009.