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Ovidiu T. Pop

VORONOVSKAJA-TYPE THEOREMS AND APPROXIMATION THEOREMS FOR A CLASS OF GBS OPERATORS

Abstract. In this paper we will demonstrate a Voronovskaja-type theorems and approximation theorems for GBS operators associated to some linear positive operators. Through parti- cular cases, we obtain statements verified by the GBS operators of Bernstein, Schurer, Durrmeyer, Kantorovich, Stancu, Bleimann-Butzer-Hahn, Mirakjan-Favard-Szász, Baskakov, Meyer-König and Zeller, Ismail-May.

Key words: linear positive operators, GBS operators, the first order modulus of smoothness, Voronovskaja-type theorem, approximation theorem.

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1. Introduction

In the paper [30] we demonstrate a Voronovskaja-type theorem and approximation theorem for a class of GBS operators associated to the linear positive operators which have the form

$$\left(L_m^\delta f\right)(x,y)=\sum_{k=0}^{p_m}\left(\delta\varphi_{m,k}(x)+(1-\delta)\varphi_{m,k}(y)\right)A^*_{m,k}(f).$$

In this paper we study the same thing for the linear positive operators which have the form

$$\left(L^*_{m,n}f\right)(x,y)=\sum_{k=0}^{p_m}\sum_{j=0}^{p_n}\varphi_{m,k}(x)\varphi_{n,j}(y)A_{m,n,k,j}(f).$$

In this section, we recall some notions and results which we will use in this paper.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For the following construction see [20].

Define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1}$$

For the real number β, we have that

$$m + \beta \geq \gamma_\beta \tag{2}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0 + \beta = \begin{cases} \max\big\{1+\beta, \{\beta\}\big\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{3}$$

For the real numbers α, β, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta \\ 1 + \dfrac{\alpha - \beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{4}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{5}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \ldots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1) - (4), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{6}$$

for any natural number $m, m \geq m_0$ and any $x \in [0,1]$.

These operators are named Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [32]. In [32], the domain of definition of the Stancu operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

The fundamental polynomials of Bernstein are defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{7}$$

for any $x \in [0,1]$, $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, m\}$.

If $\alpha = \beta = 0$, then we obtained the Bernstein operators (see [5] or [33]).

If $p \in \mathbb{N}_0$, $\alpha = 0$, $\beta = -p$, replace m by $m + p$, then $\gamma_\beta = \gamma_{-p} = 1$, $\mu^{(\alpha,\beta)} = \mu^{(0,-p)} = 1 + p$ and then we obtain the Schurer operators (see [31] or [33]).

For $m \in \mathbb{N}$, let the operators $M_n : L_1([0, 1]) \to C([0, 1])$ defined for any function $f \in L_1([0, 1])$ by

$$(M_m f)(x) = (m + 1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{8}$$

for any $x \in [0, 1]$.

These operators were introduced in 1967 by J. L. Durrmeyer in [11] and were studied in 1981 by M. M. Derriennic in [9]. The operators M_m, $m \in \mathbb{N}$ are named Durrmeyer operators.

For $m \in \mathbb{N}$, let the operators $K_m : L_1([0, 1]) \to C([0, 1])$ defined for any function $f \in L_1([0, 1])$ by

$$(K_m f)(x) = (m + 1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) dt, \tag{9}$$

for any $x \in [0, 1]$.

The operators K_m, where $m \in \mathbb{N}$, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [14] or [33]).

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [7] a sequence of linear positive operators $(L_m)_{m \geq 1}$, $L_m : C_B([0, \infty)) \to C_B([0, \infty))$, defined for any function $f \in C_B([0, \infty))$ by

$$(L_m f)(x) = \frac{1}{(1 + x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m + 1 - k}\right), \tag{10}$$

for any $x \in [0, \infty)$ and $m \in \mathbb{N}$, where $C_B([0, \infty)) = \{f \mid f : [0, \infty) \to \mathbb{R},\ f$ bounded and continuous on $[0, \infty)\}$.

Let $m \in \mathbb{N}$ and the operators $S_m : C_2([0, \infty)) \to C([0, \infty))$ defined for any function $f \in C_2([0, \infty))$ by

$$(S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right) \tag{11}$$

for any $x \in [0, \infty)$, where $C_2([0, \infty)) = \left\{ f \in C([0, \infty)) : \lim\limits_{x \to \infty} \frac{f(x)}{1 + x^2} \text{ exists and is finite} \right\}$. The oprators $(S_m)_{m \geq 1}$ are named Mirakjan-Favard-Szász operators, introduced in 1941 by G. M. Mirakjan in the paper [18]. These operators are intensive studied by J. Favard in 1944 in the paper [12] and O. Szász in 1950 in the paper [34].

Let $m \in \mathbb{N}$ and the operators $V_m : C_2([0,\infty)) \to C([0,\infty))$ defined for any function $f \in C_2([0,\infty))$ by

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right) \tag{12}$$

for any $x \in [0,\infty)$. The operators $(V_m)_{m\geq 1}$ are named Baskakov operators, introduced in 1957 by V. A. Baskakov in the paper [3].

W. Meyer-König and K. Zeller have introduced in [17] a sequence of linear and positive operators. After a slight adjustment given by E. W. Cheney and A. Sharma in [8], these operators take the form $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right), \tag{13}$$

for any $x \in [0,1)$ and $m \in \mathbb{N}$.

These operators are named Meyer-König and Zeller operators. Observe that we can consider $Z_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$.

In the paper [13], M. Ismail and C. P. May consider the operators $(R_m)_{m\geq 1}$. For $m \in \mathbb{N}$, $R_m : C([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C([0,\infty))$ by

$$(R_m f)(x) = e^{-\frac{mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m}\right), \tag{14}$$

for any $x \in [0,\infty)$.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the function sets: $B(I) = \{f | f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, let the function $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$, for any $t \in I$.

If $I \subset \mathbb{R}$ is a given interval and $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = \sup\left\{|f(x') - f(x'')| : x',\ x' \in I,\ |x' - x''| \leq \delta\right\}. \tag{15}$$

Let $I, J \subset \mathbb{R}$ intervals, $E(I \times J)$, $F(I \times J)$ which are subsets of the set of real functions defined on $I \times J$ and $L : E(I \times J) \to F(I \times J)$ be a linear positive operator. The operator $UL : E(I \times J) \to F(I \times J)$ defined for any function $f \in E(I \times J)$, any $(x, y) \in I \times J$ by

$$(ULf)(x, y) = L\left(f(x, *) + f(\cdot, y) - f(\cdot, *)\right)(x, y) \tag{16}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable (see [2]).

If $f \in E(I \times J)$ and $(x, y) \in I \times J$, let the functions $f^x = f(x, *)$, $f^y = f(\cdot, y) : I \times J \to \mathbb{R}$, $f^x(s, t) = f(x, t)$, $f^y(s, t) = f(s, y)$ for any $(s, t) \in I \times J$. Then, we can consider that f^x, f^y are functions of real variable, $f^x : J \to \mathbb{R}$, $f^x(t) = f(x, t)$ for any $t \in J$ and $f^y : I \to \mathbb{R}$, $f^y(s) = f^y(s, y)$ for any $s \in I$.

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ be a bounded function. The function $\omega_{total}(f; \cdot, *) : [0, \infty) \times [0, \infty) \to \mathbb{R}$, defined for any $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$ by

$$\omega_{total}(f; \delta_1, \delta_2) = \sup\left\{|f(x, y) - f(x', y')| : (x, y), (x', y') \in I_1 \times I_2,\ |x - x'| \le \delta_1, |y - y'| \le \delta_2\right\} \tag{17}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f (see [35]).

The first order modulus of smoothness for bivariate functions has properties similar to the properties of the first modulus of smoothness for univariate functions.

2. Preliminaries

For the following construction and results see [25], [29] and [30], where $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$.

Let $I, J \subset \mathbb{R}$ be intervals with $I \cap J \neq \emptyset$. For any $m \in \mathbb{N}$ and $k \in \{0, 1, ..., p_m\} \cap \mathbb{N}_0$ consider the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \ge 0$ for any $x \in J$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$.

Definition 1. *For $m \in \mathbb{N}$ define the operator $L_m : E(I) \to F(J)$ by*

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(f), \tag{18}$$

for any $f \in E(I)$ and $x \in J$, where $E(I)$ and $F(J)$ are subsets of the set of real functions defined on I and J, respectively.

Proposition 1. *The L_m, $m \in \mathbb{N}$ operators are linear and positive on $E(I \cap J)$.*

Definition 2. *For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, define $T^*_{m,i}$ by*

$$\left(T^*_{m,i}L_m\right)(x) = m^i \left(L_m\psi_x^i\right)(x) = m^i \sum_{k=0}^{p_m} \varphi_{m,k}(x)A_{m,k}(\psi_x^i), \tag{19}$$

for any $x \in I \cap J$.

In the following let $s \in \mathbb{N}_0$, s even and we suppose that the operators $(L_m)_{m\geq 1}$ verify the conditions: there exists, the smallest $\alpha_j \in [0,\infty)$ so that

$$\lim_{m\to\infty} \frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}} = B_j\,(x) \in \mathbb{R}, \tag{20}$$

for any $x \in I \cap J$, $j \in \{0, 2, 4, \ldots, s+2\}$ and

$$\begin{cases} \alpha_{s-2l} + \alpha_{2l} - \alpha_s \leq 0 \\ \alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2 < 0 \\ \alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4 < 0 \end{cases} \tag{21}$$

where $l \in \left\{0, 1, 2, \ldots, \frac{s}{2}\right\}$.

Remark 1. From the first and second relation from (21), for $l = 0$ it results that

$$\alpha_0 = 0 \tag{22}$$

and

$$\alpha_{s+2} < \alpha_s + 2. \tag{23}$$

For $m, n \in \mathbb{N}$, let the linear positive functionals $A_{m,n,k,j} : E(I \times I) \to \mathbb{R}$ with the properties

$$A_{m,n,k,j}\left((\cdot - x)^i(* - y)^l\right) = A_{m,k}\left((\cdot - x)^i\right)A_{n,j}\left((* - y)^l\right), \tag{24}$$

$$A_{m,n,k,j}(f_x) = A_{n,j}(f_x) \tag{25}$$

and

$$A_{m,n,k,j}(f^y) = A_{m,k}(f^y), \tag{26}$$

for any $x, y \in I$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, $j \in \{0, 1, \ldots, p_n\} \cap \mathbb{N}_0$ and $i, l \in \{0, 1, \ldots, s\}$, where " $\cdot$ " and " $*$ " stand for the first and second variable.

Remark 2. In this paper p_m, p_n are simultaneous finite or infinite, where $m, n \in \mathbb{N}$.

Definition 3. Let $m, n \in \mathbb{N}$. The operator $L^*_{m,n} : E(I \times I) \to F(J \times J)$ defined for any function $f \in E(I \times I)$ and any $(x, y) \in J \times J$ by

$$(27) \qquad \left(L^*_{m,n} f\right)(x, y) = \sum_{k=0}^{p_m} \sum_{j=0}^{p_n} \varphi_{m,k}(x) \varphi_{n,j}(y) A_{m,n,k,j}(f)$$

is named the bivariate operator of L-type.

Proposition 2. *The operators* $\left(L^*_{m,n}\right)_{m,n \geq 1}$ *are linear and positive on* $E\left((I \times I) \cap (J \times J)\right)$.

In the following, we consider that

$$(28) \qquad \left(T^*_{m,0} L_m\right)(x) = A_{m,k}(e_0) = 1,$$

for any $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, where $e_0 : I \to \mathbb{R}$, $e_0(x) = 1$ for any $x \in I$. From (28) it results immediately that

$$(29) \qquad \sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1$$

for any $x \in I$ and $m \in \mathbb{N}$.

In [30] are given the following results.

Theorem 1. *Let* $f : I \to \mathbb{R}$ *be a function. If* $x \in I \cap J$ *and* f *is a* s *times differentiable in* x *with* $f^{(s)}$ *continuous in* x*, then*

$$(30) \qquad \lim_{m \to \infty} m^{s - \alpha_s} \left[(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} \left(T^*_{m,i} L_m\right)(x) \right] = 0.$$

Assume that f *is* s *times differentiable function on* I*, with* $f^{(s)}$ *continuous on* I *and there exists an interval* $K \subset I \cap J$ *such that there exist* $m(s) \in \mathbb{N}$ *and* $k_j \in \mathbb{R}$ *depending on* K*, so that for any* $m \geq m(s)$ *and any* $x \in K$ *we have*

$$(31) \qquad \frac{\left(T^*_{m,j} L_m\right)(x)}{m^{\alpha_j}} \leq k_j$$

where $j \in \{s, s+2\}$*. Then the convergence given in* (30) *is uniform on* K *and*

$$(32) \qquad m^{s - \alpha_s} \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} \left(T^*_{m,i} L_m\right)(x) \right| \leq$$
$$\leq \frac{1}{s!} (k_s + k_{s+2}) \omega \left(f^{(s)}; \frac{1}{\sqrt{m^{2 + \alpha_s - \alpha_{s+2}}}} \right)$$

for any $x \in K$ *and* $m \geq m(s)$.

Remark 3. From (29) it results that $k_0 = 1$.

Theorem 2. *Let $f : I \to \mathbb{R}$ be a function. If $x \in I \cap J$ and f is continuous in x, then*

$$\lim_{m\to\infty} (L_m f)(x) = f(x). \tag{33}$$

Assume that f is continuous on I and there exists an interval $K \subset I \cap J$ such that there exist $m(0) \in \mathbb{N}$ and $k_2 \in \mathbb{R}$ depending on K, so that for any $m \geq m(0)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,2} L_m\right)(x)}{m^{\alpha_2}} \leq k_2. \tag{34}$$

Then the convergence given in (33) is uniform on K and

$$|(L_m f)(x) - f(x)| \leq (1 + k_2)\omega\left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{35}$$

for any $x \in K$ and any $m \in \mathbb{N}$, $m \geq m(0)$.

For the following results, see the paper [25] and [29].

Theorem 3. *Let $f : I \times I \to \mathbb{R}$ be a bivariate function. If $(x, y) \in (I \times I) \cap (J \times J)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x, y), then*

$$\lim_{m\to\infty} m^{s-\alpha_s}\Bigg[\left(L^*_{m,m} f\right)(x,y) - \sum_{i=0}^{s} \frac{1}{m^i i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)\left(T^*_{m,i-l} L^*_m\right)(x)\left(T^*_{m,l} L^*_m\right)(y)\Bigg] = 0. \tag{36}$$

If f admits partial derivatives of order s continuous on $(I \times I) \cap (J \times J)$ and there exists an interval $K \subset I \cap J$ such that there exist $m(s) \in \mathbb{N}$ and $k_{2l} \in \mathbb{R}$ depending on K, so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,2l} L^*_m\right)(x)}{m^{\alpha_{2l}}} \leq k_{2l} \tag{37}$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2} + 1\right\}$, then the convergence given in (36) is uniform on $K \times K$ and

$$m^{s-\alpha_s}\Bigg|\left(L^*_{m,m} f\right)(x,y) - \sum_{i=0}^{s} \frac{1}{m^i i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)\left(T^*_{m,i-l} L_m\right)(x)\left(T^*_{m,l} L_m\right)(y)\Bigg| \tag{38}$$

$$\leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left(k_{2l} + k_{2l+2}\right) \left(k_{s-2l} + k_{s-2l+2}\right)$$
$$\times \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \frac{1}{\sqrt{m^{\beta_s}}}, \frac{1}{\sqrt{m^{\beta_s}}} \right)$$

for any $(x, y) \in K \times K$*, any* $m \in \mathbb{N}$*,* $m \geq m(s)$*, where*

$$\beta_s = -\max\left\{\alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2, \frac{1}{2}(\alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4) : l \in \left\{0, 1, ..., \frac{s}{2}\right\}\right\}.$$

Theorem 4. *Let* $f : I \times I \to \mathbb{R}$ *be a bivariate function. If* $(x, y) \in (I \times I) \cap (J \times J)$ *and* f *is continuous in* (x, y)*, then*

$$\lim_{m \to \infty} \left(L^*_{m,m} f\right)(x, y) = f(x, y). \tag{39}$$

If f *is continuous on* $(I \times I) \cap (J \times J)$ *and there exists an interval* $K \subset I \cap J$ *such that there exist* $m(0) \in \mathbb{N}$ *and* $k_2 \in \mathbb{R}$ *depending on* K *so that for any* $m \in \mathbb{N}$*,* $m \geq m(0)$ *and any* $x \in K$ *we have*

$$\frac{\left(T^*_{m,2} L_m\right)(x)}{m^{\alpha_2}} \leq k_2, \tag{40}$$

then the convergence given in (39) *is uniform on* $K \times K$ *and*

$$\left|\left(L^*_{m,m} f\right)(x, y) - f(x, y)\right| \leq (1 + k_2)^2 \omega_{total} \left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}, \frac{1}{\sqrt{m^{2-\alpha_2}}}\right), \tag{41}$$

for any $(x, y) \in K \times K$*, any natural number* m*,* $m \geq m(0)$*.*

3. Main results

In this section, we study the GBS operators $\left(UL^*_{m,n}\right)_{m,n \geq 1}$ associated to the $\left(L^*_{m,n}\right)_{m,n \geq 1}$ operators.

Lemma. *If* $m, n \in \mathbb{N}$*, then* $UL^*_{m,n} : E(I \times I) \to F(J \times J)$ *have the form*

$$\left(UL^*_{m,n} f\right)(x, y) = (L_n f_x)(y) + (L_m f^y)(x) - \left(L^*_{m,n} f\right)(x, y) \tag{42}$$

for any $(x, y) \in J \times J$*, any* $f \in E(I \times I)$*.*

Proof. We have

$$\begin{aligned}\left(UL^*_{m,n}f\right)(x,y) &= \left(L^*_{m,n}(f(x,*)+f(\cdot,y)-f(\cdot,*)\right)(x,y)\\ &= \left(L^*_{m,n}f(x,*)\right)(x,y)+\left(L^*_{m,n}f(\cdot,y)\right)(x,y)-\left(L^*_{m,n}f\right)(x,y)\\ &= \sum_{k=0}^{p_m}\sum_{j=0}^{p_n}\varphi_{m,k}(x)\varphi_{n,j}(y)A_{m,n,k,j}(f_x)\\ &+ \sum_{k=0}^{p_m}\sum_{j=0}^{p_n}\varphi_{m,k}(x)\varphi_{n,j}(y)A_{m,n,k,j}(f^y)-\left(L^*_{m,n}f\right)(x,y)\end{aligned}$$

and taking (25), (26) into account, we obtain

$$\begin{aligned}\left(UL^*_{m,n}f\right)(x,y) &= \sum_{k=0}^{p_m}\sum_{j=0}^{p_n}\varphi_{m,k}(x)\varphi_{n,j}(y)A_{n,j}(f_x)\\ &+ \sum_{k=0}^{p_m}\sum_{j=0}^{p_n}\varphi_{m,k}(x)\varphi_{n,j}(y)A_{m,k}(f^y)-\left(L^*_{m,n}f\right)(x,y)\\ &= \left(\sum_{k=0}^{p_m}\varphi_{m,k}(x)\right)\left(\sum_{j=0}^{p_n}\varphi_{n,j}(y)A_{n,j}(f_x)\right)\\ &+ \left(\sum_{k=0}^{p_m}\varphi_{m,k}(x)A_{m,k}(f^y)\right)\left(\sum_{j=0}^{p_n}\varphi_{n,j}(y)\right)-\left(L^*_{m,n}f\right)(x,y).\end{aligned}$$

From (18) and (29), the relation (42) is obtained. ■

Theorem 5. *Let $f : I\times I\to\mathbb{R}$ be a bivariate function. If $(x,y)\in(I\times I)\cap(J\times J)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x,y), then*

$$\begin{aligned}(43)\quad \lim_{m\to\infty} m^{s-\alpha_s}\Bigg\{ &\left(UL^*_{m,m}f\right)(x,y)\\ -\sum_{i=0}^{s}\frac{1}{m^i i!}\Bigg[&\left(\frac{\partial^i f}{\partial\tau^i}(x,y)\left(T^*_{m,i}L_m\right)(y)+\frac{\partial^i f}{\partial t^i}(x,y)\left(T^*_{m,i}L_m\right)(x)\right)\\ -\sum_{l=0}^{i}\binom{i}{l}&\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T^*_{m,i-l}L_m\right)(x)\left(T^*_{m,l}L_m\right)(y)\Bigg]\Bigg\}=0.\end{aligned}$$

If f admits partial derivatives of order s continuous on $(I\times I)\cap(J\times J)$ and there exists an interval $K\subset I\cap J$ such that there exist $m(s)\in\mathbb{N}$ and

$k_{2l} \in \mathbb{R}$ *depending on* K*, so that for any* $m \in \mathbb{N}$*,* $m \geq m(s)$ *and any* $x \in K$ *we have*

$$\frac{\left(T^*_{m,2l}L_m\right)(x)}{m^{\alpha_{2l}}} \leq k_{2l}, \tag{44}$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2}+1\right\}$*, then the convergence given in* (43) *is uniform on* $K \times K$ *and*

(45)

$$\begin{aligned}
m^{s-\alpha_s}\Bigg|&(UL^*_{m,m}f)(x,y)\\
&-\sum_{i=0}^{s}\frac{1}{m^i i!}\left[\frac{\partial^i f}{\partial \tau^i}(x,y)(T^*_{m,i}L_m)(y)+\frac{\partial^i f}{\partial t^i}(x,y)(T^*_{m,i}L_m)(x)-\right.\\
&\left.-\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial \tau^i}(x,y)(T^*_{m,i-l}L_m)(x)(T^*_{m,l}L_m)(y)\right]\Bigg|\\
\leq \frac{1}{s!}\Bigg\{&(k_s+k_{s+2})\left[\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right)\right.\\
&\left.+\,\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right)\right]\\
&+\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(k_{2l}+k_{2l+2})(k_{s-2l}+k_{s-2l+2})\sum_{i=0}^{s}\binom{s}{i}\\
&\qquad\times\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i}\,\frac{1}{\sqrt{m^{\beta_s}}},\frac{1}{\sqrt{m^{\beta_s}}}\right)\Bigg\}\\
\leq \frac{1}{s!}\Bigg\{&(k_s+k_{s+2})\left[\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{\beta_s}}}\right)+\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{\beta_s}}}\right)\right]\\
&+\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(k_{2l}+k_{2l+2})(k_{s-2l}+k_{s-2l+2})\sum_{i=0}^{s}\binom{s}{i}\\
&\qquad\times\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m^{\beta_s}}},\frac{1}{\sqrt{m^{\beta_s}}}\right)\Bigg\}
\end{aligned}$$

for any $(x,y) \in K \times K$*, any* $m \in \mathbb{N}$*,* $m \geq m(s)$*, where*

$$\beta_s = -\max\left\{\alpha_{s-2l+2}+\alpha_{2l}-\alpha_s-2, \frac{1}{2}(\alpha_{s-2l+2}+\alpha_{2l+2}-\alpha_s-4) : l \in \left\{0,1,...,\frac{s}{2}\right\}\right\}.$$

Proof. We use the (30) relation from Theorem 1 for the functions f_x and f^y, the (36) relation from Theorem 3 for the function f and then we obtain the (43) relation. If we note by S the left member of (43) relation, we can write

$$S = m^{s-\alpha_s}\Bigg|\Bigg[(L_m f_x)(y) - \sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,y)\left(T^*_{m,i}L_m\right)(y)\Bigg] + \Bigg[(L_m f^y)(x) - \sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\left(T^*_{m,i}L_m\right)(x)\Bigg] + \Bigg[\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)\left(T^*_{m,i-l}L_m\right)(x)\left(T^*_{m,l}L_m\right)(y) - \left(L^*_{m,m}f\right)(x,y)\Bigg]\Bigg|$$

$$\leq m^{s-\alpha_s}\left|(L_m f_x)(y) - \sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,y)\left(T^*_{m,i}L_m\right)(y)\right| + m^{s-\alpha_s}\left|(L_m f^y)(x) - \sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\left(T^*_{m,i}L_m\right)(x)\right| + m^{s-\alpha_s}\left|\left(L^*_{m,m}f\right)(x,y) - \sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y) \times \left(T^*_{m,i-l}L_m\right)(x)\left(T^*_{m,l}L_m\right)(y)\right|$$

and taking (32), (38) relations into account we obtain the first inequality from (45). From hypothesis $\beta_s \geq -(\alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2)$ and if $l = 0$ we obtain that $\beta_s \geq \alpha_s + 2 - \alpha_{s+2}$. From the increasing monotony of the function ω, the second inequality from (45) results. From (45) the uniform convergence for (43) results. ■

Corollary 1. *Let $f : I \times I \to \mathbb{R}$ be a bivariate function. If $(x,y) \in (I \times I) \cap (J \times J)$ and f is continuous in (x,y), then*

$$\lim_{m\to\infty}\left(UL^*_{m,m}f\right)(x,y) = f(x,y). \tag{46}$$

Assume that f is continuous on $(I \times I) \cap (J \times J)$ and there exists an interval $K \subset I \cap J$ such that there exist $m(0) \in \mathbb{N}$ and $k_2 \in \mathbb{R}$ depending on K so that for any $m \in \mathbb{N}$, $m \geq m(0)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,2}L_m\right)(x)}{m^{\alpha_2}} \leq k_2. \tag{47}$$

Then the convergence given in (46) *is uniform on* $K \times K$ *and*

$$\left|\left(UL^*_{m,m}f\right)(x,y) - f(x,y)\right| \leq (1+k_2)\left[\omega\left(f_x;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right) + \omega\left(f^y;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\right] + (1+k_2)^2\omega_{total}\left(f;\frac{1}{\sqrt{m^{2-\alpha_2}}},\frac{1}{\sqrt{m^{2-\alpha_2}}}\right), \tag{48}$$

for any $(x,y) \in K \times K$ *and any* $m \in \mathbb{N}$, $m \geq m(0)$.

Proof. It results from Theorem 5 for $s = 0$ or from Theorem 2 and Theorem 4. ■

Corollary 2. *Let* $f : I \times I \to R$ *be a bivariate function. If* $(x,y) \in (I \times I) \cap (J \times J)$ *and* f *admits partial derivatives of second order continuous in a neighborhood of the point* (x,y), *then*

$$\lim_{m\to\infty} m^{2-\alpha_2}\left[\left(UL^*_{m,m}f\right)(x,y) - f(x,y) + \frac{1}{m^2}\frac{\partial^2 f}{\partial t\partial\tau}(x,y)\left(T^*_{m,1}L_m\right)(x)\left(T^*_{m,1}L_m\right)(y)\right] = 0. \tag{49}$$

If f *admits partial derivatives of second order continuous on* $(I \times I) \cap (J \times J)$ *and there exists an interval* $K \subset I \cap J$ *such that there exist* $m(2) \in \mathbb{N}$ *and* $k_{2l} \in \mathbb{R}$ *depending on* K, *so that for any* $m \in \mathbb{N}$, $m \geq m(2)$ *and any* $x \in K$ *we have*

$$\frac{\left(T^*_{m,2l}L_m\right)(x)}{m^{\alpha_{2l}}} \leq k_{2k}, \tag{50}$$

$l \in \{1,2\}$, *then the convergence given in* (49) *is uniform on* $K \times K$.

Proof. It results from Theorem 5 for $s = 2$. ■

In the following, by particularization and applying Theorem 5, Corollary 1 and Corollary 2 we can obtain Voronovskaja's type theorem and approximation theorem for some known operators. Because every application is a simple substitute in this theorem and corollaries of this section, we won't replace anything. In every application we have $\alpha_2 = 1$.

In Applications 1 - 3 let $p_m = m$, $\varphi_{m,k} = p_{m,k}$, where $k \in \{0,1,\ldots,m\}$, $m \in \mathbb{N}$ and $K = [0,1]$.

Application 1. Let $I = [0,\mu^{(\alpha,\beta)}]$, $J = [0,1]$, $E(I) = C([0,\mu^{(\alpha,\beta)}])$ and $F(J) = C([0,1])$. For any $m \in \mathbb{N}$, $m \geq m_0$, let $A_{m,k} : C([0,\mu^{(\alpha,\beta)}]) \to \mathbb{R}$,

$A_{m,k}(f) = f\left(\dfrac{k+\alpha}{m+\beta}\right)$ for any $f \in C([0, \mu^{(\alpha,\beta)}])$, any $k \in \{0, 1, \dots, m\}$.
In this case, we obtain the Stancu operators. We have $\left(T^*_{m,1} P^{(\alpha,\beta)}_m\right)(x) = \dfrac{m(\alpha - \beta x)}{m+\beta}$, $\left(T^*_{m,2} P^{(\alpha,\beta)}_m\right)(x) = \dfrac{m^2[mx(1-x) + (\alpha-\beta x)^2]}{(m+\beta)^2}$, for any $x \in [0,1]$, any $m \in \mathbb{N}$, $m \geq m_0$, $k_2 = \dfrac{5}{4}$ and $k_4 = 1$ (see [27]).

Application 2. If $I = J = [0,1]$, $E(I) = L_1([0,1])$, $F(J) = C([0,1])$, $A_{m,k}(f) = (m+1)\displaystyle\int_0^1 f(t)dt$, where $k \in \{0,1,\dots,m\}$, $m \in \mathbb{N}$ and $f \in L_1([0,1])$, then we obtain the Durrmeyer operators. We have $\left(T^*_{m,1} M_m\right)(x) = \dfrac{m(1-2x)}{m+2}$, $\left(T^*_{m,2} M_m\right)(x) = m^2 \dfrac{2(m-3)x(1-x)+2}{(m+2)(m+3)}$ for any $x \in [0,1]$, any $m \in \mathbb{N}$, $k_2 = \dfrac{3}{2}$ and $k_4 = \dfrac{7}{4}$ (see [21]).

Application 3. We consider $I = J = [0,1]$, $E(I) = L_1([0,1])$, $F(J) = C([0,1])$, $A_{m,k}(f) = (m+1)\displaystyle\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt$, where $k \in \{0,1,\dots,m\}$, $m \in \mathbb{N}$ and $f \in L_1([0,1])$.
In this case, we obtain the Kantorovich operators. We have $\left(T^*_{m,1} K_m\right)(x) = \dfrac{m}{2(m+1)}(1-2x)$, $\left(T^*_{m,2} K_m\right)(x) = \left(\dfrac{m}{m+1}\right)^2 \dfrac{(1-x)^3 + x^3 + 3mx(1-x)}{3}$ for any $x \in [0,1]$, any $m \in \mathbb{N}$, $k_2 = 1$ and $k_4 = \dfrac{3}{2}$ (see [21]).

Application 4. In this application $I = J = [0,\infty)$, $E(I) = F(J) = C_B([0,\infty))$, $K = [0,b]$, $b > 0$, $p_m = m$, $\varphi_{m,k}(x) = \dbinom{m}{k} \dfrac{x^k}{(1+x)^m}$, $A_{m,k}(f) = f\left(\dfrac{k}{m+1-k}\right)$ for any $x \in [0,\infty)$, $k \in \{0,1,\dots,m\}$ and $m \in \mathbb{N}$. We obtain the Bleimann, Butzer and Hahn operators. We have $\left(T^*_{m,1} L_m\right)(x) = -mx\left(\dfrac{x}{1+x}\right)^m$, $x \in [0,\infty)$, $m \in \mathbb{N}$, $k_2 = 4b(1+b)^2$ if $x \in [0,b]$ and $m \in \mathbb{N}$, $m \geq 24(1+b)$ (see [28]).

In Application 5 - 8 let $p_m = \infty$, for any $m \in \mathbb{N}$, in Application 5 - 6 and Application 8 let $K = [0,b]$, $b > 0$.

Application 5. We consider $I = J = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = e^{-mx}\dfrac{(mx)^k}{k!}$, $A_{m,k}(f) = f\left(\dfrac{k}{m}\right)$ for any $x \in [0,\infty)$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C_2([0,\infty))$. In this application we obtain the Mirakjan-Favard-Szász operators. We have $\left(T^*_{m,1}S_m\right)(x) = 0$, $\left(T^*_{m,2}S_m\right)(x) = mx$, $x \in [0,\infty)$, $m \in \mathbb{N}$, $k_2 = b$ and $k_4 = 3b^2 + b$ (see [23]).

Application 6. Let $I = J = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(J) = [0,\infty)$, $\varphi_{m,k}(x) = (1+x)^{-m}\dbinom{m+k-1}{k}\left(\dfrac{x}{1+x}\right)^k$, $A_{m,k}(f) = f\left(\dfrac{k}{m}\right)$ for any $x \in [0,\infty)$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C_2([0,\infty))$. In this case we obtain the Baskakov operators. We have $\left(T^*_{m,1}V_m\right)(x) = 0$, $\left(T^*_{m,2}V_m\right)(x) = mx(1+x)$, $x \in [0,\infty)$, $m \in \mathbb{N}$, $k_2 = b(1+b)$ and $k_4 = 9b^4 + 18b^3 + 10b^2 + b$ (see [23]).

Application 7. In this application $I = J = [0,1]$, $E(I) = F(J) = C([0,1])$, $K = [0,1]$, $\varphi_{m,k}(x) = \dbinom{m+k}{k}(1-x)^{m+1}x^k$, $A_{m,k}(f) = f\left(\dfrac{k}{m+k}\right)$ for any $x \in [0,1]$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C([0,1])$. We obtain the Meyer-König and Zeller operators. We have $\left(T^*_{m,1}Z_m\right)(x) = 0$, $x \in [0,1]$, $m \in \mathbb{N}$ and $k_2 = 2$ (see [23]).

Application 8. We consider $I = J = [0,\infty)$, $E(I) = F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = e^{-\frac{(m+k)x}{1+x}}\dfrac{m(m+k)^{k-1}}{k!}\left(\dfrac{x}{1+x}\right)^k$, $A_{m,k}(f) = f\left(\dfrac{k}{m}\right)$ for any $x \in [0,\infty)$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C([0,\infty))$. In this application we obtain the Ismail-May operators. We have $\left(T^*_{m,1}R_m\right)(x) = A_{m,1}(x) = 0$, $\left(T^*_{m,2}R_m\right)(x) = A_{m,2}(x) = mx(1+x)^2$ for any $x \in [0,\infty)$, $m \in \mathbb{N}$ and $k_2 = b(1+b)^2$ (see [33]).

For the operators we shape in this paper, we have $\lim\limits_{m\to\infty}\left(T^*_{m,1}L_m\right)(x) = 0$, $x \in I$ or $\lim\limits_{m\to\infty}\left(T^*_{m,1}L_m\right)(x) = B(x)$, $x \in I$, where $B(x)$ is bounded on I. It results that $\lim\limits_{m\to\infty}\dfrac{1}{m}\left(T^*_{m,1}L_m\right)(x)\left(T^*_{m,1}L_m\right)(y) = 0$ uniform on $(I\times I)\cap(J\times J)$ and then the Corollary 2 can be reformulated through Corollary 3 for the operators studied in this paper.

Corollary 3. *Let $f : I\times I \to \mathbb{R}$ be a bivariate function. If $(x,y) \in (I\times I)\cap(J\times J)$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then*

$$\lim_{m\to\infty} m\left[\left(UL^*_{m,m}f\right)(x,y) - f(x,y)\right] = 0. \tag{51}$$

If f admits partial derivatives of second order continuous on $(I\times I)\cap(J\times J)$ and there exists an interval $K \subset I\cap J$ such that there exist $m(2) \in \mathbb{N}$ and

$k_{2l} \in \mathbb{R}$ depending on K, so that for any $m \in \mathbb{N}$, $m \geq m(2)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,2l} L_m\right)(x)}{m^{\alpha_{2l}}} \leq k_{2l}, \tag{52}$$

$l \in \{1,2\}$, then the convergence given in (51) is uniform on $K \times K$.

Now we give an example.

For $m \in \mathbb{N}$, let the operators $\mathcal{O}_m : C([0,2]) \to C([0,1])$ defined for any $x \in [0,1]$ and any function $f \in C([0,2])$ by

$$(\mathcal{O}_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m} + \frac{1}{\sqrt{m}}\right). \tag{53}$$

One verifies immediately that $(\mathcal{O}_m e_0)(x) = 1$, $(\mathcal{O}_m e_1)(x) = x + \dfrac{1}{\sqrt{m}}$, $(\mathcal{O}_m e_2)(x) = x^2 + \dfrac{x(1-x)}{m} + \dfrac{2}{\sqrt{m}} x + \dfrac{1}{m}$, $\left(T^*_{m,1} \mathcal{O}_m\right)(x) = \sqrt{m}$ and $\left(T^*_{m,2} \mathcal{O}_m\right)(x) = m[x(1-x)+1]$, where $x \in [0,1]$ and $m \in \mathbb{N}$. Then, from the Corollary 2 we obtain the following proposition for the $(\mathcal{O}_m)_{m \geq 1}$ operators.

Proposition 3. *Let $f : [0,2] \times [0,2] \to \mathbb{R}$ be a bivariate function. If $(x,y) \in [0,1] \times [0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then*

$$\lim_{m \to \infty} m \left[\left(U\mathcal{O}^*_{m,m} f\right)(x,y) - f(x,y)\right] = \frac{\partial^2 f}{\partial t \partial \tau}(x,y). \tag{54}$$

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OVIDIU T. POP
NATIONAL COLLEGE "MIHAI EMINESCU"
5 MIHAI EMINESCU STREET
SATU MARE 440014, ROMANIA
e-mail: ovidiutiberiu@yahoo.com

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