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OSCILLATION AND NON-OSCILLATION CRITERIA FOR A NEUTRAL DELAY DIFFERENCE EQUATION OF FIRST ORDER

Abstract. In this paper, necessary and sufficient condition are obtained so that every bounded solution of

$$\Delta(y_n - y_{n-k}) + q_n G(y_{\sigma(n)}) = 0$$

is oscillatory, under a condition weaker than $\sum_{n=n_0}^{\infty} q_n = \infty$.

Key words: oscillatory solution, nonoscillatory solution, asymptotic behaviour, difference equation.

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1. Introduction

In this paper, necessary and sufficient conditions are obtained so that every bounded solution of

$$\Delta(y_n - y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = 0 \tag{1}$$

is oscillatory, where Δ is the forward difference operator given by $\Delta y_n = y_{n+1} - y_n$, $\{q_n\}$ are assumed to be infinite sequences of real numbers with $q_n \geq 0$, but $\not\equiv 0$. We assume $\tau(n)$, $\sigma(n)$ are unbounded increasing sequence of integers less than n and $G \in C(R, R)$. Further in this work we assume that $xG(x) > 0$ for $x \neq 0$ and G is non-decreasing.

All over the world, during the last decade or two a lot of research activity is undertaken on the study of the oscillation of neutral delay difference equations(NDDEs in short). For recent results and references see the monograph by Agarwal[1] and the papers [2, 3, 5, 9, 10] and [12]–[20] and the references cited there in. In these papers the authors have studied the oscillation and non-oscillation of solutions of the NDDE

$$\Delta(y_n - p_n y_{n-k}) + q_n G(y_{n-r}) = f_n \tag{2}$$

under the condition

$$\sum_{n=n_0}^{\infty} q_n = \infty. \tag{3}$$

However, in this work we find a necessary and sufficient condition for the oscillation of all bounded solutions of (1) under a condition weaker than (3). Thus our results improve the following Theorems, which are particular cases of some of the results in [13, 16, 20].

Theorem 1 ([20], Theorem 3.5). *Suppose* (3) *hold.Then every bounded solution of*

$$\Delta(y_n - y_{n-k}) + q_n G(y_{n-r}) = 0 \tag{4}$$

oscillates.

Theorem 2 ([13], Theorem 2.3). *Suppose that there exists a sequence* $\{F_n\}$ *such that* $F_n \to 0$ *as* $n \to \infty$ *and* $\Delta F_n = f_n$. *Further assume that*

$$\sum_{j=0}^{\infty} q_{n_j} = \infty, \quad \textit{for every subsequence } n_j \textit{ of } n. \tag{5}$$

Then every bounded solution of (4) *oscillates or tends to zero as* $n \to \infty$.

Theorem 3 ([16], Theorem 3.3). *Suppose that* $G(u)/u > \gamma > 0$. *Assume that*

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} q_i > \frac{k^{k+1}}{\gamma(k+1)^{k+1}}. \tag{6}$$

Then every bounded solution of (4) *oscillates or tends to zero as* $n \to \infty$.

Let n_0 be a fixed nonnegative integer. Let $\rho = \min\{\tau(n_0), \sigma(n_0)\}$. By a solution of (1) we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geq \rho$ and satisfies (1) for $n \geq n_0$. Clearly if the initial condition

$$y_n = a_n \quad \text{for} \quad \rho \leq n \leq n_0 \tag{7}$$

is given then the equation (1) has a unique solution satisfying the given initial condition (7). A solution $\{y_n\}$ of (1) is said to be oscillatory if for every positive integer $n_0 > 0$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

2. Main results

In this section we prove that every bounded solution of

$$\Delta(y_n - y_{n-k}) + q_n G(y_{\sigma(n)}) = 0 \tag{8}$$

oscillates if

$$\sum_{n=n_0}^{\infty} n q_n = \infty. \tag{9}$$

We need the following Lemma,which generalizes the Lemma [12, Lemma2.1] and can be easily proved.

Lemma 1. *Let $\{f_n\}$ and $\{g_n\}$ be sequences of real numbers for $n \geq 0$ such that*

$$f_n = g_n - p g_{\tau(n)}, \quad n \geq n_0$$

where $p \in \mathbb{R}$, $p \neq 1$ and $\tau(n) \leq n, \forall n$, with $\lim_{n\to\infty} \tau(n) = \infty$. Suppose that $\lim\limits_{n\to\infty} f_n = \lambda \in \mathbb{R}$ exists. Then the following statements hold.

(i) If $\liminf\limits_{n\to\infty} g_n = a \in \mathbb{R}$ then $\lambda = (1-p)a$.

(ii) If $\limsup\limits_{n\to\infty} g_n = b \in \mathbb{R}$, then $\lambda = (1-p)b$.

Theorem 4. *Suppose that $f_n \leq 0$ for every n and*

$$\left| \sum_{i=1}^{\infty} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} f_j \right| < \infty. \tag{10}$$

holds. Then the neutral equation

$$\Delta(y_n - y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n \tag{11}$$

admits a positive bounded solution if and only if

$$\sum_{i=1}^{\infty} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} q_j < \infty, \tag{12}$$

holds.

Proof. Suppose that (11) admits a positive bounded solution. From (10), we obtain

$$\left| \sum_{j=\tau_{-1}^{i}(n_0)}^{\infty} f_j \right| < \infty, \text{ for every } i \geq 0.$$

Hence $\left|\sum_{j=n_0}^{\infty} f_j\right| < \infty$. If we set $F_n = -\sum_{j=n}^{\infty} f_j$ for $n \geq n_0$, then $\Delta F_n = f_n$ and

$$F_n \geq 0 \text{ and } \lim_{n\to\infty} F_n = 0. \tag{13}$$

Let $\{y_n\}$ be a positive bounded solution of (11) such that $y_n > 0$, $y_{\tau(n)}$ and $y_{\sigma(n)} > 0$ for $n \geq n_0$. Setting

$$z_n = y_n - y_{\tau(n)} \text{ and } w_n = z_n - F_n \tag{14}$$

for $n \geq n_0$, we obtain

$$\Delta w_n = -q_n G(y_{\sigma(n)}) \leq 0. \tag{15}$$

Since w_n is bounded,then $\lim_{n\to\infty} w_n = l$ exists. From (13) and (14), we obtain $\lim_{n\to\infty} z_n = l$. From Lemma 1, it follows that $l = 0$. Hence $w_n > 0$ for $n \geq n_1 \geq n_0$ as it is decreasing. From (14) we obtain $y_n > y_{\tau(n)}$ for $n \geq n_1$ because $F_n > 0$. This implies $\liminf_{n\to\infty} y_n > 0$. Thus there exists $\gamma > 0$ such that $y_n > \gamma$ for $n \geq n_2 \geq n_1$. Summing (15) from n to ∞, we obtain for $n \geq n_3 \geq n_2$,

$$w_n = \sum_{i=n}^{\infty} q_i G(y_{\sigma(i)}).$$

That is,

$$y_{\tau(n)} < y_n - G(\gamma)\sum_{i=n}^{\infty} q_i + \sum_{i=n}^{\infty} f_i. \tag{16}$$

Replacing n by $\tau_{-1}(n)$ in (16), we get

$$y_n < y_{\tau_{-1}(n)} - G(\gamma) \sum_{i=\tau_{-1}(n)}^{\infty} q_i + \sum_{i=\tau_{-1}(n)}^{\infty} f_i. \tag{17}$$

From (16) and (17) it follows that,

$$y_{\tau(n)} < y_{\tau_{-1}(n)} - G(\gamma)\sum_{i=0}^{1} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} q_j + \sum_{i=0}^{1} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} f_j. \tag{18}$$

Hence repeating the above process k times, we obtain

$$y_{\tau(n)} < y_{\tau_{-1}^{k}(n)} - G(\gamma)\sum_{i=0}^{k} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} q_j + \sum_{i=0}^{k} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} f_j. \tag{19}$$

Hence

$$(20) \qquad G(\gamma)\sum_{i=0}^{k}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} q_j < y_{\tau_{-1}^{k}(n)} - y_{\tau(n)} + \sum_{i=0}^{k}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} f_j.$$

Taking limit $k \to \infty$ and using (10) and that y_n is bounded, we obtain (12). Conversely,if (12) holds then Let $\mu = \max\{|G(x)| : 2 \leq x \leq 6\}$. Then from (12), and (10) one can find $N_1 > 0$ such that for $n \geq N_1$ we have

$$\mu\sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} q_j < 1,$$

and

$$\left|\sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} f_j\right| < 1.$$

Let

$$S = \{y \in X : 2 \leq y_n \leq 6, n \geq N_1\}.$$

Choose $N_2 > N_1$ such that $k \geq N_1$, where $k = \min\{\tau(N_2), \sigma(N_2)\}$. Then define the mapping

$$(21) \qquad (By)_n = \begin{cases} (By)_{N_2}, & N_1 \leq n \leq N_2, \\ 4 - \sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} q_j G(y_{\sigma(j)}) & \\ \quad + \sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} f_j, & n \geq N_2. \end{cases}$$

Then we may easily establish that (i) $By \in S$ for $y \in S$ (ii) BS is relatively compact. Then by [6] **Schauder's Fixed Point Theorem:** there is a fixed point y^0 in S such that $By_n^0 = y_n^0$. For $n \geq N_2$,writing y_n for y_n^0 we obtain

$$y_n = 4 - \sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} q_j G(y_{\sigma(j)}) + \sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty} f_j.$$

Then

$$y_n - y_{\tau(n)} = \sum_{j=n}^{\infty} q_j G(y_{\sigma(j)} - \sum_{j=n}^{\infty} f_j.$$

Then applying Δ bothsides we find that y_n is the required positive and bounded solution of (11) for $n \geq N_2$. Hence the theorem is proved. ■

Corollary. *Assume that* $f_n \geq 0$ *for every* n, *and* (10) *holds. The neutral equation* (11) *admits a negative bounded solution if and only if* (12) *holds.*

Proof. The proof is similar to the proof of the above theorem, hence omitted. ■

Theorem 5. *Every bounded solution of* (1) *oscillates if and only if* (12) *holds.*

Proof. It follows from the proof of the Theorem 4 for $f_n \equiv 0$. ■

Next, our objective is to show that the conditions (12) and (9) are equivalent. For that we need the following definition and there after a useful lemma.

Definition. *Define the factorial function(See[8, page-20]) by*

$$n^{(k)} := n(n-1)\dots(n-k+1),$$

where $k \leq n$ and $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Note that $n^{(k)} = 0$, if $k > n$.

Then we have

$$\Delta n^{(k)} = kn^{(k-1)}, \tag{22}$$

where $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and Δ is the forward difference operator. One can show, by summing up (22) that

$$\sum_{i=m}^{n-1} i^{(k)} = \frac{1}{k+1}\left(n^{(k+1)} - m^{(k+1)}\right) \tag{23}$$

holds. Now set

$$b_k(n,m) := \begin{cases} 1, & k = 0 \\ \sum\limits_{j=m}^{n} b_{k-1}(n,j), & k \in \mathbb{N}. \end{cases} \tag{24}$$

Here, we evaluate b_k by recursion. Clearly, for $k = 1$ in (24), we have

$$b_1(n,m) = \sum_{j=m}^{n} b_0(n,j) = \sum_{j=m}^{n} 1 = (n+1-m) = (n+1-m)^{(1)}.$$

By (23) and for $k = 2$ in (24), we get

$$\begin{aligned} b_2(n,m) &= \sum_{j=m}^{n} b_1(n,j) = \sum_{j=m}^{n} (n+1-j)^{(1)} \\ &= \sum_{i=1}^{n+1-m} i^{(1)} = \frac{1}{2}(n+2-m)^{(2)} - \frac{1}{2}1^{(2)} = \frac{1}{2}(n+2-m)^{(2)}. \end{aligned}$$

Note that $1^{(2)} = 0$. By (23) and for $k = 3$ in (24), we get

$$\begin{aligned} b_3(n,m) &= \sum_{j=m}^{n} b_2(n,j) = \frac{1}{2}\sum_{j=m}^{n}(n+2-j)^{(2)} \\ &= \frac{1}{2}\sum_{i=2}^{n+2-m} i^{(2)} = \frac{1}{6}\left[(n+3-m)^{(3)} - 2^{(3)}\right] = \frac{1}{3!}(n+3-m)^{(3)}. \end{aligned}$$

Using a simple induction, we obtain

$$b_k(n,m) = \frac{1}{k!}(n+k-m)^{(k)}. \tag{25}$$

Lemma 2. *Let $p \in \mathbb{N}$ and $x(n)$ be a non oscillatory sequence which is positive for large n. If there exists an integer $p_0 \in \{0,1,\ldots,p-1\}$ such that $\Delta^{p_0}w(\infty)$ exits(finite) and $\Delta^i w(\infty) = 0$ for all $i \in \{p_0+1,\ldots,p-1\}$. Then*

$$\Delta^p w(n) = -x(n) \tag{26}$$

implies

$$\begin{aligned} \Delta^{p_0}w(n) &= \Delta^{p_0}w(\infty) \\ &\quad + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!}\sum_{i=n}^{\infty}(i+p-p_0-1-n)^{(p-p_0-1)}x(i). \end{aligned} \tag{27}$$

for all sufficiently large n.

Proof. Summing up (26) from n to ∞, we get

$$\Delta^{p-1}w(\infty) - \Delta^{p-1}w(n) = -\sum_{i=n}^{\infty}x(i)$$

or simply

$$\Delta^{p-1}w(n) = \sum_{i=n}^{\infty}x(i) = \sum_{i=n}^{\infty}b_0(i,n)x(i). \tag{28}$$

Summing up (28) from n to ∞, we get

$$\begin{aligned} \Delta^{p-2}w(n) &= \Delta^{p-2}w(\infty) - \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}b_0(j,i)x(j) \\ &= -\sum_{j=n}^{\infty}\sum_{i=n}^{j}b_0(j,i)x(j) \\ &= -\sum_{j=n}^{\infty}b_1(j,n)x(j) = -\sum_{i=n}^{\infty}b_1(i,n)x(i). \end{aligned} \tag{29}$$

Again summing up (29) from n to ∞, we obtain

$$\Delta^{p-3} w(n) = \sum_{j=n}^{\infty} \sum_{i=j}^{\infty} b_1(i,j)\, x(i) = \sum_{i=n}^{\infty} \sum_{j=n}^{i} b_1(i,j)\, x(i)$$
$$= \sum_{i=n}^{\infty} b_2(i,n)\, x(i).$$

By the emerging pattern, we have

$$\Delta^{j} w(n) = (-1)^{p-j-1} \sum_{i=n}^{\infty} b_{p-j-1}(i,n)\, x(i), \quad j \in \{p_0+1, \ldots p-1\}.$$

Then by letting $j = p_0 + 1$, we get

$$\Delta^{p_0+1} w(n) = (-1)^{p-p_0-2} \sum_{i=n}^{\infty} b_{p-p_0-2}(i,n)\, x(i). \tag{30}$$

Summing up (30) from n to ∞ and arranging we get

$$\Delta^{p_0} w(n) = \Delta^{p_0} w(\infty) + (-1)^{p-p_0-1} \sum_{i=n}^{\infty} b_{p-p_0-1}(i,n)\, x(i). \tag{31}$$

From (25) and (31) it follows that

$$\Delta^{p_0} w(n) = \Delta^{p_0} w(\infty) + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!} \sum_{i=n}^{\infty} (i+p-p_0-1-n)^{(p-p_0-1)}\, x(i).$$

Hence the Lemma is proved. ∎

Theorem 6. *Consider the delay difference equation*

$$\Delta^2 x_n + q_n x_{\sigma(n)} = 0. \tag{32}$$

Then the following conditions are equivalent.

(a) *every solution of* (32) *oscillates.*

(b) *The condition* (9) *holds.*

(c) $\sum_{i=0}^{\infty} \sum_{j=n_0+ik}^{\infty} q_j = \infty$, *for any fixed positive integer k and $n_0 > 0$.*

Proof. We show that $(a) \Leftrightarrow (c)$ and $(a) \Leftrightarrow (b)$. Hence $(b) \Leftrightarrow (c)$. First let us prove $(a) \Leftrightarrow (c)$. Suppose that (a) holds.For the sake of contradiction, assume that (c) does not hold. Then

$$\sum_{i=0}^{\infty} \sum_{j=n_0+ik}^{\infty} q_j < \infty.$$

Hence we can find an integer $n_1 > 0$ such that

$$k\sum_{i=n_1}^{\infty}\sum_{j=n_0+ik}^{\infty} q_j < 1/3. \tag{33}$$

Let $n_2 = n_0 + n_1 k$. Then from (33), we obtain

$$k\sum_{i=0}^{\infty}\sum_{j=n+ik}^{\infty} q_j < 1/3 \quad \text{for} \quad n \geq n_2. \tag{34}$$

Choose $N_0 \geq n_2$ and $N_1 > N_0$ such that $\sigma(N_1) \geq N_0$. Let $X = l_\infty^{N_0}$ be the Banach space of bounded real sequences $x = \{x_n\}, n \geq N_0$ with supremum norm $||x|| = \sup\{|x_n| : n \geq N_0\}$. Define S to be a closed subset of X such that $S = \{y \in X : 1 \leq y_n \leq 3/2, n \geq N_0\}$. Then S is a metric space, where the metric is induced by the norm on X. For $x \in S$, define

$$Ax_n = \begin{cases} 1, & N_0 \leq n \leq N_1, \\ 1 + \sum\limits_{i=N_1}^{n-1}\sum\limits_{j=i}^{\infty} q_j x_{\sigma(j)}, & n \geq N_1. \end{cases}$$

Then for $n \geq N_0$,

$$\begin{aligned} 1 \leq Ax_n &< 1 + \sum_{i=N_1}^{\infty}\sum_{j=i}^{\infty} q_j x_{\sigma(j)} \\ &\leq 1 + \sum_{p=0}^{\infty}\sum_{i=N_1+pk}^{N_1+pk+k-1}\sum_{j=i}^{\infty} q_j x_{\sigma(j)} \\ &\leq 1 + k\sum_{p=0}^{\infty}\sum_{j=N_1+pk}^{\infty} q_j x_{\sigma(j)} \\ & 1 + \frac{3}{2}k\sum_{p=0}^{\infty}\sum_{j=N_1+pk}^{\infty} q_j \leq 1 + 1/2 \leq 3/2. \end{aligned}$$

Hence $AS \subset S$. Further, it may be shown that, for $x, y \in S$, $||Ax - Ay|| \leq \frac{1}{3}||x - y||$. Hence A is a contraction. Consequently A has a unique fixed point x in S. It is a positive bounded solution of (32) for $n \geq N_2$, a contradiction. Hence $(a) \Rightarrow (c)$ holds.

Next, suppose that (c) holds.Let $x = \{x_n\}$ be a bounded non-oscillatory solution of (32). We may take, with out any loss of generality $x_n > 0$, $x_{\sigma(n)} > 0$ for $n \geq n_0 > 0$. Then $\Delta^2 x_n \leq 0$ for $n \geq n_1 \geq n_0$. Hence x_n and Δx_n are monotonic and is of constant sign for $n \geq n_2 \geq n_1$. Since

x_n is bounded, $\Delta x_n > 0$ and $\lim_{n\to\infty} x_n = l > 0$ and $\lim_{n\to\infty} \Delta x_n = 0$. Let $x_n > \alpha > 0$ for $n \geq n_3 \geq n_2$. Summing (32) from n to ∞, we obtain $\Delta x_n = \sum_{i=n}^{\infty} q_i x_{\sigma(i)}$. Hence

$$\sum_{p=0}^{j} \sum_{n=n_3+pk}^{n_3+pk+k-1} \Delta x_n = \sum_{p=0}^{j} \sum_{n=n_3+pk}^{n_3+pk+k-1} \sum_{i=n}^{\infty} q_i x_{\sigma(i)}. \tag{35}$$

This implies

$$\begin{aligned} x_{n_3+(j+1)k} - x_{n_3} &= \sum_{p=0}^{j} \sum_{n=n_3+pk}^{n_3+pk+k-1} \sum_{i=n}^{\infty} q_i x_{\sigma(i)} \\ &\geq \alpha k \sum_{p=0}^{j} \sum_{i=n_3+pk+k-1}^{\infty} q_i. \end{aligned}$$

Since $\{x_n\}$ is bounded, $\sum_{p=0}^{\infty} \sum_{i=n_3+pk+k-1}^{\infty} q_i < \infty$, a contradiction. Hence $(c) \Rightarrow (a)$ is proved.

Next, to show $(a) \Rightarrow (b)$. Suppose that (a) holds.For the sake of contradiction, assume (b) does not hold.That is $\sum_{i=0}^{\infty} iq_i < \infty$. Hence for any $n \geq n_0$, we have $\sum_{i=n}^{\infty}(i-n+1)q_i < \infty$. Then proceeding as in the proof of the case $(a) \Rightarrow (c)$, we find N_0 such that $n \geq N_0$ implies $\sum_{i=n}^{\infty}(i-n+1)q_i < 1/4$. Let $N_1 > N_0$ such that $\sigma(N_1) \geq N_0$. Set $S = \{x_n \in X : 3/4 \leq x_n \leq 1, n \geq N_0\}$ and for $x \in S$,

$$Ax_n = \begin{cases} Ax_{N_1}, & N_0 \leq n \leq N_1 \\ 1 - \sum\limits_{i=n}^{\infty} (i-n+1) q_i x_{\sigma(i)}, & n \geq N_1. \end{cases}$$

Clearly, $3/4 \leq A(x_n) \leq 1$. Hence $A(S) \subset S$ and A is a contraction. Hence A has a unique fixed point in S which is a positive bounded solution of (32), a contradiction. Hence (a) implies (b).

Next, suppose (b) holds. Let $\{x_n\}$ be a bounded non-oscillatory solution of (32) for $n \geq n_0 > 0$. Proceeding as in the proof of the case $(c) \Rightarrow (a)$, we obtain $\lim_{n\to\infty} x_n = l > 0$ exists and $\lim_{n\to\infty} \Delta x_n = 0$. From (32), using Lemma 2 for $p = 2$ and $p^* = 0$, we get $x_n = l - \sum_{i=n}^{\infty}(i-n+1)q_i x_{\sigma(i)}$. This implies $\sum_{i=n}^{\infty}(i-n+1)q_i x_{\sigma(i)} < \infty$. On the other hand,

$$\sum_{i=n}^{\infty}(i-n+1)q_i x_{\sigma(i)} > \frac{l}{2}\sum_{i=n}^{\infty}(i-n+1)q_i = \infty.$$

a contradiction. Hence $(b) \Rightarrow (a)$. Thus the theorem is completely proved. ■

Remark 1. In the condition (12), if we substitute $\tau(n) = n - k$ i.e $\tau_{-1}(n) = n + k$ and $\tau_{-1}^{i}(n) = n + ik$, then from Theorem 6, it follows that (12) $\Leftrightarrow$ (9).

From Remark 1 and Theorem 5 the following result follows.

Theorem 7. *Every bounded solution of* (8) *oscillates if and only if* (9) *holds.*

Remark 2. The above theorem improves Theorems 1, 2 and 3.

The following example illustrates the above Theorem 7.

Example. Consider the neutral difference equation

$$\Delta(y_n - y_{n-1}) + \frac{(n-2)(4n^6 + 6n^2 - 2)}{n^3(n-1)^3(n+1)^3} y_{n-2}^{1/3} = 0, \quad n \geq 1. \tag{36}$$

Here, $q_n \approx \frac{1}{n^2}$. Hence $\sum_{n=1}^{\infty} q_n < \infty$. However, $\sum_{n=1}^{\infty} nq_n = \infty$. Hence all the conditions of Theorem 7 are satisfied. Hence all it's solutions are oscillatory. In particular $y_n = \frac{(-1)^n}{n^3}$ is an oscillatory solution of (36). But the results of [13, 16, 20], i.e. Theorems 1, 2 and 3 fail to give any conclusion, because (3) is not satisfied. Note that both the conditions (5) and (6) required for the Theorems 2 and 3 respectively, independently impliy (3).

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