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Binod C. Tripathy and Hemen Dutta

SOME DIFFERENCE PARANORMED SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

Abstract. In this paper we introduce the difference paranormed sequence spaces $c_0(M, \Delta_m^n, p)$, $c(M, \Delta_m^n, p)$ and $\ell_\infty(M, \Delta_m^n, p)$ respectively. We study their different properties like completeness, solidity, monotonicity, symmetricity etc. We also obtain some relations between these spaces as well as prove some inclusion results.

Key words: difference sequence, Orlicz function, paranormed space, completeness, solidity, symmetricity, convergence free, monotone space.

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1. Introduction

Throughout the paper w, ℓ_∞, c and c_0 denote the spaces of *all*, *bounded*, *convergent* and *null* sequences $x = (x_k)$ with complex terms respectively. The zero sequence is denoted by θ=(0, 0, . . .).

The notion of difference sequence space was introduced by Kizmaz [2], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [1] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [13], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$. Tripathy, Esi and Tripathy [14] generalized the above notions and unified these as follows:

Let m, n be non-negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv} \, .$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [1]. Taking n=1, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [13]. Taking m=n=1, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [2].

An Orlicz function is a function M:$[0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0)$=0, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [5] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \{(x_k) \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \quad \text{for some } \ \rho > 0\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|(x_k)\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}.$$

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, $D = \max\{1, 2^{G-1}\}$. Then for all a_k, $b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}).$$

The studies on paranormed sequence spaces were initiated by Nakano [8] and Simons [11]. Later on it was further studied by Maddox [6], Nanda [9], Lascarides [3], Lascarides and Maddox [4], Tripathy and Sen [15] and many others. Parasar and Choudhary [10], Mursaleen, Khan and Qamaruddin [7] and many others studied paranormed sequence spaces using Orlicz functions.

2. Definitions and preliminaries

A sequence space E is said to be *solid* (*or normal*) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space E is said to be *monotone* if it contains the canonical preimages of all its step spaces.

A sequence space E is said to be *symmetric* if $(x_{\pi(k)}) \in E$, where π is a permutation on N.

A sequence space E is said to be *convergence free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $y_k = 0$ whenever $x_k = 0$.

A sequence space E is said to be a *sequence algebra* if $(x_k y_k) \in E$ whenever $(x_k) \in E$ and $(y_k) \in E$.

Let $p = (p_k)$ be any bounded sequence of positive real numbers. Then we define the following sequence spaces for an Orlicz function M:

$c_0(M, \Delta_m^n, p) = \{x = (x_k) : \lim_{k\to\infty} (M(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} = 0, \text{ for some } \rho > 0\}$,

$c(M, \Delta_m^n, p) = \{x = (x_k) : \lim_{k\to\infty} (M(\frac{|\Delta_m^n x_k - L|}{\rho}))^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } L \in C\}$,

$\ell_\infty(M, \Delta_m^n, p) = \{x = (x_k) : \sup_{k\geq 1}(M(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} < \infty, \text{ for some } \rho > 0\}$,

when $p_k = p$, a constant, for all k, then $c_0(M, \Delta_m^n, p) = c_0(M, \Delta_m^n)$, $c(M, \Delta_m^n, p) = c(M, \Delta_m^n)$ and $\ell_\infty(M, \Delta_m^n, p) = \ell_\infty(M, \Delta_m^n)$.

Lemma 1. *If a sequence space E is solid, then E is monotone.*

3. Main results

In this section we prove the results of this article. The proof of the following result is easy, so omitted.

Proposition 1. *The classes of sequences $c_0(M, \Delta_m^n, p)$, $c(M, \Delta_m^n, p)$ and $\ell_\infty(M, \Delta_m^n, p)$ are linear spaces.*

Theorem 1. *For $Z = \ell_\infty$, c and c_0, the spaces $Z(M, \Delta_m^n, p)$ are paranormed spaces, paranormed by*

$$g(x) = \sum_{k=1}^{nm} |x_k| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n x_k|}{\rho}) \leq 1\},$$

where

$$H = \max(1, \sup_k p_k).$$

Proof. Clearly $g(-x) = g(x)$, $g(\theta) = 0$. Let (x_k) and (y_k) be any two sequences belong to any one of the spaces $Z(M, \Delta_m^n, p)$, for $Z = c_0$, c and ℓ_∞. Then we have ρ_1, $\rho_2 > 0$ such that

$$\sup_k M(\frac{|\Delta_m^n x_k|}{\rho_1}) \leq 1$$

and

$$\sup_k M(\frac{|\Delta_m^n y_k|}{\rho_2}) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by convexity of M, we have

$$\sup_k M(\frac{|\Delta_m^n(x_k + y_k)|}{\rho}) \le (\frac{\rho_1}{\rho_1 + \rho_2}) \sup_k M(\frac{|\Delta_m^n x_k|}{\rho_1}) + (\frac{\rho_2}{\rho_1 + \rho_2}) \sup_k M(\frac{|\Delta_m^n y_k|}{\rho_2}) \le 1.$$

Hence we have,

$$\begin{aligned} g(x+y) &= \sum_{k=1}^{mn} |x_k + y_k| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n(x_k + y_k)|}{\rho}) \le 1\} \\ &\le \sum_{k=1}^{mn} |x_k| + \inf\{\rho_1^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n x_k|}{\rho_1}) \le 1\} \\ &\quad + \sum_{k=1}^{mn} |y_k| + \inf\{\rho_2^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n y_k|}{\rho_2}) \le 1\}. \end{aligned}$$

This implies that

$$g(x+y) \le g(x) + g(y).$$

The continuity of the scalar multiplication follows from the following inequality:

$$\begin{aligned} g(\lambda x) &= \sum_{k=1}^{mn} |\lambda x_k| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n \lambda x_k|}{\rho}) \le 1\} \\ &= |\lambda| \sum_{k=1}^{mn} |x_k| + \inf\{(t|\lambda|)^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n x_k|}{t}) \le 1\}, \quad \text{where } t = \frac{\rho}{|\lambda|}. \end{aligned}$$

Hence the space $Z(M, \Delta_m^n, p)$, for $Z = c_0$, c and ℓ_∞ are paranormed spaces, paranormed by g. ■

Theorem 2. *For $Z = \ell_\infty$, c and c_0, the spaces $Z(M, \Delta_m^n, p)$ are complete paranormed spaces, paranormed by*

$$g(x) = \sum_{k=1}^{nm} |x_k| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n x_k|}{\rho}) \le 1\},$$

where

$$H = \max(1, \sup_k p_k).$$

Proof. We prove for the space $\ell_\infty(M, \Delta_m^n, p)$ and for the other spaces it will follow on applying similar arguments.

Let (x^i) be any Cauchy sequence in $\ell_\infty(M, \Delta_m^n, p)$. Let $x_0 > 0$ be fixed and $t > 0$ be such that for a given $0 < \varepsilon < 1$, $\frac{\varepsilon}{x_0 t} > 0$, and $x_0 t \geq 1$. Then there exists a positive integer n_0 such that

$$g(x^i - x^j) < \frac{\varepsilon}{x_0 t}, \text{ for all } i, j \geq n_0.$$

Using the definition of paranorm, we get

$$(1) \quad \sum_{k=1}^{mn} |x_k^i - x_k^j| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n(x_k^i - x_k^j)|}{\rho}) \leq 1\} < \frac{\varepsilon}{x_0 t}, \quad \text{for all } i, j \geq n_0.$$

Hence we have,

$$\sum_{k=1}^{mn} |x_k^i - x_k^j| < \varepsilon, \text{ for all } i, j \geq n_0.$$

This implies

$$|x_k^i - x_k^j| < \varepsilon, \text{ for all } i, j \geq n_0 \text{ and } 1 \leq k \leq nm.$$

Thus (x_k^i) is a Cauchy sequence in C for $k = 1, 2, ..., nm$. Hence (x_k^i) is convergent in C for $k = 1, 2, ..., nm$.

$$(2) \qquad \text{Let } \lim_{i \to \infty} x_k^i = x_k, \text{ say for } k = 1, 2, \dots, nm.$$

Again from (1) we have,

$$\inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n(x_k^i - x_k^j)|}{\rho}) \leq 1\} < \varepsilon, \text{ for all } i, j \geq n_0.$$

Hence we get

$$\sup_k M(\frac{|\Delta_m^n(x_k^i - x_k^j)|}{g(x^i - x^j)}) \leq 1, \text{ for all } i, j \geq n_0.$$

It follows that $M(\frac{|\Delta_m^n(x_k^i - x_k^j)|}{g(x^i - x^j)}) \leq 1$, for each $k \geq 1$ and for all $i, j \geq n_0$.

For $t > 0$ with $M(\frac{tx_0}{2}) \geq 1$, we have

$$M(\frac{|\Delta_m^n(x_k^i - x_k^j)|}{g(x^i - x^j)}) \leq M(\frac{tx_0}{2}).$$

This implies
$$|\Delta_m^n x_k^i - \Delta_m^n x_k^j| \le \frac{tx_0}{2}\frac{\varepsilon}{tx_0} = \frac{\varepsilon}{2}.$$
Hence $(\Delta_m^n x_k^i)$ is a Cauchy sequence in C for all $k \in N$.

This implies that $(\Delta_m^n x_k^i)$ is convergent in C for all $k \in N$. Let $\lim_{i\to\infty} \Delta_m^n x_k^i = y_k$ for each $k \in N$.

Let $k = 1$. Then we have

$$\lim_{i\to\infty} \Delta_m^n x_1^i = \lim_{i\to\infty} \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{1+mv}^i = y_1. \tag{3}$$

We have by (2) and (3) $\lim_{i\to\infty} x_{mn+1}^i = x_{mn+1}$, exists. Proceeding in this way inductively, we have $\lim_{i\to\infty} x_k^i = x_k$ exists for each $k \in N$.

Now we have for all $i, j \ge n_0$,

$$\sum_{k=1}^{mn} |x_k^i - x_k^j| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n(x_k^i - x_k^j)|}{\rho}) \le 1\} < \varepsilon.$$

This implies that

$$\lim_{j\to\infty}\{\sum_{k=1}^{mn} |x_k^i - x_k^j| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n(x_k^i - x_k^j)|}{\rho}) \le 1\}\} < \varepsilon,$$

for all $i \ge n_0$. Using the continuity of M, we have

$$\sum_{k=1}^{nm} |x_k^i - x_k| + \inf\{\rho^{\frac{p_k}{H}} : \sup_k M(\frac{|\Delta_m^n x_k^i - \Delta_m^n x_k|}{\rho}) \le 1\} < \varepsilon,$$

for all $i \ge n_0$. It follows that $(x^i - x) \in \ell_\infty(M, \Delta_m^n, p)$. Since $(x^i) \in \ell_\infty(M, \Delta_m^n, p)$ and $\ell_\infty(M, \Delta_m^n, p)$ is a linear space, so we have $x = x^i - (x^i - x) \in \ell_\infty(M, \Delta_m^n, p)$.

This completes the proof of the Theorem. ■

Theorem 3. *If $0 < p_k \le q_k < \infty$ for each k, then $Z(M, \Delta_m^n, p) \subseteq Z(M, \Delta_m^n, q)$, for $Z = c_0$ and c.*

Proof. We prove the result for the case $Z = c_0$ and for the other case it will follow on applying similar arguments.

Let $(x_k) \in c_0(M, \Delta_m^n, p)$. Then there exists some $\rho > 0$ such that

$$\lim_{k\to\infty} (M(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} = 0.$$

This implies that $M(\frac{|\Delta_m^n x_k|}{\rho}) < \varepsilon (0 < \varepsilon < 1)$ for sufficiently large k.

Hence we get

$$\lim_{k\to\infty}(M(\frac{|\Delta_m^n x_k|}{\rho}))^{q_k} \leq \lim_{k\to\infty}(M(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} = 0.$$

This implies that $(x_k) \in c_0(M, \Delta_m^n, q)$. This completes the proof. ∎

The following result is a consequence of Theorem 4.

Corollary. (**a**) If $0 < \inf p_k \leq p_k \leq 1$, for each k, then $Z(M, \Delta_m^n, p) \subseteq Z(M, \Delta_m^n)$, for $Z = c_0$ and c.

(**b**)If $1 \leq p_k \leq \sup p_k < \infty$, for each k, then $Z(M, \Delta_m^n) \subseteq Z(M, \Delta_m^n, p)$, for $Z = c_0$ and c.

Theorem 4. *If M_1 and M_2 be two Orlicz functions. Then*

(*i*) $Z(M_1, \Delta_m^n, p) \subseteq Z(M_2 \circ M_1, \Delta_m^n, p)$,

(*ii*) $Z(M_1, \Delta_m^n, p) \cap Z(M_2, \Delta_m^n, p) \subseteq Z(M_1 + M_2, \Delta_m^n, p)$, *for* $Z = \ell_\infty$, c *and* c_0.

Proof. We prove this part for $Z = \ell_\infty$ and the rest of the cases will follow similarly.

Let $(x_k) \in \ell_\infty(M_1, \Delta_m^n, p)$. Then there exists $0 < U < \infty$ such that

$$(M_1(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} \leq U, \text{ for all } k \in N.$$

Let $y_k = M_1(\frac{|\Delta_m^n x_k|}{\rho})$. Then $y_k \leq U^{\frac{1}{p_k}} \leq V$, say for all $k \in N$.

Hence we have

$$((M_2 \circ M_1)(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} = (M_2(y_k))^{p_k} \leq (M_2(V))^{p_k} < \infty, \text{ for all} k \in N.$$

Hence $\sup_k((M_2 \circ M_1)(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} < \infty$. Thus $(x_k) \in \ell_\infty(M_2 \circ M_1, \Delta_m^n, p)$.

(*ii*) We prove the result for the case $Z = c_0$ and for the other cases it will follow on applying similar arguments.

Let $(x_k) \in c_0(M_1, \Delta_m^n, p) \cap c_0(M_2, \Delta_m^n, p)$. Then there exist some ρ_1, $\rho_2 > 0$ such that

$$\lim_{k\to\infty}(M_1(\frac{|\Delta_m^n x_k|}{\rho_1}))^{p_k} = 0 \text{ and } \lim_{k\to\infty}(M_2(\frac{|\Delta_m^n x_k|}{\rho_2}))^{p_k} = 0.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned}((M_1 + M_2)(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k} \ &\leq \ D[\frac{\rho_1}{\rho_1 + \rho_2} M_1(\frac{\Delta_m^n x_k}{\rho_1})]^{p_k} \\ &\quad + D[\frac{\rho_2}{\rho_1 + \rho_2} M_2(\frac{\Delta_m^n x_k}{\rho_2})]^{p_k}.\end{aligned}$$

This implies that

$$\lim_{k\to\infty}((M_1+M_2)(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k}=0.$$

Thus $(x_k)\in c_0(M_1+M_2,\Delta_m^n,p)$. This completes the proof. ■

Theorem 5. *$Z(M,\Delta_m^{n-1},p)\subset Z(M,\Delta_m^n,p)$ (in general $Z(M,\Delta_m^i,p)\subset Z(M,\Delta_m^n,p)$, for $i=1,2,\ldots,n-1$, for $Z=\ell_\infty$, c and c_0.*

Proof. We prove the result for $Z=\ell_\infty$ and for the other cases it will follow on applying similar arguments.

Let $x=(x_k)\in\ell_\infty(M,\Delta_m^{n-1},p)$. Then we can have $\rho>0$ such that

$$(M(\frac{|\Delta_m^{n-1}x_k|}{\rho}))^{p_k}<\infty,\ \text{ for all }\ k\in N \tag{4}$$

On considering 2ρ and using the convexity of M, we have

$$M(\frac{|\Delta_m^n x_k|}{2\rho})\le\frac{1}{2}M(\frac{|\Delta_m^{n-1}x_k|}{\rho})+\frac{1}{2}M(\frac{|\Delta_m^{n-1}x_{k+m}|}{\rho}).$$

Hence we have

$$\left(M\left(\frac{|\Delta_m^n x_k|}{2\rho}\right)\right)^{p_k}\le D\left\{\left(\frac{1}{2}M\left(\frac{|\Delta_m^{n-1}x_k|}{\rho}\right)\right)^{p_k}+\left(\frac{1}{2}M\left(\frac{|\Delta_m^{n-1}x_{k+m}|}{\rho}\right)\right)^{p_k}\right\}.$$

Then using (4), we have

$$(M(\frac{|\Delta_m^n x_k|}{\rho}))^{p_k}<\infty,\ \text{ for all }\ k\in N.$$

Thus $\ell_\infty(M,\Delta_m^{n-1},p)\subset\ell_\infty(M,\Delta_m^n,p)$.

The inclusion is strict follows from the following example.

Example 1. Let $m=3$, $n=2$, $M(x)=x^2$, for all $x\in[0,\infty)$ and $p_k=4$ for all k odd and $p_k=3$ for all k even. Consider the sequence $x=(x_k)=(k)$. Then $\Delta_3^2x_k=0$, for all $k\in N$. Hence x belongs to $c_0(M,\Delta_3^2,p)$. Again we have $\Delta_3^1x_k=-3$, for all $k\in N$. Hence x does not belong to $c_0(M,\Delta_3^1,p)$. Thus the inclusion is strict. ■

Theorem 6. *Let M be an Orlicz function. Then*

$$c_0(M,\Delta_m^n,p)\subset c(M,\Delta_m^n,p)\subset\ell_\infty(M,\Delta_m^n,p).$$

The inclusions are proper.

Proof. It is obvious that $c_0(M,\Delta_m^n,p) \subset c(M,\Delta_m^n,p)$. We shall prove that $c(M,\Delta_m^n,p) \subset \ell_\infty(M,\Delta_m^n,p)$.

Let $(x_k) \in c(M,\Delta_m^n,p)$. Then there exists some $\rho > 0$ and $L \in C$ such that

$$\lim_{k\to\infty}(M(\frac{|\Delta_m^n x_k - L|}{\rho}))^{p_k} = 0.$$

On taking $\rho_1 = 2\rho$, we have

$$\begin{aligned}(M(\frac{|\Delta_m^n x_k|}{\rho_1}))^{p_k} &\leq D[\frac{1}{2}(M(\frac{|\Delta_m^n x_k - L|}{\rho}))]^{p_k} + D[\frac{1}{2}M(\frac{|L|}{\rho})]^{p_k}\\ &\leq D(\frac{1}{2})^{p_k}[M(\frac{|\Delta_m^n x_k - L|}{\rho})]^{p_k} + D(\frac{1}{2})^{p_k}\max(1,(M(\frac{|L|}{\rho}))^H),\end{aligned}$$

where $H = \max(1, \sup p_k)$. Thus we get $(x_k) \in \ell_\infty(M,\Delta_m^n,p)$.

The inclusions are strict follow from the following examples.

Example 2. Let $m = 2$, $n = 2$, $M(x) = x^4$, for all $x \in [0,\infty)$ and $p_k = 1$, for all $k \in N$. Consider the sequence $x = (x_k) = (k^2)$. Then x belongs to $c(M,\Delta_2^2,p)$, but x does not belong to $c_0(M,\Delta_2^2,p)$.

Example 3. Let $m = 2$, $n = 2$, $M(x) = x^2$, for all $x \in [0,\infty)$ and $p_k = 2$, for all k odd and $p_k = 3$, for all k even. Consider the sequence $x = (x_k) = \{1,3,2,4,5,7,6,8,9,11,10,12,...\}$. Then x belongs to $\ell_\infty(M,\Delta_2^2,p)$, but x does not belong to $c(M,\Delta_2^2,p)$. ■

Theorem 7. *The spaces $\ell_\infty(M,\Delta_m^n,p)$, $c(M,\Delta_m^n,p)$ and $c_0(M,\Delta_m^n,p)$ are not monotone and as such are not solid in general.*

Proof. The proof follows from the following example.

Example 4. Let $n = 2$, $m = 3$, $p_k = 1$, for all k odd and $p_k = 2$, for all k even and $M(x) = x^2$, for all $x \in [0,\infty)$. Then $\Delta_3^2 x_k = x_k - 2x_{k+3} + x_{k+6}$, for all $k \in N$. Consider the J^{th} step space of a sequence space E defined by $(x_k),(y_k) \in E^J$ implies that $y_k = x_k$, for k odd and $y_k = 0$, for k even. Consider the sequence (x_k) defined by $x_k = k$, for all $k \in N$. Then (x_k) belongs to $Z(M,\Delta_3^2,p)$, for $Z = \ell_\infty, c$ and c_0, but its J^{th} canonical pre-image does not belong to $Z(M,\Delta_3^2,p)$, for $Z = \ell_\infty$, c and c_0. Hence the spaces $\ell_\infty(M,\Delta_m^n,p), c(M,\Delta_m^n,p)$ and $c_0(M,\Delta_m^n,p)$ are not monotone and as such are not solid in general. ■

Theorem 8. *The spaces $\ell_\infty(M,\Delta_m^n,p)$, $c(M,\Delta_m^n,p)$ and $c_0(M,\Delta_m^n,p)$ are not symmetric in general.*

Proof. The proof follows from the following example.

Example 5. Let $n = 2$, $m = 2$, $p_k = 2$, for all k odd and $p_k = 3$, for all k even and $M(x) = x^2$, for all $x \in [0, \infty)$. Then $\Delta_2^2 x_k = x_k - 2x_{k+2} + x_{k+4}$, for all $k \in N$. Consider the sequence (x_k) defined by $x_k = k$, for k odd and $x_k = 0$, for k even. Then $\Delta_2^2 x_k = 0$, for all $k \in N$. Hence (x_k) belongs to $Z(M, \Delta_2^2, p)$, for $Z = \ell_\infty$, c and c_0. Consider the rearranged sequence, (y_k) of (x_k) defined by

$$(y_k) = (x_1, x_3, x_2, x_4, x_5, x_7, x_6, x_8, x_9, x_{11}, x_{10}, x_{12}, ...).$$

Then (y_k) does not belong to $Z(M, \Delta_2^2, p)$, for $Z = \ell_\infty$, c and c_0. Hence the spaces $\ell_\infty(M, \Delta_m^n, p)$, $c(M, \Delta_m^n, p)$ and $c_0(M, \Delta_m^n, p)$ are not symmetric in general. ■

Theorem 9. *The spaces* $\ell_\infty(M, \Delta_m^n, p)$, $c(M, \Delta_m^n, p)$ *and* $c_0(M, \Delta_m^n, p)$ *are not convergence free in general.*

Proof. The proof follows from the following example.

Example 6. Let $m = 3$, $n = 1$, $p_k = 6$, for all $k \in N$ and $M(x) = x^3$, for all $x \in [0, \infty)$. Then $\Delta_3^1 x_k = x_k - x_{k+3}$, for all $k \in N$. Consider the sequences (x_k) and (y_k) defined by $x_k = 4$ for all $k \in N$ and $y_k = k^2$, for all $k \in N$. Then (x_k) belongs to $Z(M, \Delta_3^1, p)$, but (y_k) does not belong to $Z(M, \Delta_3^1, p)$, for $Z = \ell_\infty$, c and c_0. Hence the spaces $\ell_\infty(M, \Delta_m^n, p)$, $c(M, \Delta_m^n, p)$ and $c_0(M, \Delta_m^n, p)$ are not convergence free in general. ■

Theorem 10. *The spaces* $\ell_\infty(M, \Delta_m^n, p)$, $c(M, \Delta_m^n, p)$ *and* $c_0(M, \Delta_m^n, p)$ *are not sequence algebra in general.*

Proof. The proof follows from the following examples.

Example 7. Let $n = 2$, $m = 1$, $p_k = 1$, for all $k \in N$ and $M(x) = x^3$, for all $x \in [0, \infty)$. Then $\Delta_1^2 x_k = x_k - 2x_{k+1} + x_{k+2}$, for all $k \in N$. Let $x = (k)$ and $y = (k^2)$. Then x, y both belong to $Z(M, \Delta_1^2, p)$, for $Z = \ell_\infty$ and c, but xy does not belong to $Z(M, \Delta_1^2, p)$, for $Z = \ell_\infty$ and c. Hence the spaces $\ell_\infty(M, \Delta_m^n, p)$ and $c(M, \Delta_m^n, p)$ are not sequence algebra in general.

Example 8. Let $n = 2$, $m = 1$, $p_k = 7$, for all $k \in N$ and $M(x) = x^7$, for all $x \in [0, \infty)$. Then $\Delta_1^2 x_k = x_k - 2x_{k+1} + x_{k+2}$, for all $k \in N$. Let $x = (k)$ and $y = (k)$. Then x, y both belong to $c_0(M, \Delta_1^2, p)$, but xy does not belong to $c_0(M, \Delta_1^2, p)$. Hence the space $c_0(M, \Delta_m^n, p)$ is not sequence algebra in general. ■

References

[1] ET M., COLOK R., On generalized difference sequence spaces, *Soochow J. Math.,* 21(4)(1995), 377-386.

[2] Kizmaz H., On certain sequence spaces, *Canad. Math. Bull.,* 24(2)(1981), 169-176.

[3] Lascarides C.G., A study of certain sequence spaces of Maddox and generalization of a theorem of Iyer, *Pacific Jour. Math.,* 38(2)(1971), 487-500.

[4] Lascarides C.G., Maddox I.J., Matrix transformation between some classes of sequences, *Proc. Camb. Phil. Soc.,* 68(1970), 99-104.

[5] Lindenstrauss J., Tzafriri L., On Orlicz sequence spaces, *Israel J. Math.,* 10(1971), 379-390.

[6] Maddox I.J., Paranormed sequence spaces generated by infinite matrices, *Proc. Camb. Phil. Soc.,* 64(1968), 335-340.

[7] Mursaleen, Khan A., Qamaruddin, Difference sequence spaces defined by Orlicz functions, *Demonstratio Mathematica,* XXXII(1)(1999), 145-150.

[8] Nakano H., Modular sequence spaces, *Proc. Japan Acad.,* 27(1951), 508-512.

[9] Nanda S., Some sequence spaces and almost convergence, *J. Austral. Math. Soc. Ser. A,* 22(1976), 446-455.

[10] Parasar S.D., Choudhary B., Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math,* 25(4)(1994), 419-428.

[11] Simons S., The sequence spaces $\ell(p_v)$ and $m(p_v)$, *Proc. London Math. Soc.,* 15(1965), 422-436.

[12] Tripathy B.C., A class of difference sequences related to the p-normed space ℓ^p, *Demonstratio Math.,* 36(4)(2003), 867-872.

[13] Tripathy B.C., Esi A., A new type of difference sequence spaces, *Internat. J. Sci. Technol.,* (1)(2006), 11-14.

[14] Tripathy B.C., Esi A., Tripathy B.K., On a new type of generalized difference Cesàro sequence spaces, *Soochow J. Math.,* 31(3)(2005), 333-340.

[15] Tripathy B.C., Sen M., On generalized statistically convergent sequence spaces, *Indian J. Pure Appl. Math.,* 32(11)(2001), 1689-1694.

Binod Chandra Tripathy
Mathematical Sciences Division
Institute of Advanced Study in Science and Technology
Paschim Boragaon, Garchuk, Guwahati-781035; India
e-mail: tripathybc@yahoo.com *or* tripathybc@rediffmail.com

Hemen Dutta
Department of Mathematics
A.D.P. COLLEGE, Nagaon-782002, Assam, India
e-mail: hemen_dutta@rediffmail.com

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