F A S C I C U L I M A T H E M A T I C I

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ON A CLASS OF RATIONAL DIFFERENCE EQUATIONS

Abstract. In this paper we study the behavior of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-(3k+2)}}{1 + x_{n-k}x_{n-(2k+1)}}, \quad n = 0, 1, 2, ...$$

where the initial values $x_{-(3k+2)}, x_{-(3k+1)}, \ldots, x_{-1}, x_0 \in (0, \infty)$ and $k = 0, 1, 2, \ldots$.

Key words: difference equation, positive solutions, period $(3k+3)$ solution.

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1. Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, etc. So, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, for examples [1-10].

In [1-5] Cinar studied the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}},$$
$$x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}, \quad x_{n+1} = \frac{a_n + b_n x_n}{c_n x_{n-1}}$$

where the positive initial values.

In [9] Stevic studied the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

where $x_{-1}, x_0 \in (0,\infty)$. Also, this result was generalized to the equation of the following form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \quad \text{for} \quad n = 0,1,2,\ldots$$

where $x_{-1}, x_0 \in (0,\infty)$.

Moreover in [6], Simsek et al. solved the following problem for the positive initial values

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}} \quad \text{for} \quad n = 0,1,2,\ldots.$$

Also in [7], Simsek et al. studied the behavior of the positive solutions of the difference equation

$$x_{n+1} = \frac{x_{n-(5k+9)}}{1 + x_{n-4}x_{n-9}...x_{n-(5k+4)}}, \quad n = 0,1,2,\ldots$$

where $x_{-(5k+9)}, x_{-(5k+8)}, ..., x_{-1}, x_0 \in (0,\infty)$.

In this paper we study the behavior of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-(3k+2)}}{1 + x_{n-k}x_{n-(2k+1)}}, \quad n = 0,1,2,\ldots \tag{1}$$

where the initial values $x_{-(3k+2)}, x_{-(3k+1)}, ..., x_{-1}, x_0 \in (0,\infty)$ and $k = 0,1,2\ldots.$

To the best of our knowledge, the Eq.(1) has not been investigated so far. Therefore, it is meaningful to study the qualitative properties of the Eq.(1).

2. Main result

In this section we consider the behavior of the positive solutions of Eq.(1). The following theorem describes the behavior of the positive solutions of Eq.(1).

Theorem 1. *Consider the difference Eq.(1). Then the following statements are true:*

(a) The sequences $(x_{(3k+3)n-(3k+2)}), (x_{(3k+3)n-(3k+1)}), \ldots, (x_{(3k+3)n})$ *are decreasing and there exist* $a_1, a_2, \ldots, a_{3k+3} \geqslant 0$ *such that*

$$\lim_{n\to\infty} x_{(3k+3)n-(3k+2)} = a_1,$$
$$\lim_{n\to\infty} x_{(3k+3)n-(3k+1)} = a_2,$$
$$\ldots,$$
$$\lim_{n\to\infty} x_{(3k+3)n} = a_{3k+3}$$

(*b*) $(a_1, a_2, \ldots, a_{3k+3}, a_1, a_2, \ldots, a_{3k+3}, \ldots)$ *is a solution with period* $(3k+3)$ *of Eq.(1) .*

(*c*) $a_m \cdot a_{m+(k+1)} \cdot a_{m+(2k+2)} = 0$ *for* $m = 1, 2, \ldots, (k+1)$

(*d*) *If there exists* $n_0 \in N$ *such that*

$$x_{n-(2k+1)} \geq x_{n+1}$$

for all $n \geqslant n_0$*, then*

$$\lim_{n\to\infty} x_n = 0.$$

(*e*) *The following formulas*

$$x_{(3k+3)n+b} = x_{b-(3k+3)}(1 - \frac{x_{b-(k+1)}x_{b-(2k+2)}}{1 + x_{b-(k+1)}x_{b-(2k+2)}} \times \sum_{j=0}^{n} \prod_{i=1}^{3j} \frac{1}{1 + x_{(k+1)i+b-(k+1)}x_{(k+1)i+b-(2k+2)}})$$

$$x_{(3k+3)n+(k+1)+b} = x_{b-(2k+2)}(1 - \frac{x_{b-(k+1)}x_{b-(3k+3)}}{1 + x_{b-(k+1)}x_{b-(2k+2)}} \times \sum_{j=0}^{n} \prod_{i=1}^{3j+1} \frac{1}{1 + x_{(k+1)i+b-(k+1)}x_{(k+1)i+b-(2k+2)}})$$

$$x_{(3k+3)n+b+(2k+2)} = x_{b-(k+1)}(1 - \frac{x_{b-(2k+2)}x_{b-(3k+3)}}{1 + x_{b-(k+1)}x_{b-(2k+2)}} \times \sum_{j=0}^{n} \prod_{i=1}^{3j+2} \frac{1}{1 + x_{(k+1)i+b-(k+1)}x_{(k+1)i+b-(2k+2)}})$$

hold for $b = 1, 2, \ldots, k+1$.

Proof. (*a*) Firstly, from the Eq.(1), we obtain

$$x_{n+1}(1 + x_{n-k}x_{n-(2k+1)}) = x_{n-(3k+2)}.$$

Since $x_{n-k}, x_{n-(2k+1)} \in (0, \infty)$, then $(1 + x_{n-k}x_{n-(2k+1)}) \in (1, \infty)$. So,

$$x_{n+1} < x_{n-(3k+2)}.$$

This implies that

$$0 < ... < x_{6k+7} < x_{3k+4} < x_1 < x_{-(3k+2)}$$
$$0 < ... < x_{6k+8} < x_{3k+5} < x_2 < x_{-(3k+1)}$$
$$\ldots$$
$$0 < ... < x_{9k+8} < x_{6k+5} < x_{3k+2} < x_{-1}$$
$$0 < ... < x_{9k+9} < x_{6k+6} < x_{3k+3} < x_0$$

Hence, we obtain that there exist

$$\begin{aligned}\lim_{n\to\infty} x_{(3k+3)n-(3k+2)} &= a_1,\\ \lim_{n\to\infty} x_{(3k+3)n-(3k+1)} &= a_2,\\ &\ldots,\\ \lim_{n\to\infty} x_{(3k+3)n} &= a_{3k+3}.\end{aligned}$$

(b) $(a_1, a_2, \ldots, a_{3k+3}, a_1, a_2, \ldots, a_{3k+3}, \ldots)$ is a solution with period $(3k+3)$ of Eq.(1).

(c) In view of the Eq.(1), we obtain

$$x_{(3k+3)n+m} = \frac{x_{(3k+3)n-(3k+3-m)}}{1 + x_{(3k+3)n-(k+1-m)}x_{(3k+3)n-(2k+2-m)}}$$

for $m = 1, 2, 3, ..., (k+1)$.

Take the limits on both sides of the above equality

$$\lim_{n\to\infty} x_{(3k+3)n+m} = \lim_{n\to\infty} \frac{x_{(3k+3)n-(3k+3-m)}}{1 + x_{(3k+3)n-(k+1-m)}x_{(3k+3)n-(2k+2-m)}}$$

from the above equality, we obtain

$$a_m \cdot a_{m+(k+1)} \cdot a_{m+(2k+2)} = 0.$$

(d) If there exists $n_0 \in N$ such that

$$x_{n-(2k+1)} \geq x_{n+1} \quad \text{for all} \quad n \geqslant n_0$$

then we get

$$\begin{aligned}&a_1 \geq a_{2k+3} \geq a_{k+2} \geq a_1 \geq a_{2k+3},\\ &a_2 \geq a_{2k+4} \geq a_{k+3} \geq a_2 \geq a_{2k+4},\\ &\vdots\\ &a_k \geq a_{3k+2} \geq a_{2k+1} \geq a_k \geq a_{3k+2},\\ &a_{k+1} \geq a_{3k+3} \geq a_{2k+2} \geq a_{k+1} \geq a_{3k+3}.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} x_n = 0.$$

(e) Subracting $x_{n-(3k+2)}$ from both the left and right-hand sides of the Eq.(1) we obtain

$$x_{n+1} - x_{n-(3k+2)} = \frac{1}{1 + x_{n-k}x_{n-(2k+1)}}(x_{n-k} - x_{n-(4k+3)})$$

and the following formula

$$x_{(k+1)n+b} - x_{(k+1)n+b-(3k+3)} = (x_b - x_{b-(3k+3)}) \times \prod_{i=1}^{n} \frac{1}{1 + x_{(k+1)i+b-(k+1)} x_{(k+1)i+b-(2k+2)}} \tag{2}$$

hold for $n \geqslant 0$ and $b = 1, 2, \ldots, (k+1)$. From the Eq.(2), we obtain

$$x_{(3k+3)n+b} - x_{b-(3k+3)} = (x_b - x_{b-(3k+3)}) \times \sum_{j=0}^{n} \prod_{i=1}^{3j} \frac{1}{1 + x_{(k+1)i+b-(k+1)} x_{(k+1)i+b-(2k+2)}} \tag{3}$$

$$x_{(3k+3)n+(k+1)+b} - x_{b-(2k+2)} = (x_b - x_{b-(3k+3)}) \times \sum_{j=0}^{n} \prod_{i=1}^{3j+1} \frac{1}{1 + x_{(k+1)i+b-(k+1)} x_{(k+1)i+b-(2k+2)}} \tag{4}$$

$$x_{(3k+3)n+(2k+2)+b} - x_{b-(k+1)} = (x_b - x_{b-(3k+3)}) \times \sum_{j=0}^{n} \prod_{i=1}^{3j+2} \frac{1}{1 + x_{(k+1)i+b-(k+1)} x_{(k+1)i+b-(2k+2)}} \tag{5}$$

for $n \geq 0$

Now, we obtain from the Eq.(3), Eq.(4) and Eq.(5)

$$x_{(3k+3)n+b} = x_{b-(3k+3)}\Big(1 - \frac{x_{b-(k+1)} x_{b-(2k+2)}}{1 + x_{b-(k+1)} x_{b-(2k+2)}} \times \sum_{j=0}^{n} \prod_{i=1}^{3j} \frac{1}{1 + x_{(k+1)i+b-(k+1)} x_{(k+1)i+b-(2k+2)}}\Big)$$

$$x_{(3k+3)n+(k+1)+b} = x_{b-(2k+2)}\Big(1 - \frac{x_{b-(k+1)} x_{b-(3k+3)}}{1 + x_{b-(k+1)} x_{b-(2k+2)}} \times \sum_{j=0}^{n} \prod_{i=1}^{3j+1} \frac{1}{1 + x_{(k+1)i+b-(k+1)} x_{(k+1)i+b-(2k+2)}}\Big)$$

and

$$x_{(3k+3)n+(2k+2)+b} = x_{b-(k+1)}\Big(1 - \frac{x_{b-(2k+2)} x_{b-(3k+3)}}{1 + x_{b-(k+1)} x_{b-(2k+2)}} \times \sum_{j=0}^{n} \prod_{i=1}^{3j+2} \frac{1}{1 + x_{(k+1)i+b-(k+1)} x_{(k+1)i+b-(2k+2)}}\Big)$$

Then, the proof is complete. ■

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