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COMMON FIXED POINT RESULTS FOR SIX MAPS ON CONE METRIC SPACES WITH SOME WEAKER CONDITIONS

Abstract. The existence of coincidence points and common fixed points for six mappings satisfying certain contractive conditions without exploiting the notion of continuity in cone metric spaces is established. Our results generalize, improve and extend several well known comparable results in the literature.

Key words: cone metric space, weakly compatible mappings, common fixed point.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

Sessa [22] initiated the tradition of improving commutativity given by Jungck [9] in fixed point theory by introducing the notion of weakly commuting maps in metric spaces. Jungck [10] soon enlarged this concept to compatible maps.

Initiated the study of noncompatible maps Pant [17] introduced R- weakly commuting maps and proved common fixed point theorems, assuming the continuity of at least one mapping.

Jungck and Rhoades [12] introduced the notion of weakly compatible maps, which is weaker than compatibility.

Kannan [15] was the first who proved the existence of a fixed point for a map that can have a discontinuity in a domain, however the maps involved in every case were continuous at the fixed point.

In the recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to Beg, Abbas [4], Jungck [11], Pant [17], Rhoades [21] and references contained therein.

Many authors proved common fixed point theorems in metric spaces and Banach spaces for noncompatible mappings without assuming continuity of any mapping including Sharma and Deshpande [23] - [25].

Nonconvex analysis, especially ordered normed spaces, normal cones and topical functions [2], [6]- [8], [16], [18], [20] have some applications in optimization theory. In these cases, an order is introduced by using vector spaces cones.

Guang and Xian [6] used this approach and they replaced the set of real numbers by an ordered Banach space and defined cone metric space which is generalization of metric space. They obtained some fixed point theorems for mappings satisfying the different contractive conditions. Fxed point theorems in cone metric spaces have been studied by [1], [3], [5]-[8], [13], [18], [20] and many others.

In this paper, we prove coincidence point results for six mappings which satisfy generalized contractive condition. Common fixed point results for weakly compatible maps which are more general than compatible mappings are obtained in the settings of cone metric spaces without exploiting the notion of continuity. Our results generalize, improve and extend the results of Guang and Xian [6], Jungck [9], Kannan [15], Pant [17] and Abbas and Jungck [1].

Consistent with Guang and Xian [6] the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{0\}$

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ imply $ax + by \in P$

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $x - y \in P$. A cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{I}$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \text{int } P$ (interior of P).

Definition 1. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:*

($d1$) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

($d2$) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

($d3$) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space.

Definition 2. *Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:*

(i) a Cauchy sequence if for every c in E with $c \gg 0$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for every c in E with $0 \ll c$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence is unique provided P is a normal cone with normal constant K (see Guang and Xian [6]).

Definition 3. *Let A and B be self maps of a nonempty set X. If $w = Ax = Bx$ for some x in X, then x is called a coincidence point of A and B and w is called a point of coincidence of A and B.*

Definition 4. *Let A and B be self maps of a set X. If $w = A(x) = B(x)$ for some x in X, then x is called a coincidence point of A and B and w is called a point of coincidence of coincidence of A and B. Self maps A and B are said to be weakly compatible if they commute at their coincidence point, that is if $Ax = Bx$ for some x in X then $ABx = BAx$.*

Remark 1. [26] Let E be an ordered Banach space, then c is an interior point of P if and only if $[-c, c]$ is a neighbourhood of 0.

Corollary 1. [[13](see, e.g., [19] without proof)]
(i) If $a \le b$ and $b \ll c$, then $a \ll c$.
(ii) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(iii) If $0 \le u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.

Remark 2. [26] If E is a real Banach space with cone P and if $a \le \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

2. Main results

Theorem 1. *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings $A, B, S, T, L, Q : X \to X$ satisfy*

$$L(X) \sqsubseteq ST(X), \quad Q(X) \sqsubseteq AB(X), \tag{1}$$

$$d(Lx, Qy) \le k \max\{d(ABx, Lx), d(STy, Qy), d(ABx, STy), d(STy, Lx), d(ABx, Qy)\} \tag{2}$$

for all $x, y \in X$ where $k \in (0, \frac{1}{2})$ is a constant,

(3) *one of $L(X)$, $Q(X)$, $AB(X)$, $ST(X)$ is a complete subspace of X then*

(i) Q and ST have a coincidence point,
(ii) L and AB have a coincidence point.
Further if

(4) $$AB = BA, \quad ST = TS, \quad LB = BL, \quad QT = TQ.$$

(5) *the pairs $\{L, AB\}$ and $\{Q, ST\}$ are weakly compatible then*

(iii) A, B, S, T, L and Q have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. From condition (1) there exists $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$. Inductively we can construct sequence $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$.

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (2), we get

$$\begin{aligned} d(Lx_{2n}, Qx_{2n+1}) \leq k \max\{&d(ABx_{2n}, Lx_{2n}), d(STx_{2n+1}, Qx_{2n+1}), \\ &d(ABx_{2n}, STx_{2n+1}), d(STx_{2n+1}, Lx_{2n}), d(ABx_{2n}, Qx_{2n+1}))\}, \end{aligned}$$

$$\begin{aligned} d(y_{2n}, y_{2n+1}) \leq k \max\{&d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \\ &d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}. \end{aligned}$$

Case 1. If

$$\begin{aligned} \max\{&d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \\ &d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\} = d(y_{2n-1}, y_{2n}) \end{aligned}$$

then

$$d(y_{2n}, y_{2n+1}) \leq k d(y_{2n-1}, y_{2n}).$$

Case 2. If

$$\begin{aligned} \max\{&d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \\ &d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\} \\ &= d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) \end{aligned}$$

then

$$d(y_{2n}, y_{2n+1}) \leq k\{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\},$$

therefore

$$d(y_{2n}, y_{2n+1}) \leq k_1 d(y_{2n-1}, y_{2n}), \quad \text{for all } n \text{ where } k_1 = \frac{k}{1-k} < 1.$$

Let $h = \max(k, k_1)$. Then

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\le hd(y_{2n-1}, y_{2n}) \\ &\le h^2 d(y_{2n-2}, y_{2n-1}), \\ &\quad \dots \\ &\le h^{2n} d(y_1, y_0). \end{aligned}$$

Thus for $n > m$

$$\begin{aligned} d(y_{2n}, y_{2m}) &\le d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n-2}) + \ldots + d(y_{2m+1}, y_{2m}), \\ &\le (h^{2n-1} + h^{2n-2} + \ldots + h^{2m}) d(y_1, y_0), \\ &\le \frac{h^{2m}}{1-h} d(y_1, y_0). \end{aligned}$$

From (I) we have

$$\|d(y_{2n}, y_{2m})\| \le \frac{h^{2m}}{1-h} K \|d(y_1, y_0)\|,$$

which implies that $d(y_{2n}, y_{2m}) \to 0$ as $n, m \to \infty$ hence $\{y_{2n}\}$ is a Cauchy sequence. Suppose $ST(X)$ is complete. Note that the subsequence $\{y_{2n}\}$ is contained in $ST(X)$ and has a limit in $ST(X)$. Call it z. Let $u \in ST^{-1}z$. Then $STu = z$. We shall use the fact that the subsequence $\{y_{2n+1}\}$ also converges to z. By (2) we have

$$\begin{aligned} d(STu, Qu) &\le d(STu, Lx_{2n}) + d(Lx_{2n}, Qu) = d(STu, y_{2n}) + d(y_{2n}, Qu) \\ &\le d(STu, y_{2n}) + k \max\{d(ABx_{2n}, Lx_{2n}), d(STu, Qu), d(ABx_{2n}, STu), \\ &\qquad d(STu, Lx_{2n}), d(ABx_{2n}, Qu)\} \\ &\le d(STu, y_{2n}) + k \max\{d(y_{2n-1,} y_{2n}), d(STu, Qu), d(y_{2n-1}, STu), \\ &\qquad d(STu, y_{2n}), d(y_{2n-1,} y_{2n}) + d(y_{2n}, STu) + d(STu, Qu)\}. \end{aligned}$$

Thus

$$d(STu, Qu) \le k\{d(y_{2n-1,} y_{2n}) + d(y_{2n}, STu) + d(STu, Qu)\}.$$

Therefore

$$d(STu, Qu) \le \frac{1+k}{1-k} d(STu, y_{2n}) + \frac{k}{1-k} d(y_{2n-1}, y_{2n}).$$

Let $0 \ll c$ then for infinitely many n, we have

$$d(STu, Qu) \ll \frac{1+k}{1-k} \frac{c(1-k)}{2(1+k)} + \frac{k}{1-k} \frac{c(1-k)}{2k}.$$

Thus $d(STu, Qu) \ll c$ for each $c \in \text{int}p$, using Corollary 1 (iii), it follows that $d(STu, Qu) = 0$ or $STu = Qu = z$. This proves (i).

Since $Q(X) \subseteq AB(X)$, $Qu = z$ implies that $z \in AB(X)$. Let $v \in (AB)^{-1}z$ then $ABv = z$. By (2), we have

$$\begin{aligned} d(Lv, ABv) &= d(Lv, Qx_{2n+1}) + d(Qx_{2n+1}, ABv) \\ &\le k \max\{d(Lv, ABv), d(STx_{2n+1}, Qx_{2n+1}), d(ABv, STx_{2n+1}), \\ d(STx_{2n+1}, Lv), & d(ABv, Qx_{2n+1})\} + d(Qx_{2n+1}, ABv) \\ &\le k \max\{d(ABv, Lv), d(y_{2n}, y_{2n+1}), d(ABv, y_{2n}), \\ &\qquad d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, ABv) + d(ABv, Lv), d(ABv, y_{2n+1})\} \\ &\quad + d(y_{2n+1}, ABv), \\ &= k\{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, ABv) + d(ABv, Lv)\} + d(y_{2n+1}, ABv). \end{aligned}$$

Thus

$$d(Lv, ABv) \le \frac{k}{1-k} d(y_{2n}, y_{2n+1}) + \frac{1+k}{1-k} d(y_{2n+1}, ABv).$$

Let $0 \ll c$ then for infinitely many n, we have

$$d(Lv, ABv) \ll \frac{k}{1-k} \frac{c(1-k)}{2k} + \frac{1+k}{1-k} \frac{c(1-k)}{2(1+k)}.$$

So $d(Lv, ABv) \ll c$ for each $c \in \text{int } p$, using Corollary 1 (iii), it follows that $d(Lv, ABv) = 0$ or $ABv = Lv = z$ that is v is coincidence point of L and AB. This proves (ii). The remaining two cases pertain essentially to the pervious cases. Indeed if $L(X)$ or $Q(X)$ is complete then by (1), $z \in L(X) \subseteq ST(X)$ or $z \in Q(X) \subseteq AB(X)$. Thus (i) and (ii) are completely established.

Since the pair (Q, ST) is weakly compatible, Therefore Q and ST commute at their coincidence point that is $Q(ST)u = (ST)Qu$ or $Qz = STz$. Similarly $L(ABv) = (AB)Lv$ or $Lz = ABz$.

Now we can prove that $Qz = z$. By (2) we have,

$$\begin{aligned} d(z, Qz) &= d(Lv, Qz) \\ &\le k \max\{d(ABv, Lv), d(STz, Qz), d(ABv, STz), \\ &\qquad d(STz, Lv), d(ABv, Qz)\}. \end{aligned}$$

Thus $d(z, Qz) \le kd(z, Qz)$, thus $Qz = z = STz$. Now we show that $Lz = z$. By (2), we have

$$\begin{aligned} d(Lz, z) &= d(Lz, Qz), \\ &\le k \max\{d(ABz, Lz), d(STz, Qz), d(ABz, STz), \\ &\qquad d(STz, Lz), d(ABz, Qz)\}, \\ &= k \max\{d(Lz, Lz), 0, d(Lz, z), d(z, Lz), d(Lz, z)\}, \end{aligned}$$

Thus $d(Lz, z) \leq kd(Lz, z)$, which gives $Lz = z$. So $Lz = z = ABz = STz = z$.

Putting $x = z$, $y = Tz$ in (2) and using (4), we have

$$\begin{aligned} d(z, Tz) &= d(Lz, T(Qz)) = d(Lz, Q(Tz)) \\ &\leq k \max\{d(ABz, Lz), d(ST(Tz), Q(Tz)), d(ABz, ST(Tz)), \\ &\qquad d(ST(Tz), Lz), d(ABz, Q(Tz))\}, \\ &\leq k \max\{d(z, z), d(Tz, Tz), d(z, Tz), d(Tz, z), d(z, Tz)\}, \end{aligned}$$

which gives $d(z, Tz) \leq kd(z, Tz)$, therefore $d(z, Tz) = 0$ and thus $Tz = z$. Since $STz = z$ therefore $Sz = z$. Putting $x = Bz$ and $y = z$ in (2) and using (4), we have

$$\begin{aligned} d(L(Bz), Qz) &\leq k \max\{d(AB(Bz), L(Bz)), d(STz, Qz), \\ &\qquad d(AB(Bz), STz), d(STz, L(Bz)), d(AB(Bz), Qz)\}, \\ &= kd(Bz, z), \end{aligned}$$

which gives $Bz = z$. Since $ABz = z$, we have $Az = z$. thus $Az = Bz = Sz = Tz = Lz = Qz = z$ that is z is a common fixed point of A, B, S, T, L and Q. Now, we show that z is the unique common fixed point of A, B, S, T, L and Q. For this assume that there exists another fixed point w in X such that $Aw = Bw = Sw = Tw = Lw = w$. Now by (2), we have

$$\begin{aligned} d(z, w) &= d(Lz, Qw) \\ &\leq k \max\{d(ABz, Lz), d(STw, Qw), d(ABz, STw), \\ &\qquad d(STw, Lz), d(ABz, Qw)\}, \end{aligned}$$

which gives $d(z, w) \leq kd(z, w)$, therefore $z = w$. This completes the proof. ■

Put $L = Q$ in Theorem 1, we have the following:

Corollary 2. *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings $A, B, S, T, L : X \to X$ satisfy*

(6) $$L(X) \sqsubseteq ST(X), \quad L(X) \sqsubseteq AB(X),$$

(7) $$\begin{aligned} d(Lx, Ly) \leq k \max\{&d(ABx, Lx), d(STy, Ly), d(ABx, STy), \\ &d(STy, Lx), d(ABx, Ly)\} \end{aligned}$$

for all $x, y \in X$ where $k \in (0, \frac{1}{2})$ is a constant,

(8) *one of $L(X), AB(X), ST(X)$ is a complete subspace of X then*

(i) L and ST have a coincidence point,
(ii) L and AB have a coincidence point.
Further if

(9) $$AB = BA, \quad ST = TS, \quad LB = BL, \quad LT = TL.$$

(10) *the pairs* $\{L, AB\}$ *and* $\{L, ST\}$ *are weakly compatible then*

(iii) A, B, S, T and L have a unique common fixed point in X.

If we put $B = T = I_X$ (the identity map on X) in Theorem 1 then (4) is satisfied trivially and we have the following:

Corollary 3. *Let* (X, d) *be a cone metric space and* P *a normal cone with normal constant* K. *Suppose mappings* $A, S, L, Q : X \to X$ *satisfy*

(11) $$L(X) \sqsubseteq S(X), \quad Q(X) \sqsubseteq A(X),$$

(12) $$d(Lx, Qy) \leq k \max\{d(Ax, Lx), d(Sy, Qy), d(Ax, Sy), d(Sy, Lx), d(Ax, Qy)\}$$

for all $x, y \in X$ *where* $k \in (0, \frac{1}{2})$ *is a constant,*

(13) *one of* $L(X), Q(X), A(X), S(X)$ *is a complete subspace of* X *then*

(i) Q and S have a coincidence point,
(ii) L and A have a coincidence point.
Further if

(14) *the pairs* $\{L, A\}$ *and* $\{Q, S\}$ *are weakly compatible then*

(iii) A, S, L and Q have a unique common fixed point in X.

If we put $L = Q$ in Corollary 3, we have the following:

Corollary 4. *Let* (X, d) *be a cone metric space and* P *a normal cone with normal constant* K. *Suppose mappings* A, S *and* $Q : X \to X$ *satisfy*

(15) $$Q(X) \sqsubseteq S(X), \quad Q(X) \sqsubseteq A(X),$$

(16) $$d(Qx, Qy) \leq k \max\{d(Ax, Qx), d(Sy, Qy), d(Ax, Sy), d(Sy, Qx), d(Ax, Qy)\}$$

for all $x, y \in X$ *where* $k \in (0, \frac{1}{2})$ *is a constant,*

(17) *one of* $Q(X), A(X), S(X)$ *is a complete subspace of* X *then*

(i) Q and S have a coincidence point,
(ii) Q and A have a coincidence point.
Further if

(18) *the pairs $\{Q, A\}$ and $\{Q, S\}$ are weakly compatible then*

(iii) A, S and Q have a unique common fixed point in X.

If we put $S = A$ in Corollary 4 we have the following:

Corollary 5. *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose mappings $A, Q : X \to X$ satisfy*

$$Q(X) \sqsubseteq A(X), \tag{19}$$

$$d(Qx, Qy) \leq k \max\{d(Ax, Qx), d(Ay, Qy), d(Ax, Ay), d(Ay, Qx), d(Ax, Qy)\} \tag{20}$$

for all $x, y \in X$ where $k \in (0, \frac{1}{2})$ is a constant,

(21) *one of $Q(X), A(X)$ is a complete subspace of X then*

(i) Q and A have a coincidence point.
Further if

(22) *the pair $\{Q, A\}$ is weakly compatible then*

(iii) A and Q have a unique common fixed point in X.

Example. Let $X = [0, \infty)$, $E = X^2$, $P = \{(x, y) \in E : x, y \geq 0\} \sqsubset X^2$, $d : X \times X \to E$ such that $d(x, y) = (|x - y|, 2(|x - y|)$. Define $A, S, L, Q : X \to X$ as follows:

$$Lx = \begin{cases} \frac{x}{2}, & x \neq 0 \\ 1, & x = 0 \end{cases} \qquad Sx = \begin{cases} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$$Qx = \begin{cases} \frac{x}{4}, & x \neq 0 \\ 1, & x = 0 \end{cases} \qquad Ax = \begin{cases} 2x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

We can see that conditions (11) and (12) of Corollary 3 hold. Q and S have a coincidence point $0 \in X$. Also L and A have a coincidence point $0 \in X$.

In the above example L and A do not commute at the coincidence point 0 and therefore are not weakly compatible. Also Q and S do not commute at the coincidence point 0 so Q and S are not weakly compatible.

Thus this example demonstrates the crucial role of weak compatibility in our results.

We can observe that the pairs $\{L, A\}$ and $\{Q, S\}$ are not compatible and all the four mappings involved in this example are discontinuous.

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