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EXISTENCE AND DATA DEPENDENCE OF FIXED POINTS FOR MULTIVALUED WEAKLY φ-CONTRACTIVE OPERATORS

Abstract. In this paper we give a data dependence theorem for multivalued weakly φ-contractive operators. Then, we define the concept of ball with respect to w-distance and we present fixed point results for multivalued contractive type operators using this ball.

Key words: w-distance, fixed point, multivalued operator.

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1. Introduction

A. Granas and M. Frigon proved in [1] the following principle for multivalued contractions.

Theorem 1. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ an a-contraction operator such that $D(x_0, T(x_0)) < (1 - a)r$. Then $FixT \neq \emptyset$.*

In 1996 the Japanese mathematicians O. Kada, T. Suzuki and W. Takahashi introduced the w-distance (see[3]) and discussed some properties of this new distance. Later, T. Suzuki and W. Takahashi starting by the above definition, gave some fixed points result for a new class of operators, the so-called weakly contractive operators (see[6]). An extension of this concept was introduced in [2].

The purpose of this paper is to present a data dependence result for a fixed point theorem, Theorem 4.2 from [2], to define the notion of ball with respect to w-distance and, using the contraction principle, to give some fixed point results for multivalued operators.

2. Preliminaries

Let (X, d) be a complete metric space. We will use the following notations:

$P(X)$ – *the set of all nonempty subsets of metric space* X;
$\mathcal{P}(X) = P(X) \bigcup \{\emptyset\}$
$P_{cl}(X)$ – *the set of all nonempty closed subsets of the metric space* X;
$P_b(X)$ – *the set of all nonempty bounded subsets of the metric space* X;
$\mathbb{N}$ – *the set of natural numbers*;
$\mathbb{N}^* = \mathbb{N} \setminus \{0\}$

For two subsets $A, B \in P_b(X)$ we recall the following functionals.

$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, D(Z,Y) = inf\{d(x,y) : x \in Z \ , y \in Y\}$, $Z \subset X$ – *the gap functional.*

$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A,B) := sup\{d(a,b) | x \in A, b \in B\}$ – *the diameter functional*;

$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \rho(A,B) := sup\{D(a,B) | a \in A\}$ – *the excess functional*;

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A,B) := max\{\sup_{a\in A} \inf_{b\in B} d(a,b), \sup_{b\in B} \inf_{a\in A} d(a,b)\}$ – *the Pompeiu-Hausdorff functional*;

$FixF := \{x \in X \mid x \in F(x)\}$ – *the set of the fixed points of* F;

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see[3]) as follows:

Let (X,d) be a metric space, $w : X \times X \to [0,\infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x,z) \le w(x,y) + w(y,z)$, for any $x, y, z \in X$;
2. for any $x \in X : w(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z,x) \le \delta$ and $w(z,y) \le \delta$ implies $d(x,y) \le \varepsilon$.

Let us give some examples of w-distance (see [3])

Example 1. Let (X,d) be a metric space. Then the metric "d" is a w-distance on X.

Example 2. Let X be a normed linear space with norm $||\cdot||$. Then the function $w : X \times X \to [0,\infty)$ defined by $w(x,y) = ||x|| + ||y||$ for every $x, y \in X$ is a w-distance.

Example 3. Let (X,d) be a metric space and let $g : X \to X$ a continuous mapping. Then the function $w : X \times Y \to [0,\infty)$ defined by:

$$w(x,y) = \max\{d(g(x),y), d(g(x),g(y))\}$$

for every $x, y \in X$ is a w-distance.

For the proof of the main results we need the following crucial result for w-distance (see [6]).

Lemma 1. *Let* (X,d) *be a metric space, and let* w *be a* w*-distance on* X. *Let* (x_n) *and* (y_n) *be two sequences in* X, *let* (α_n), (β_n) *be sequences in* $[0,+\infty[$ *converging to zero and let* $x, y, z \in X$. *Then the following hold:*

(a) *If* $w(x_n, y) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$.
(b) *If* $w(x_n, y_n) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* (y_n) *converges to* z *in* (X, d).
(c) *If* $w(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* (x_n) *is a Cauchy sequence in* (X, d).
(d) *If* $w(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* (x_n) *is a Cauchy sequence.*

From [6] we undertake the definition of the weakly contractive multivalued operators as follows.

Definition 1. *Let* (X, d) *be a metric space and* $Y \subseteq X$. *A multivalued operator* $T : Y \to P(X)$ *is called a-weakly contractive if there exists a* w*-distance* w *on* X *and* $a \in [0, 1)$ *such that for any* $x_1, x_2 \in Y$ *and* $y_1 \in T(x_1)$ *there is* $y_2 \in T(x_2)$ *with* $w(y_1, y_2) \leq aw(x_1, x_2)$.

In [2] we have the definition of weakly φ-contractive operators as follows.

Definition 2. *Let* (X, d) *be a metric space and* $Y \subseteq X$. *Then* $T : Y \to P(X)$ *is called weakly* φ*-contractive if there exists a* w*-distance,* w*, on* X *and a function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for every* x_1, x_2 *and* $y_1 \in T(x_1)$ *there is* $y_2 \in T(x_2)$ *with* $w(y_1, y_2) \leq \varphi(w(x_1, x_2))$.

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function (see [5]) if it is increasing and $\varphi^k(t) \to 0$, as $k \to \infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$.

Also recall the notion of strict comparison function. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function (see [5]) if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$, for each $t > 0$.

3. Data dependence for weakly φ-contractive multivalued operators

In [2] we find the following fixed points result with respect to weakly φ contractive operators (Theorem 4.2, [2]).

Theorem 2. *Let* (X, d) *be a complete metric space,* $w : X \times X \to \mathbb{R}_+$ *a* w*-distance on* X*,* $T : X \to P_{cl}(X)$ *a multivalued operator and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a function such that are accomplish the following conditions:*
(a) T *are weakly* φ*-contractive operator;*
(b) *The function* φ *is a monotone increasing function such that*

$$\sigma(t) := \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for every } t \in \mathbb{R}_+ \setminus \{0\}.$$

Then there exists $x^* \in X$ *such that* $x^* \in T(x^*)$ *and* $w(x^*, x^*) = 0$.

The main result of this section is the following data dependence theorem with respect to the above theorem.

Theorem 3. *Let (X, d) be a complete metric space, $T_1, T_2 : X \to P_{cl}(X)$ be two weakly φ-contractive multivalued operators. Then the following are true:*

(a) $FixT_1 \neq \emptyset \neq FixT_2$;

(b) We suppose that there exists $\eta > 0$ such that for every $u \in T_1(x)$ there exists $v \in T_2(x)$ such that $w(u, v) \leq \eta$, (respectively for every $v \in T_2(x)$ there exists $u \in T_1(x)$ such that $w(v, u) \leq \eta$).

Then for every $u^* \in FixT_1$ there exists $v^* \in FixT_2$ such that

$$w(u^*, v^*) \leq \sigma(\eta)$$

(respectively for every $v^* \in FixT_2$ there exists $u^* \in FixT_1$ such that

$$w(v^*, u^*) \leq \sigma(\eta)).$$

Proof. Let $u_0 \in FixT_1$, then $u_0 \in T_1(u_0)$. Using the hypothesis (2) we have that there exists $u_1 \in T_2(u_0)$ such that $w(u_0, u_1) \leq \eta$.

Since T_1, T_2 are weakly φ-contractive, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function we have that for every $u_0, u_1 \in X$ with $u_1 \in T_2(u_0)$ there exists $u_2 \in T_2(u_1)$ such that

$$w(u_1, u_2) \leq \varphi(w(u_0, u_1))$$

For $u_1 \in X$ and $u_2 \in T_2(u_1)$ there exists $u_3 \in T_2(u_2)$ such that

$$w(u_2, u_3) \leq \varphi(w(u_1, u_2)) \leq \varphi^2(w(u_0, u_1))$$

By induction we obtain a sequence $(u_n)_{n \in \mathbb{N}} \in X$ such that

(a) $u_{n+1} \in T_2(u_n)$, for every $n \in \mathbb{N}$;

(b) $w(u_n, u_{n+1}) \leq \varphi^n(w(u_0, u_1))$

For $n, m \in \mathbb{N}$, with $m > n$ we have the inequality

$$\begin{aligned} w(u_n, u_m) &\leq w(u_n, u_{n+1}) + w(u_{n+1}, u_{n+2}) + \cdots + w(u_{m-1}, u_m) \\ &\leq < \varphi^n w(u_0, u_1) + \varphi^{n+1} w(u_0, u_1) + \cdots + \varphi^{m-1} w(u_0, u_1) \\ &\leq \sum_{k=n}^{\infty} \varphi^k(w(x_0, x_1)) \leq \sigma(w(x_0, x_1)) \end{aligned}$$

By the Lemma 1 (c) we have that the sequence $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space we have that there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$.

By the lower semicontinuity of $w(x,\cdot) : X \to [0,\infty)$ we have

$$(1) \qquad w(u_n, v^*) \leq \lim_{m\to\infty} \inf w(u_n, u_m) \leq \sigma(\varphi^n(w(x_0, x_1)))$$

For $u_{n-1}, v^* \in X$ and $u_n \in T_2(u_{n-1})$ there exists $z_n \in T_2(v^*)$ such that, using relation (1), we have

$$(2) \qquad w(u_n, z_n) \leq \varphi(w(u_{n-1}, v^*)) \leq ... \leq \sigma(\varphi^{n-1}(w(x_0, x_1)))$$

Applying Lemma 1 (b), from relations (1) and (2) we have that $z_n \xrightarrow{d} v^*$.

Then, we know that $z_n \in T_2(v^*)$ and $z_n \xrightarrow{d} v^*$. In this case, by the closure of T_2 result that $v^* \in T_2(v^*)$. Then, by $w(u_n, v^*) \leq \sigma(\varphi^n(w(x_0, x_1)))$, with $n \in \mathbb{N}$, for $n = 0$ we obtain

$$w(u_0, v^*) \leq \sigma(w(x_0, x_1)) \leq \sigma(\eta)$$

which complete the proof. ∎

4. Existence of fixed points for non-self multivalued weakly contractive operators

Let (X, d) be a metric space, w be a w-distance on X $x_0 \in X$ and $r > 0$. Let us define:

$B_w(x_0; r) := \{x \in X | w(x_0, x) < r\}$ the open ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}^d(x_0; r)$- the closure in (X, d) of the set $B_w(x_0; r)$.

Theorem 4. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(X, d)$ be a multivalued operator such that:*

(i) T is a-weakly contractive;

(ii) $D_w(x_0, T(x_0)) < (1 - a)r$;

Then T has a fixed point.

Proof. Let $0 < s < r$ such that $B_w(x_0; s) \subset B_w(x_0; r)$ and $D_w(x_0, T(x_0)) < (1 - a)s < (1 - a)r$.

Then there exists $x_1 \in T(x_0)$ such that $w(x_0, x_1) < (1 - a)s \leq s$. Hence $x_1 \in B_w(x_0; s)$.

For $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$w(x_1, x_2) \leq aw(x_0, x_1) \leq a(1 - a)s.$$

Then $w(x_0, x_2) \leq w(x_0, x_1)+w(x_1, x_2) < (1-a)s+a(1-a)s = (1-a^2)s \leq s$. Hence $x_2 \in B_w(x_0; s)$.

For $x_1 \in B_w(x_0; s)$ and $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that

$$w(x_2, x_3) \leq aw(x_1, x_2) \leq a^2 w(x_0, x_1) \leq a^2(1-a)s$$

Then $w(x_0, x_3) \leq w(x_0, x_2) + w(x_2, x_3) < (1-a^2)s + a^2(1-a)s = (1-a)(1+a+a^2)s = (1-a^3)s < s$. Hence $x_3 \in B_w(x_0; s)$.

By induction we obtain in this way a sequence $(x_n)_{n\in\mathbb{N}} \in B_w(x_0; s)$ with the following properties:

(*a*) $x_n \in T(x_{n-1})$, for each $n \in \mathbb{N}$;

(*b*) $w(x_n, x_{n+1}) \leq a^n(1-a)s$, for each $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} w(x_n, x_m) &\leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + ... + w(x_{m-1}, x_m) \\ &\leq a^n(1-a)s + a^{n+1}(1-a)s + ... + a^{m-1}(1-a)s \\ &\leq \frac{a^n}{1-a}(1-a)s = a^n s. \end{aligned}$$

Using Lemma 1 (*c*) we have that $(x_n)_{n\in\mathbb{N}} \in B_w(x_0; s)$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit $x^* \in \widetilde{B_w}^d(x_0; s)$.

Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}} \in B_w(x_0; s)$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous we have

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq a^n s, \text{ for every } n \in \mathbb{N}.$$

For $x^* \in \widetilde{B_w}^d(x_0; s)$ and $x_n \in T(x_{n-1})$, $n \in \mathbb{N}^*$, there exists $u_n \in T(x^*)$ such that

$$w(x_n, u_n) \leq aw(x_{n-1}, x^*) \leq ... \leq aa^{n-1}s = a^n s, \text{ for every } n \in \mathbb{N}.$$

So, we have the following two relations:

$$w(x_n, x^*) \leq a^n s, \text{ for every } n \in \mathbb{N}.$$
$$w(x_n, u_n) \leq a^n s, \text{ for every } n \in \mathbb{N}.$$

Then, by Lemma 1 (*b*) we obtain that $u_n \xrightarrow{d} x^*$. As $u_n \in T(x^*)$ for each $n \in \mathbb{N}$ and using that T has closed values in (X, d) it follows that $x^* \in T(x^*)$ Then T has a fixed point. ∎

Let us present another fixed point result. For a related result, see [4].

Theorem 5. *Let (X, d) be a complete metric space, $T : \widetilde{B_w}^d(x_0; r) \to P_{cl}(X, d)$ be a multivalued operator such that:*

(i) T is weakly φ-contractive operator, where $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a strict comparison function such that the function $\psi(t):=t-\varphi(t)$ is strictly increasing, continuous in r and $\sum_{n=1}^{\infty}\varphi^n(\psi(s))\le\varphi(s)$, for each $s\in]0,r[$;

(ii) $D_w(x_0,T(x_0))<r-\varphi(r)$;

Then T has a fixed point.

Proof. Let $0<s<r$ such that $B_w(x_0;s)\subset B_w(x_0;r)$ and $D_w(x_0,T(x_0))<s-\varphi(s)<r-\varphi(r)$. For $D_w(x_0,T(x_0))<s-\varphi(s)$ there exists $x_1\in T(x_0)$ such that $w(x_0,x_1)<s-\varphi(s)=\psi(s)$.

For $x_1\in T(x_0)$ there exists $x_2\in T(x_1)$ such that

$$w(x_1,x_2)\le\varphi(w(x_0,x_1))\le\varphi(\psi(s)).$$

Then $w(x_0,x_2)\le w(x_0,x_1)+w(x_1,x_2)<\psi(s)+\varphi(\psi(s))<s$. Hence $x_2\in B_w(x_0;s)$.

For $x_1\in B_w(x_0;s)$ and $x_2\in T(x_1)$ there exists $x_3\in T(x_2)$ such that

$$w(x_2,x_3)\le\varphi(w(x_1,x_2))\le\varphi^2(w(x_0,x_1))\le\varphi^2(\psi(s)).$$

Then $w(x_0,x_3)\le w(x_0,x_2)+w(x_2,x_3)<\psi(s)+\varphi(\psi(s))+\varphi^2(\psi(s))<s$. Hence $x_3\in B_w(x_0;s)$.

By induction we obtain in this way a sequence $(x_n)_{n\in\mathbb{N}}\in B_w(x_0;s)$ with the following properties:

(1) $x_n\in T(x_{n-1})$, for each $n\in\mathbb{N}$;

(2) $w(x_n,x_{n+1})\le\varphi^n(w(x_0,x_1))<\varphi^n(\psi(s))$, for each $n\in\mathbb{N}$.

For $m,n\in\mathbb{N}$ with $m>n$ we have

$$\begin{aligned}
w(x_n,x_m) &\le w(x_n,x_{n+1})+w(x_{n+1},x_{n+2})+...+w(x_{m-1},x_m)\\
&\le \varphi^n(w(x_0,x_1))+\varphi^{n+1}(w(x_0,x_1))+...+\varphi^{m-1}(w(x_0,x_1))\\
&< \varphi^n(\psi(s))+\varphi^{n+1}(\psi(s))+...+\varphi^{m-1}(\psi(s))\\
&\le \sum_{k=n}^{\infty}\varphi^k(\psi(s))
\end{aligned}$$

Letting $n\to\infty$ we have

$$\lim_{n\to\infty}w(x_n,x_m)\le\sum_{k=1}^{\infty}\varphi^k(\psi(s))-\lim_{n\to\infty}\sum_{k=1}^{n}\varphi^k(\psi(s))=0.$$

Using Lemma 1 (c) we have that $(x_n)_{n\in\mathbb{N}}\in B_w(x_0;s)$ is a Cauchy sequence in (X,d). Since (X,d) is a complete metric space we have that the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit $x^*\in\widetilde{B_w}^{d}(x_0;s)$.

Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}} \in B_w(x_0; s)$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous we have

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \lim_{n\to\infty} (\sum_{k=n}^{\infty} \varphi^k(\psi(s))) = 0$$

For $x^* \in \widetilde{B_w}^d(x_0; s)$ and $x_n \in T(x_{n-1})$, $n \in \mathbb{N}^*$ there exists $u_n \in T(x^*)$ such that

$$w(x_n, u_n) \leq \varphi(w(x_{n-1}, x^*)) \leq ... \leq \varphi^n(w(x_0, x_1)) < \varphi^n(\psi(s))$$

So, we have that:

$$\begin{aligned} w(x_n, x^*) &< \sum_{k=n}^{\infty} \varphi^k(\psi(s)) \\ w(x_n, u_n) &< \varphi^n(\psi(s)). \end{aligned}$$

Then, by Lemma 1 (b) we obtain that $u_n \xrightarrow{d} x^*$. As $u_n \in T(x^*)$ for each $n \in \mathbb{N}$ and using that T has closed values in (X, d) it follows that $x^* \in T(x^*)$. Then T has a fixed point. ■

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